Configuration spaces and Operads

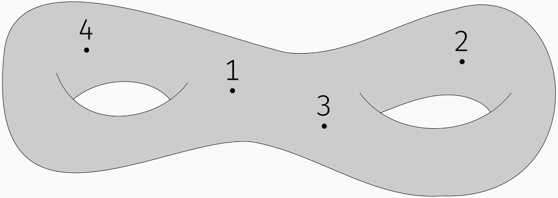
Najib Idrissi (in part j/w Campos, Ducoulombier, Lambrechts, Willwacher)

January 2019 @ Higher Structures, CIRM

Configuration spaces

M: n-manifold

$$\mathrm{Conf}_r(M) := \{(x_1, \ldots, x_r) \in M^r \mid \forall i \neq j,\ x_i \neq x_j\}$$



(name dropping)

- Braid groups
- Loop spaces
- Moduli spaces of curves
- Particles in movement [physics]
- Motion planning [robotics]

Question

Does the homotopy type of M determine the homotopy type of $\mathrm{Conf}_r(M)$? How to compute homotopy invariants of $\mathrm{Conf}_r(M)$?

Non-compact manifolds

False: $\mathrm{Conf}_2(\mathbb{R}) \not\sim \mathrm{Conf}_2(\{0\})$ even though $\mathbb{R} \sim \{0\}$.

Closed manifolds

Longoni–Salvatore (2005): counter-example (lens spaces)... but not simply connected.

Simply connected closed manifolds

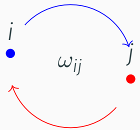
Homotopy invariance is still open.

We can also localize: $M \simeq_{\mathbb{Q}} N \implies \mathrm{Conf}_r(M) \simeq_{\mathbb{Q}} \mathrm{Conf}_r(N)$?

Presentation of $H^*(\mathrm{Conf}_r(\mathbb{R}^n))$ [Arnold, Cohen]

- Generators: ω_{ij} of degree $n-1$ (for $1 \leq i \neq j \leq r$)
- Relations:

$$\omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$$



Theorem (Arnold 1969)

Formality: $H^*(\mathrm{Conf}_r(\mathbb{C})) \sim_{\mathbb{C}} \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(\mathbb{C}))$, $\omega_{ij} \mapsto \mathrm{d}\log(z_i - z_j)$.

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

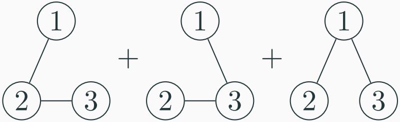
$H^*(\mathrm{Conf}_r(\mathbb{R}^n)) \sim_{\mathbb{R}} \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_r(\mathbb{R}^n))$ for all $r \geq 0$ and $n \geq 2$.

Corollary

The cohomology of $\mathrm{Conf}_r(\mathbb{R}^n)$ determines its rational homotopy type.

Kontsevich's graph complexes

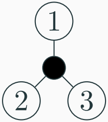
Arnold relations: $R_{123} =$



$\omega_{12}\omega_{23}$ $\omega_{23}\omega_{31}$ $\omega_{31}\omega_{12}$

$\implies H^*(\mathrm{Conf}_r(\mathbb{R}^n)) = \mathbb{R}\langle\text{graphs with } r \text{ vertices}\rangle/(R_{ijk})$

$\rightsquigarrow$ add "internal" vertices and a differential which contracts edges incident to these new vertices:



$\overset{d}{\longmapsto} R_{123}$

Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 1)

We get a quasi-free CDGA $\mathtt{Graphs}_n(r)$ and a quasi-isomorphism $\mathtt{Graphs}_n(r) \xrightarrow{\sim} H^*(\mathrm{Conf}_r(\mathbb{R}^n))$.

The relations R_{ijk} are only satisfied up to homotopy in $\Omega^*(\mathrm{Conf}_r(\mathbb{R}^n))$.
How to find representatives to get $\mathtt{Graphs}_n(r) \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_r(\mathbb{R}^n))$?

Let $\varphi \in \Omega^{n-1}(\mathrm{Conf}_2(\mathbb{R}^n))$ be the volume form.
For $\Gamma \in \mathtt{Graphs}_n(r)$ with i internal vertices:

$$\omega(\Gamma) := \int_{\mathrm{Conf}_{r+i}(\mathbb{R}^n) \to \mathrm{Conf}_r(\mathbb{R}^n)} \bigwedge_{(ij) \in E_\Gamma} \varphi_{ij}.$$

Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 2)

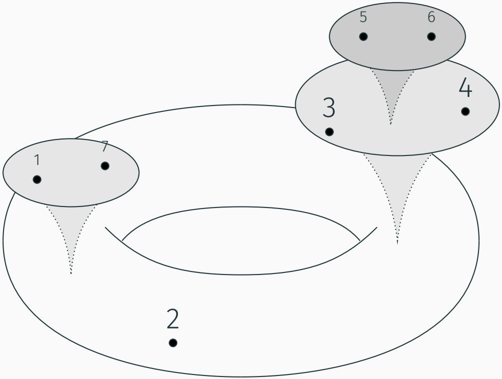
We get a quasi-isomorphism $\omega : \mathtt{Graphs}_n(r) \xrightarrow{\sim} \Omega(\mathrm{Conf}_r(\mathbb{R}^n))$.

⚠ I'm cheating! We have to compactify $\mathrm{Conf}_r(\mathbb{R}^n)$ to make sure $\int$ converges and to apply the Stokes formula correctly.

Compactification

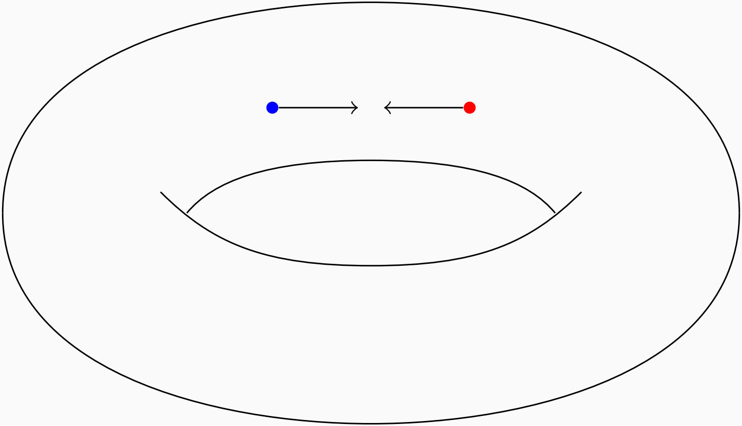
Problem: $\mathrm{Conf}_r(\mathbb{R}^n)$ is not compact.

Fulton–MacPherson compactification $\mathrm{Conf}_r(M) \xrightarrow{\sim} \mathtt{FM}_M(r)$

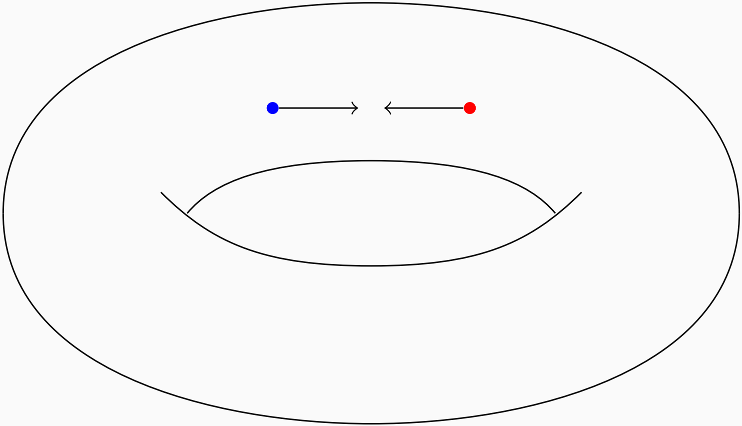


M closed manifold $\implies$ semi-algebraic stratified manifold $\dim = nr$

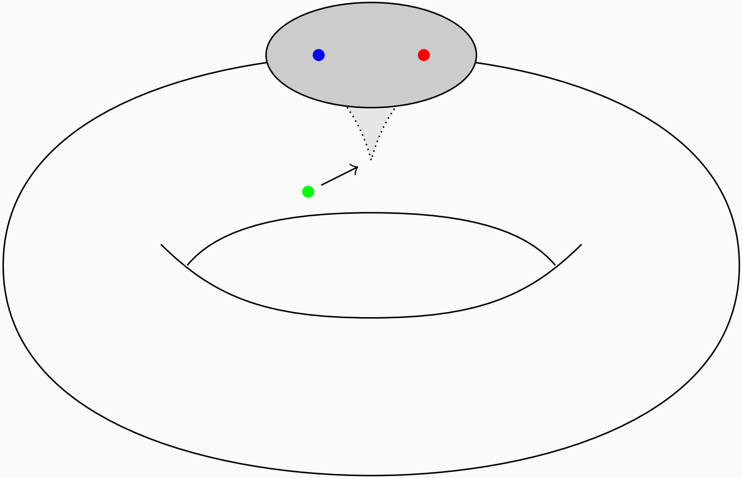
Animation #1



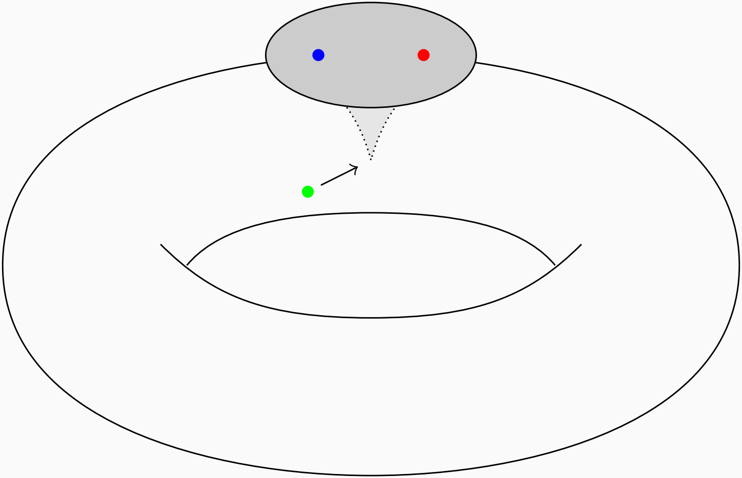
Animation #1



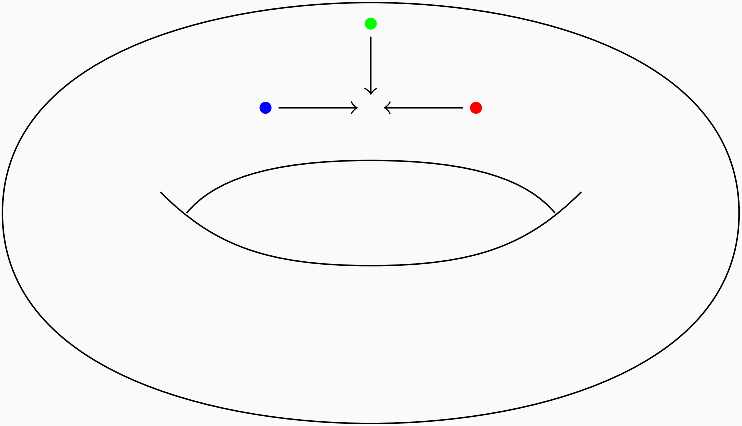
Animation #2



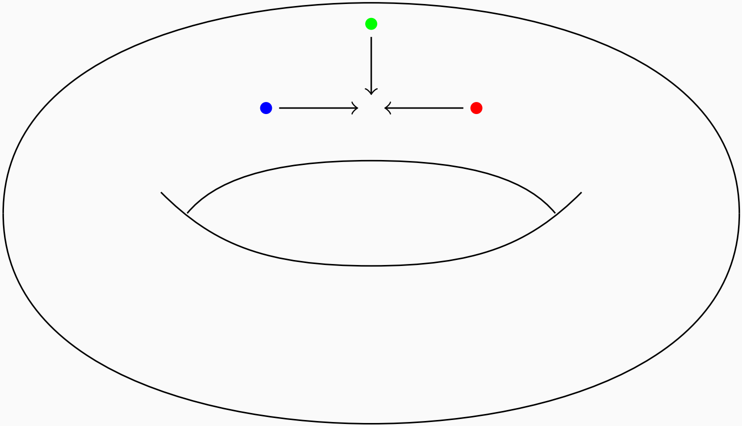
Animation #2



Animation #3



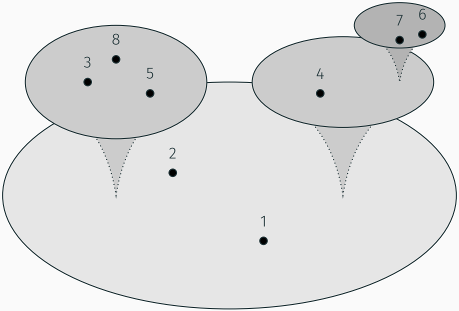
Animation #3



Compactification of $\mathrm{Conf}_r(\mathbb{R}^n)$

We have to “normalize” $\mathrm{Conf}_r(\mathbb{R}^n)$ to mitigate the non-compacity of $\mathbb{R}^n$:

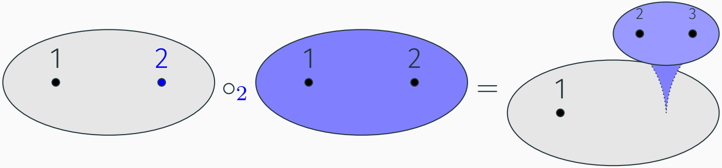
$$\mathrm{Conf}_r(\mathbb{R}^n) \xrightarrow{\sim} \mathrm{Conf}_r(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_{>0}) \overset{\sim}{\hookrightarrow} \mathsf{FM}_n(r)$$



$\implies$ semi-algebraic stratified manifold $\dim = nr - n - 1$

Operad

We see a new structure on FM_n: an operad! We can “insert” an infinitesimal configuration in another one:



$$\mathsf{FM}_n(k) \times \mathsf{FM}_n(l) \xrightarrow{\circ_i} \mathsf{FM}_n(k+l-1), \quad 1 \leq i \leq k$$

Remark

Weakly equivalent to the “little disks operad”.

Functoriality $\implies H^*(\mathtt{FM}_n) = H^*(\mathrm{Conf}_\bullet(\mathbb{R}^n))$ and $\Omega^*(\mathtt{FM}_n)$ are Hopf cooperads; $\mathtt{Graphs}_n$ is one too, and:

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

The operad $\mathtt{FM}_n$ is formal over $\mathbb{R}$:

$$\Omega^*(\mathtt{FM}_n) \xleftarrow[\omega]{\sim} \mathtt{Graphs}_n \xrightarrow{\sim} H^*(\mathtt{FM}_n).$$

Formality has important applications, e.g. Deligne conjecture, deformation quantization of Poisson manifolds, etc.

(Note: $H_*(\mathtt{FM}_n)$ governs Poisson n-algebras for $n \geq 2$.)

The Lambrechts–Stanley model

M: oriented closed manifold

$A \sim \Omega(M)$: Poincaré duality model of M

$\mathsf{G}_A(r)$: (conjectural) model of $\mathrm{Conf}_r(M) = M^{\times r} \setminus \bigcup_{i \neq j} \Delta_{ij}$ ↪ $:= \{x_i = x_j\}$

- "Generators": $A^{\otimes r}$ and the ω_{ij} from $\mathrm{Conf}_r(\mathbb{R}^n)$
- Arnold relations + symmetry
- $d\omega_{ij}$ kills the dual of $[\Delta_{ij}]$.

Examples:

- $\mathsf{G}_A(0) = \mathbb{R}$ is a model of $\mathrm{Conf}_0(M) = \{\varnothing\}$ ✓
- $\mathsf{G}_A(1) = A$ is a model of $\mathrm{Conf}_1(M) = M$ ✓
- $\mathsf{G}_A(2) \sim A^{\otimes 2}/(\Delta_A)$ should be a model of $\mathrm{Conf}_2(M) = M^2 \setminus \Delta$?
- $r \geq 3$: more complicated.

Brief history of G_A

1969 [Arnold, Cohen] $H^*(\mathrm{Conf}_r(\mathbb{R}^n)) = \mathsf{G}_{H^*(\mathbb{R}^n)}(r)$

1978 [Cohen–Taylor] spectral sequence $E^2 = \mathsf{G}_{H^*(M)}(k) \Rightarrow H^*(\mathrm{Conf}_k(M))$

1994 For smooth projective complex manifolds ($\implies$ Kähler):

- [Kříž] $\mathsf{G}_{H^*(M)}(r)$ is a model of $\mathrm{Conf}_r(M)$;
- [Totaro] the Cohen–Taylor SS collapses.

2004 [Lambrechts–Stanley] model for $r = 2$ if $\pi_{\leq 2}(M) = 0$

2004 [Félix–Thomas, Berceanu–Markl–Papadima] relation with Bendersky–Gitler spectral sequence

2008 [Lambrechts–Stanley] $H^i(\mathsf{G}_A(r)) \cong_{\Sigma_r\text{-Vect}} H^i(\mathrm{Conf}_r(M))$

2015 [Cordova Bulens] model for $r = 2$ if $\dim M = 2m$

First part of the theorem

By generalizing the proof of Kontsevich & Lambrechts–Volić:

Theorem (I.)

Let M be a closed simply connected smooth manifold and A be any Poincaré duality model of M. Then $\mathsf{G}_A(r)$ is a real model of $\mathrm{Conf}_r(M)$.

Corollary (cf. Campos–Willwacher)

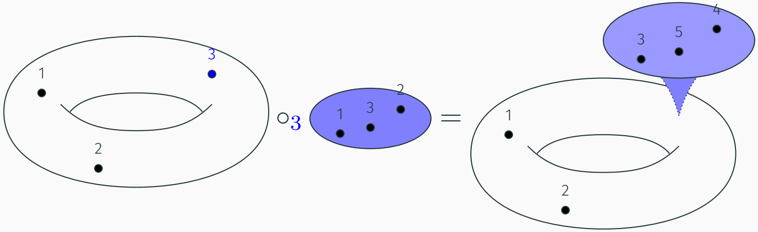
$M \sim_{\mathbb{R}} N \implies \mathrm{Conf}_r(M) \sim_{\mathbb{R}} \mathrm{Conf}_r(N)$ for all r.

We can "compute everything" over $\mathbb{R}$ for $\mathrm{Conf}_r(M)$.

Remark

$\dim M \leq 3$: only spheres (Poincaré conjecture) and we know that G_A is a model anyway, but adapting the proof is problematic!

M parallelized $\implies$ $\mathsf{FM}_M = \{\mathsf{FM}_M(r)\}_{r\geq 0}$ is a right FM_n-module :



We can rewrite:

$$\mathsf{G}_A(r) = (A^{\otimes r} \otimes H^*(\mathsf{FM}_n(r))/\text{relations}, d)$$

A bit of abstract nonsense:

Proposition

$\chi(M) = 0 \implies \mathsf{G}_A = \{\mathsf{G}_A(r)\}_{r\geq 0}$ is a Hopf right $H^*(\mathsf{FM}_n)$-comodule.

Theorem (I. 2018)

M: closed simply connected smooth manifold, $\dim M \geq 4$

$$\begin{array}{ccccc} \mathsf{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R & \overset{\sim}{\dashrightarrow} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M) \\ \circlearrowleft^{\dagger} & & \circlearrowleft^{\dagger} & & \circlearrowleft^{\ddagger} \\ H^*(\mathsf{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n) \end{array}$$

† if $\chi(M) = 0$

‡ if M is parallelized.

$A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(M)$

Conclusion

Not only do we have a model of each $\mathrm{Conf}_r(M)$, but also of their richer structure if we look at them all at once.

Space of embeddings: $\mathrm{Emb}(M, N) = \{f : M \hookrightarrow N\}$.

Goodwillie–Weiss manifold calculus [Arone, Boavida, Turchin, Weiss...]: for parallelized manifolds of codimension ≥ 3,

$$\mathrm{Emb}(M, N) \simeq \mathrm{Mor}^h_{\mathrm{Conf}_\bullet(\mathbb{R}^n)}(\mathrm{Conf}_\bullet(M), \mathrm{Conf}_\bullet(N)).$$

LS model is small and explicit $\implies$ hope: computations are tractable

Remark

Requires to compare $\mathrm{Mor}^h_{\mathrm{Conf}_\bullet(\mathbb{R}^n)}(\mathrm{Conf}_\bullet(M), \mathrm{Conf}_\bullet(N))^{\mathbb{R}}$ with $\mathrm{Mor}^h_{\mathrm{Conf}_\bullet(\mathbb{R}^n)^{\mathbb{R}}}(\mathrm{Conf}_\bullet(M)^{\mathbb{R}}, \mathrm{Conf}_\bullet(N)^{\mathbb{R}})$

Application 2: Factorization homology

Factorization homology = homology where $\otimes$ replaces $\oplus$ + homotopy commutative coefficients.

For an E_n-algebra $\mathscr{A}$,

$$\int_M \mathscr{A} = \operatorname{hocolim}_{(D^n)^{\sqcup r} \hookrightarrow M} \mathscr{A}^{\otimes r}.$$

Alternate description: $\int_M \mathscr{A} \sim \mathrm{Conf}_\bullet(M) \otimes^h_{\mathrm{Conf}_\bullet(\mathbb{R}^n)} \mathscr{A}$ [Francis].

Theorem (I. 2018, cf. Markarian '17, Döppenschmidt–Willwacher '18)

M closed simply connected smooth manifold ($\dim \geq 4$),
$\mathscr{A} := \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \implies \int_M \mathscr{A} \sim_{\mathbb{R}} \mathbb{R}$.

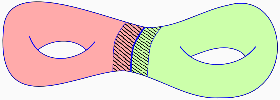
Theorem (Campos–I.–Lambrechts–Willwacher 2018)

For manifolds with boundary: homotopy invariance of $\mathrm{Conf}_r(-)$, generalization of the Lambrechts–Stanley model (and more); under good conditions, including $\dim M \geq \dots$

Remark

Poincaré duality models $\rightsquigarrow$ Poincaré–Lefschetz duality models.

Allows to compute Conf_r by "induction":



M: oriented manifold $\rightsquigarrow$ framed configuration space

$$\mathrm{Conf}_r^{\mathrm{fr}}(M) := \{(x \in \mathrm{Conf}_r(M), B_1, \ldots, B_r) \mid B_i\text{: orth. basis of } T_{x_i}M\}.$$

Natural action of the framed little disks operad on $\{\mathrm{Conf}_\bullet^{\mathrm{fr}}(M)\}$.

Theorem (Campos–Ducoulombier–I.–Willwacher 2018)

Real model of this module based on graph complexes.

First step towards embedding spaces of non-parallelized manifolds. (Not enough: need partially framed configurations for the larger manifold N.)

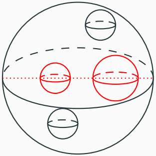
WIP: Complements of submanifolds

Goal: $\mathrm{Conf}(N \setminus M)$ where $\dim N - \dim M \geq 2$.

Motivation: work of Ayala, Francis, Rozenblyum, Tanaka

Knot complement $\rightsquigarrow$ colored Jones polynomial.

There exists an operad VSC_{mn} which models the local situation $\mathbb{R}^n \setminus \mathbb{R}^m$:

 $\in \mathsf{VSC}_{13}(2, 2) \subset \mathsf{D}_3(2 + 2)$

Theorem (I. 2018)

The operad VSC_{mn} is formal over $\mathbb{R}$ for $n - m \geq 2$.

Thank you for your attention!

These slides: `https://idrissi.eu`