CONFIGURATION SPACES AND OPERADS

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Configuration spaces

$M$: $n$-manifold

$$\text{Conf}_r(M) := \{(x_1, \ldots, x_r) \in M^r \mid \forall i \neq j, \ x_i \neq x_j\}$$

- Braid groups
- Loop spaces
- Moduli spaces of curves
- Particles in movement [physics]
- Motion planning [robotics]
**Open question**

**Question**
Does the homotopy type of $M$ determine the homotopy type of $\text{Conf}_r(M)$? How to compute homotopy invariants of $\text{Conf}_r(M)$?

**Non-compact manifolds**
False: $\text{Conf}_2(\mathbb{R}) \not\sim \text{Conf}_2(\{0\})$ even though $\mathbb{R} \sim \{0\}$.

**Closed manifolds**
Longoi–Salvatore (2005): counter-example (lens spaces)... but not simply connected.

**Simply connected closed manifolds**
Homotopy invariance is still open.

We can also localize: $M \simeq_{\mathbb{Q}} N \implies \text{Conf}_r(M) \simeq_{\mathbb{Q}} \text{Conf}_r(N)$?
Configurations in a Euclidean spaces

Presentation of $H^*(\text{Conf}_r(\mathbb{R}^n))$ [Arnold, Cohen]

- Generators: $\omega_{ij}$ of degree $n - 1$ (for $1 \leq i \neq j \leq r$)
- Relations:

\[
\omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0
\]

Theorem (Arnold 1969)

Formality: $H^*(\text{Conf}_r(\mathbb{C})) \sim \mathbb{C} \Omega^*_{dR}(\text{Conf}_r(\mathbb{C})), \omega_{ij} \mapsto d \log(z_i - z_j)$.

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

$H^*(\text{Conf}_r(\mathbb{R}^n)) \sim \mathbb{R} \Omega^*_{dR}(\text{Conf}_r(\mathbb{R}^n))$ for all $r \geq 0$ and $n \geq 2$.

Corollary

The cohomology of $\text{Conf}_r(\mathbb{R}^n)$ determines its rational homotopy type.
Arnold relations: \[ R_{123} = \omega_{12}\omega_{23} + \omega_{23}\omega_{31} + \omega_{31}\omega_{12} \]

\[ \implies H^\ast(\text{Conf}_r(\mathbb{R}^n)) = \mathbb{R}\langle \text{graphs with } r \text{ vertices} \rangle/(R_{ijk}) \]

\[ \leadsto \text{add “internal” vertices and a differential which contracts edges incident to these new vertices:} \]

Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 1)

We get a quasi-free CDGA \( \text{Graphs}_n(r) \) and a quasi-isomorphism \( \text{Graphs}_n(r) \overset{\sim}{\longrightarrow} H^\ast(\text{Conf}_r(\mathbb{R}^n)). \)
The relations $R_{ijk}$ are only satisfied up to homotopy in $\Omega^*(\text{Conf}_r(\mathbb{R}^n))$. How to find representatives to get $\text{Graphs}_n(r) \xrightarrow{\sim} \Omega^*(\text{Conf}_r(\mathbb{R}^n))$?

Let $\varphi \in \Omega^{n-1}(\text{Conf}_2(\mathbb{R}^n))$ be the volume form. For $\Gamma \in \text{Graphs}_n(r)$ with $i$ internal vertices:

$$\omega(\Gamma) := \int_{\text{Conf}_{r+i}(\mathbb{R}^n) \to \text{Conf}_r(\mathbb{R}^n)} \bigwedge_{(ij) \in E_\Gamma} \varphi_{ij}.$$ 

**Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 2)**

We get a quasi-isomorphism $\omega : \text{Graphs}_n(r) \xrightarrow{\sim} \Omega(\text{Conf}_r(\mathbb{R}^n))$.

⚠️ I’m cheating! We have to compactify $\text{Conf}_r(\mathbb{R}^n)$ to make sure $\int$ converges and to apply the Stokes formula correctly.
Problem: \( \text{Conf}_r(\mathbb{R}^n) \) is not compact.

Fulton–MacPherson compactification \( \text{Conf}_r(M) \hookrightarrow \text{FM}_M(r) \)

\( M \) closed manifold \( \rightarrow \) semi-algebraic stratified manifold \( \dim = nr \)
Animation #2
Animation #3
Compactification of $\text{Conf}_r(\mathbb{R}^n)$

We have to “normalize” $\text{Conf}_r(\mathbb{R}^n)$ to mitigate the non-compacity of $\mathbb{R}^n$:

$$\text{Conf}_r(\mathbb{R}^n) \sim \text{Conf}_r(\mathbb{R}^n)/(\mathbb{R}^n \times \mathbb{R}_{>0}) \sim \text{FM}_n(r)$$

$\implies$ semi-algebraic stratified manifold $\text{dim} = nr - n - 1$
We see a new structure on $\text{FM}_n$: an operad! We can “insert” an infinitesimal configuration in another one:

\[
\text{FM}_n(k) \times \text{FM}_n(l) \xrightarrow{\circ_i} \text{FM}_n(k + l - 1), \quad 1 \leq i \leq k
\]

**Remark**

Weakly equivalent to the “little disks operad”.
Functoriality \( \implies H^*(\text{FM}_n) = H^*(\text{Conf}_n(\mathbb{R}^n)) \) and \( \Omega^*(\text{FM}_n) \) are Hopf cooperads; \( \text{Graphs}_n \) is one too, and:

**Theorem (Kontsevich 1999, Lambrechts–Volić 2014)**

The operad \( \text{FM}_n \) is formal over \( \mathbb{R} \):

\[
\Omega^*(\text{FM}_n) \xleftarrow{\omega} \text{Graphs}_n \xrightarrow{\sim} H^*(\text{FM}_n).
\]

Formality has important applications, e.g. Deligne conjecture, deformation quantization of Poisson manifolds, etc.

(Note: \( H_*(\text{FM}_n) \) governs Poisson \( n \)-algebras for \( n \geq 2 \).)
The Lambrechts–Stanley model

\( M \): oriented closed manifold
\( A \sim \Omega(M) \): Poincaré duality model of \( M \)

\( G_A(r) \): (conjectural) model of \( \text{Conf}_r(M) = M^{\times r} \setminus \bigcup_{i \neq j} \Delta_{ij} \)

- “Generators”: \( A^{\otimes r} \) and the \( \omega_{ij} \) from \( \text{Conf}_r(\mathbb{R}^n) \)
- Arnold relations + symmetry
- \( d\omega_{ij} \) kills the dual of \([\Delta_{ij}]\).

Examples:

- \( G_A(0) = \mathbb{R} \) is a model of \( \text{Conf}_0(M) = \{\emptyset\} \)  ✓
- \( G_A(1) = A \) is a model of \( \text{Conf}_1(M) = M \)  ✓
- \( G_A(2) \sim A^{\otimes 2}/(\Delta_A) \) should be a model of \( \text{Conf}_2(M) = M^2 \setminus \Delta ? \)
- \( r \geq 3 \): more complicated.
**Brief history of $G_A$**

1969 [Arnold, Cohen] $H^*(\text{Conf}_r(\mathbb{R}^n)) = G_{H^*(\mathbb{R}^n)}(r)$

1978 [Cohen–Taylor] spectral sequence $E^2 = G_{H^*(M)}(k) \Rightarrow H^*(\text{Conf}_k(M))$

1994 For smooth projective complex manifolds (Kähler):
- [Kříž] $G_{H^*(M)}(r)$ is a model of $\text{Conf}_r(M)$;

2004 [Lambrechts–Stanley] model for $r = 2$ if $\pi_{\leq 2}(M) = 0$


2008 [Lambrechts–Stanley] $H^i(G_A(r)) \cong \Sigma^r_{\text{-Vect}} H^i(\text{Conf}_r(M))$

2015 [Cordova Bulens] model for $r = 2$ if $\dim M = 2m$
By generalizing the proof of Kontsevich & Lambrechts–Volić:

**Theorem (I.)**

Let $M$ be a closed simply connected smooth manifold and $A$ be any Poincaré duality model of $M$. Then $G_A(r)$ is a real model of $\text{Conf}_r(M)$.

**Corollary (cf. Campos–Willwacher)**

$M \sim_\mathbb{R} N \implies \text{Conf}_r(M) \sim_\mathbb{R} \text{Conf}_r(N)$ for all $r$.

We can “compute everything” over $\mathbb{R}$ for $\text{Conf}_r(M)$.

**Remark**

$\dim M \leq 3$: only spheres (Poincaré conjecture) and we know that $G_A$ is a model anyway, but adapting the proof is problematic!
$M$ parallelized $\Rightarrow$ $FM_M = \{FM_M(r)\}_{r \geq 0}$ is a right $FM_n$-module:

We can rewrite:

$$G_A(r) = (A^\otimes r \otimes H^*(FM_n(r))/\text{relations, } d)$$

A bit of abstract nonsense:

**Proposition**

$\chi(M) = 0 \implies G_A = \{G_A(r)\}_{r \geq 0}$ is a Hopf right $H^*(FM_n)$-comodule.
## Complete version of the theorem

### Theorem (I. 2018)

Let $M$ be a closed simply connected smooth manifold with $\dim M \geq 4$. Then

\[
G_A \xleftarrow{\sim} \text{Graphs}_R \xrightarrow{\sim} \Omega_{PA}^*(FM_M)
\]

\[
H^*(FM_n) \xleftarrow{\sim} \text{Graphs}_n \xrightarrow{\sim} \Omega_{PA}^*(FM_n)
\]

\[\dagger\text{ if } \chi(M) = 0\]
\[\ddagger\text{ if } M \text{ is parallelized.}\]

\[A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega_{PA}^*(M)\]

### Conclusion

Not only do we have a model of each $\text{Conf}_r(M)$, but also of their richer structure if we look at them all at once.
Space of embeddings: \( \text{Emb}(M, N) = \{ f : M \hookrightarrow N \} \).

Goodwillie–Weiss manifold calculus [Arone, Boavida, Turchin, Weiss...]: for parallelized manifolds of codimension \( \geq 3 \),

\[
\text{Emb}(M, N) \simeq \text{Mor}^h_{\text{Conf}^\bullet(\mathbb{R}^n)}(\text{Conf}^\bullet(M), \text{Conf}^\bullet(N)).
\]

LS model is small and explicit \( \implies \) hope: computations are tractable

**Remark**

Requires to compare \( \text{Mor}^h_{\text{Conf}^\bullet(\mathbb{R}^n)}(\text{Conf}^\bullet(M), \text{Conf}^\bullet(N))^\mathbb{R} \) with \( \text{Mor}^h_{\text{Conf}^\bullet(\mathbb{R}^n)^\mathbb{R}}(\text{Conf}^\bullet(M)^\mathbb{R}, \text{Conf}^\bullet(N)^\mathbb{R}) \)
Factorization homology = homology where $\otimes$ replaces $\oplus$ + homotopy commutative coefficients.

For an $E_n$-algebra $\mathcal{A}$,

$$\int_M \mathcal{A} = \text{hocolim} (D^n) \sqcup r \to M \mathcal{A} \otimes r.$$

Alternate description: $\int_M \mathcal{A} \sim \text{Conf}_\bullet(M) \otimes^h_{\text{Conf}_\bullet(\mathbb{R}^n)} \mathcal{A}$ [Francis].


$M$ closed simply connected smooth manifold ($\text{dim} \geq 4$),

$\mathcal{A} := \mathcal{O}_{\text{poly}}(T^*\mathbb{R}^d[1-n]) \implies \int_M \mathcal{A} \sim_{\mathbb{R}} \mathbb{R}.$
**Generalization 1: Manifolds with boundary**

**Theorem (Campos–I.–Lambrechts–Willwacher 2018)**

For manifolds with boundary: homotopy invariance of $\text{Conf}_r(-)$, generalization of the Lambrechts–Stanley model (and more); under good conditions, including $\dim M \geq \ldots$

**Remark**

Poincaré duality models $\sim$ Poincaré–Lefschetz duality models.

Allows to compute $\text{Conf}_r$ by “induction”: 

![Diagram](image-url)
Generalization 2: Oriented manifolds

$M$: oriented manifold $\sim$ framed configuration space

$$\text{Conf}^{fr}_r(M) := \{(x \in \text{Conf}_r(M), B_1, \ldots , B_r) \mid B_i: \text{orth. basis of } T_{x_i}M\}.$$ 

Natural action of the framed little disks operad on $\{\text{Conf}^{fr}_\bullet(M)\}$.


Real model of this module based on graph complexes.

First step towards embedding spaces of non-parallelized manifolds. (Not enough: need partially framed configurations for the larger manifold $N$.)
Goal: $\text{Conf}(N \setminus M)$ where $\dim N - \dim M \geq 2$.

Motivation: work of Ayala, Francis, Rozenblyum, Tanaka
Knot complement $\leadsto$ colored Jones polynomial.

There exists an operad $\text{VSC}_{mn}$ which models the local situation $\mathbb{R}^n \setminus \mathbb{R}^m$:

\[ \in VSC_{13}(2, 2) \subset D_3(2 + 2) \]

Theorem (I. 2018)
The operad $\text{VSC}_{mn}$ is formal over $\mathbb{R}$ for $n - m \geq 2$. 
Thank you for your attention!

These slides: https://idrissi.eu