Homotopy II : Exam

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Duration : 3 hours. Printed or handwritten lecture notes are allowed. Electronic devices are forbidden. Please read the whole exam before starting.

Exercice 1. Let C be a category and $\mathcal{W}$ a class of morphisms. One says that $\mathcal{W}$ satisfies property "2 out of 6" (2P6) if, given three composable morphisms f, g, and h,

$$\{h \circ g, g \circ f\} \subseteq \mathcal{W} \implies \{f, g, h, h \circ g \circ f\} \subseteq \mathcal{W}.$$

(1A) Prove that if $\mathcal{W}$ satisfies 2P6 and $\forall X, \mathrm{id}_X \in \mathcal{W}$, then $\mathcal{W}$ contains all isomorphisms.

(1B) Prove that if $\mathcal{W}$ satisfies 2P6 then it satisfies MC2 ("2 out of 3").

(1C) Prove that if $\mathcal{W}$ satisfies MC2 and $\{h \circ g, g \circ f\} \subseteq \mathcal{W} \implies g \in \mathcal{W}$, then $\mathcal{W}$ satisfies 2P6.

(1D) Prove that the class of isomorphisms in an arbitrary category satisfies 2P6.

(1E) Deduce that the weak equivalences of a model category satisfy 2P6.

Exercice 2. Let C be a category equipped with two model structures $(\mathcal{W}_1, \mathcal{C}_1, \mathcal{F}_1)$ et $(\mathcal{W}_2, \mathcal{C}_2, \mathcal{F}_2)$. Assume that $\mathcal{W}_1 \subseteq \mathcal{W}_2$ et $\mathcal{F}_1 \subseteq \mathcal{F}_2$. We call "mixed structure" $(\mathcal{W}_m, \mathcal{C}_m, \mathcal{F}_m)$ defined by $\mathcal{W}_m = \mathcal{W}_2$ et $\mathcal{F}_m = \mathcal{F}_1$. The mixed cofibrations, $\mathcal{C}_m$, are defined by a lifting property.

(2A) Prove that $\mathcal{C}_2 \subseteq \mathcal{C}_m \subseteq \mathcal{C}_1$.

(2B) Prove that $\mathcal{C}_m \cap \mathcal{W}_m = \mathcal{C}_1 \cap \mathcal{W}_1$. (Indication : MC3+MC5.)

(2C) Prove that the mixed structure is a model structure.

(2D) One says that f is a *special* mixed cofibration if there exists $i \in \mathcal{C}_2$ and $j \in \mathcal{C}_1 \cap \mathcal{W}_1$ such that $f = j \circ i$. Prove that any special mixed cofibration is a mixed cofibration, and that any mixed cofibration is a retract of a special mixed cofibration.

(2E) One says that a model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. Deduce from (2D) that if structure 2 is left proper, then so is the mixed structure.

(2F) Between which ones of the three model structures is id_{C} a left or right Quillen adjoint ?

Exercice 3. A Reedy category is a category R equipped with two subcategories $\vec{\mathsf{R}}$ and $\overleftarrow{\mathsf{R}}$ that contain all objects and a map $\deg : \mathrm{ob}\,\mathsf{R} \to \mathbb{N}$ such that :

- if $f \in \vec{\mathsf{R}}(\alpha, \beta)$, then ($\alpha = \beta$ and $f = \mathrm{id}_\alpha$) or $\deg \beta > \deg \alpha$;
- if $f \in \overleftarrow{\mathsf{R}}(\alpha, \beta)$, then ($\alpha = \beta$ and $f = \mathrm{id}_\alpha$) or $\deg \beta < \deg \alpha$;
- any morphism f factors uniquely as $\vec{f} \circ \overleftarrow{f}$ where $\vec{f} \in \vec{\mathsf{R}}$ and $\overleftarrow{f} \in \overleftarrow{\mathsf{R}}$.

(3A) Let $\mathsf{R}_{\leq n}$ be the subcategory of objects of degree $\leq n$. Prove that $\mathsf{R}_{\leq 0}$ is discrete.

(3B) Prove that a finite partially ordered set is a Reedy category. Prove that the simplex category Δ is Reedy, where $\vec{\Delta}$ is composed of injections, $\overleftarrow{\Delta}$ of surjections, and $\deg = \mathrm{id}_{\mathbb{N}}$. Prove that the opposite of a Reedy category is Reedy. Montrer que la catégorie opposée d'une catégorie de Reedy est de Reedy.

Let $\alpha \in \mathsf{R}$. The *latching* category $L_\alpha\mathsf{R}$ has as objects the morphisms $f \in \vec{\mathsf{R}}(\beta, \alpha)$ où $\beta \neq \alpha$. If $f : \beta \to \alpha$, $f' : \beta' \to \alpha$, then $\mathrm{Hom}_{L_\alpha\mathsf{R}}(f, f') = \{g \in \vec{\mathsf{R}}(\beta, \beta') \mid f'g = f\}$. Dually, the objects of the *matching* category $M_\alpha\mathsf{R}$ are the morphisms $f \in \overleftarrow{\mathsf{R}}(\alpha, \beta)$ où $\beta \neq \alpha$.

(3C) Describe $L_{[2]}\Delta^{\mathrm{op}}$ and $R_{[2]}\Delta^{\mathrm{op}}$.

Let $X \in \mathsf{C}^{\mathsf{R}}$ be a diagram indexed by R, where C is a (co)complete category. We define the *latching* objects by the colimits $L_\alpha X := \mathrm{colim}_{f:\beta\to\alpha\in L_\alpha\mathsf{R}} X_\beta$. Dually, its *matching* objects are $M_\alpha X := \lim_{f:\alpha\to\beta\in M_\alpha\mathsf{R}} X_\beta$. (The empty colimit is the initial object, the empty limit is the terminal object.)

(3D) Let $X_\bullet \in \mathsf{Set}^{\Delta^{\mathrm{op}}}$ be a simplicial set. Describe $M_{[n]}X_\bullet$ and $L_{[n]}X_\bullet$ for $n \leq 2$.

(3E) Let $X : \mathsf{R}_{\leq n-1} \to \mathsf{C}$ be a diagram (where $n \geq 1$) and $\alpha \in \mathsf{C}$ an element of degree n. Check that $L_\alpha X$ and $M_\alpha X$ are still well-defined and construct morphisms $c_\alpha : L_\alpha X \to M_\alpha X$ natural in α.

(3F) Let $X : \mathsf{R}_{\leq n-1} \to \mathsf{C}$ be a diagram (where $n \geq 1$). Prove that the data of an extension of X to $\mathsf{R}_{\leq n}$ is equivalent to the data of objects X_α for each α of degree n and of morphisms $l_\alpha : L_\alpha X \to X_\alpha$ and $m_\alpha : X_\alpha \to M_\alpha X$ such that $m_\alpha l_\alpha = c_\alpha$.

(3G) Let $X, Y \in \mathsf{C}^{\mathsf{R}}$ be two diagrams and $\varphi : X \Rightarrow Y$ a natural transformation. Construct morphisms $L^{\mathrm{rel}}_\alpha\varphi : X_\alpha \cup_{L_\alpha X} L_\alpha Y \to Y_\alpha$ and $M^{\mathrm{rel}}_\alpha\varphi : X_\alpha \to M_\alpha X \times_{M_\alpha Y} Y_\alpha$ natural in α. (One can use the morphisms m_α, l_α constructed in (3F).)

We now assume that C is a model category. We define a model structure (called the Reedy structure) on C^{R} by setting that a natural transformation φ is : a Reedy equivalence if each φ_α is a weak equivalence ; a Reedy cofibration if each $L^{\mathrm{rel}}_\alpha\varphi$ is a cofibration ; a Reedy fibration if each $M^{\mathrm{rel}}_\alpha\varphi$ is a fibration.

(3H) Let $\phi : X \to Y$ be a Reedy cofibration. Prove that the induced morphism $L_\alpha X \to L_\alpha Y$ is a cofibration. (Hint : prove that it has the LLP with respect to acyclic fibrations by construction the lift by induction.)

(3I) Deduce that if $\varphi : X \to Y$ is a Reedy cofibration, then φ_α is a cofibration for every α. (Hint : use the fact that φ_α factors as $X_\alpha \to X_\alpha \cup_{L_\alpha X} L_\alpha Y \xrightarrow{L^{\mathrm{rel}}_\alpha\varphi} Y_\alpha$.)

(3J) Deduce the existence of Quillen adjunctions between the Reedy structure and projective/injective structures, if they exist.

(3K) Prove that a Quillen adjunction $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ induces Quillen adjunctions between the Reedy structures.

Exercice 4. Let A be a 1-connected CDGA and $\alpha, \beta, \gamma \in H^*(A)$ be three classes such that $\alpha\beta = \beta\gamma = 0$. We consider the set of elements of the form $vz - (-1)^{|x|}xw \in A$ where $\alpha = [x]$, $\beta = [y]$, $\gamma = [z]$ (for cocycles x, y, z), $dv = xy$ and $dw = yz$.

(4A) Prove that $vz - (-1)^{|x|}xw$ is a cocycle and that its class in $H^*(A)/I$, where $I = (\alpha, \gamma)$ is the ideal generated by α and γ, is independent of the choices of x, y, z, v, w.

(4B) Assume that A is quasi-isomorphic to $H^*(A)$. let M be the minimal model of A. Why does there exist a direct quasi-isomorphism $\phi : M \to H^*(A)$?

(4C) Use ϕ to prove that the class of $vz - (-1)^{|x|}xw$ is zero in $H^*(A)/I$.

(4D) Deduce from this an example of two CDGAs with the same cohomology that are not quasi-isomorphic.