CURVED KOSZUL DUALITY FOR ALGEBRAS OVER UNITAL OPERADS

Najib Idrissi
May 2019 @ Higher Algebras in Topology, MPIM Bonn

Université de Paris  IMJ-PRG
Goal

Find resolutions of “algebras”.

Why?

• Compute derived invariants: derived tensor product, derived mapping space...
• Define homotopy algebras over operads.

Tool of choice: Koszul duality.
Quadratic algebras – Koszul duals

Starting data: quadratic algebra $A = T(E)/(R)$, $R \subset E \otimes E$

$\leadsto$ Koszul dual $A^!$: cofree coalgebra on $\Sigma E$ with “corelations” $\Sigma^2 R$
(Usually easier to understand $A^! = F(E^*)/(R^\perp)$)

Examples

1. $A = T(E)$, $R = 0 \implies A^! = E^*$ with trivial multiplication;

2. $A = S(E) = T(E)/(xy - yx) \implies A^! = T(E^*)/(x^*y^* + y^*x^*) = \Lambda(E^*)$.

$\implies$ Koszul complex $K_A := (A \otimes A^!, d_\kappa(\Sigma e) = e)$; $A$ is Koszul if $K_A$ is acyclic

Example

$T(E)$ and $S(E)$ are both Koszul.
Bar/cobar adjunction:
\[ \Omega : \{ \text{coaug.coalgebras} \} \leftrightarrows \{ \text{aug.algebras} \} : B \]
where \( BA = (T^c(\Sigma \bar{A}), d_B) \) and \( \Omega C = (T(\Sigma^{-1} \bar{C}), d_\Omega) \).

Canonical morphism \( \Omega BA \xrightarrow{\sim} A \) is always a cofibrant resolution...but big!

A quadratic \( \implies \exists \) canonical morphism \( \Omega A^i \to A \)

**Theorem (Priddy '70s)**

A is Koszul \( \iff \Omega A^i \xrightarrow{\sim} A \).

Much smaller resolution!

**Examples**

- \( A = T(E) \implies \Omega A^i = A = T(E) \) versus \( \Omega BA = TT^c F(E) \)
- \( A = S(E) \implies \Omega A^i = T \Lambda^c(E) \) versus \( \Omega BA = TT^c S(E) \).
**Quadratic-linear-constant** algebra: \( A = T_+(E)/(R) \) with \( R \subset E \otimes^2 \oplus E \oplus \mathbb{R} \)

Koszul dual \( A^i = (qA_i, d_{A_i}, \theta_{A_i}) \): curved dg-coalgebra

\[
r = r_{(2)} + r_{(1)} + r_{(0)} \quad 1 \in R \subset E \otimes^2 \oplus E \oplus \mathbb{R}.
\]

- quadratic \( \sim qA := T(E)/(qR) \) where \( qR := \text{proj}_{E \otimes^2}(R) \);
- linear \( \sim d_{A_i} : qA_i \rightarrow qA_i \) is a coderivation;
- constant \( \sim \theta_{A_i} : qA_i \rightarrow \mathbb{R} \) s.t. \( d^2 = (\theta \otimes \text{id} \mp \text{id} \otimes \theta)\Delta \) and \( \theta d = 0 \).

**Example**

\( A = U(\mathfrak{g}) = uF(\mathfrak{g})/(xy - yx - [x, y]) \sim qA = T(\mathfrak{g})/(xy - yx) = S(\mathfrak{g}) \)

\( d_{A_i} = \) coderivation induced by \( d(x \wedge y) = [x, y] \sim A^i = C^*_{CE}(\mathfrak{g}) \)
QLC algebras – resolutions

Bar/cobar adjunction:
\[ \Omega : \{ \text{curved dg-coalgebras} \} \cong \{ \text{semi.aug.algebras} \} : B \]
where \( BA = (T^c(\Sigma \bar{A}), d_2 + d_1, \theta) \) and \( \Omega(C) = (T_+(\Sigma^{-1}C), d_2 + d_1 + d_0) \).

Theorem (Polischuck, Positselski)
If \( qA \) is Koszul then \( \Omega A_i \xrightarrow{\sim} A \) is a cofibrant resolution.

Example
\[
A = U(\mathfrak{g}) \implies qA = S(\mathfrak{g}) \text{ is Koszul} \implies \Omega C^*_{CE}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}).
\]

Goal: do this for more general types of unital algebras.
What are “more general types of algebras”?

**Operad** \( P = \{P(n)\}_{n \geq 0} \): combinatorial object that encodes a type of algebra.

![Diagram](image)

**Examples**

The “three graces”: \( \text{Ass} \) = associative algebras; \( \text{Com} \) = commutative algebras; \( \text{Lie} \) = Lie algebras.

\( E_n \) = homotopy associative and commutative (for \( n \geq 2 \)) algebras.

\( e_n := H_*(E_n) = \text{Com} \circ \text{Lie}_n, \ n \geq 2 \) = Poisson \( n \)-algebras.
**Quadratic operad:** $P = \text{FOp}(E)/(R)$ where $E$ is a generating set of operations and $R \subseteq E \circ_{(1)} E$ is a set of quadratic relations.

**Example**

$\text{Com} = \text{FOp}(\mu)/(\mu(\mu(x,y), z) = \mu(x, \mu(y,z)))$ is quadratic.

Formally similar definitions: Koszul dual cooperad $P^! = \text{FOp}^c(\Sigma E, \Sigma^2 R)$ and its linear dual $P^! = \text{FOp}(E^*)/(R^\perp)$.

**Examples**

$\text{Ass}^! = \text{Ass}; \text{Com}^! = \text{Lie}, \text{Lie}^! = \text{Com}; e^!_n = e_n\{-n\}$. 
Formally similar definitions: bar/cobar adjunction

\[ \Omega : \{ \text{coaug.cooperads} \} \leftrightarrow \{ \text{aug.operads} \} : B \]

Canonical morphism \( \Omega BP \rightleftarrows P \) always a resolution, but very big

**Theorem (Ginzburg–Kapranov ’94, Getzler–Jones ’94, Getzler ’95...)**

If \( P \) is quadratic and Koszul, then \( P_\infty := \Omega P_i \rightleftarrows P \).

In this case, \( P_\infty \)-algebras = “homotopy \( P \)-algebras”.

**Examples**

\( \text{Ass}_\infty = A_\infty \)-algebras, \( \text{Com}_\infty = C_\infty \)-algebras, \( \text{Lie}_\infty = L_\infty \)-algebras...
\( \mathbf{P} = \mathbf{F}\text{Op}(E)/(R) \) Koszul quadratic operad \( \rightsquigarrow \) bar/cobar adjunction:

\[ \Omega_{\kappa} : \{ \text{coaug. } \mathbf{P}^i\text{-coalgebras} \} \rightleftarrows \{ \text{aug. } \mathbf{P}\text{-algebras} \} : \mathbf{B}_{\kappa}, \]

where \( \Omega_{\kappa}C = (\mathbf{P}(\Sigma^{-1}\tilde{C}), d) \) and \( \mathbf{B}_{\kappa}A = (\mathbf{P}^i(\Sigma\tilde{A}), d) \).

\( \rightsquigarrow \) resolution of \( \mathbf{P}\text{-algebras} \): \( \Omega_{\kappa}\mathbf{B}_{\kappa}(-) \), but very big.

**Example**

For a Lie algebra \( \mathfrak{g} \), \( \Omega_{\kappa}\mathbf{B}_{\kappa}\mathfrak{g} = (\mathcal{L}(C_{\ast-1}^{CE}(\mathfrak{g})), d) \).
Recall $\mathcal{P} = \text{FOp}(E)/(R)$.

**Monogenic $\mathcal{P}$-algebras:** $A = \mathcal{P}(V)/(S)$, $S \subset E(V)$.

(Monogenic = quadratic for binary $\mathcal{P}$)

Koszul dual: $A^i := \mathcal{P}^i(\Sigma V, \Sigma^2 S)$, $A^! = \mathcal{P}(V^*)/(S^\perp)$.

Koszul complex: $K_A = (A \otimes A^i, d_\kappa(\Sigma v) = v)$.

**Theorem (Millès ’12)**

If $\mathcal{P}$ is quadratic Koszul and if $A$ is a Koszul monogenic algebra, then $\Omega_\kappa A^i \simto A$ is a resolution of $A$.

**Examples**

$\mathcal{P} = \text{Ass}$: recovers the classical Koszul duality of associative algebras.

$A$: quadratic **Com**-algebra $\implies U(A^!) = (A_{\text{Ass}})^!$ [Löfwall].
Curved KD for QLC operads

Operads with QLC relations $uP = F\text{Op}(E)/(R)$, $R \subset E \circ_{(1)} E \oplus E \oplus \mathbb{R} \text{id}$

Koszul dual curved cooperad: $uP^i = (quP^i, d_{Ai}, \theta_{Ai})$

- quadratic $\leadsto quP$: “quadratization” of $uP$;
- linear $\leadsto d_{Ai}: quP^i \to quP^i$ coderivation;
- constants $\leadsto \theta_{Ai}: quP^i \to \mathbb{R} \text{id}$ s.t. $d^2 = (\theta \circ \text{id} \mp \text{id} \circ \theta) \Delta$ and $\theta d = 0$

Example

$u\text{Com} = F\text{Op}(\mu, \uparrow)/(\mu(\mu(x,y),z) = \mu(x,\mu(y,z)), \mu(\uparrow, x) = x)$

$u\text{Com}^i = (\text{Com}^i \oplus \uparrow^c, d = 0, \theta(\mu^c \circ_1 \uparrow^c) = -1)$

Bar/cobar extends to the curved setting

Theorem (Hirsh–Millès ’12)

If $quP$ is Koszul, then $uP_\infty := \Omega(uP^i) \leadsto uP$: resolution of $uP$
Consider $P = \text{FO}_{\mathfrak{p}}(E)/(R)$: binary quadratic operad

$\rightsquigarrow$ unital version $uP = \text{FO}_{\mathfrak{p}}(E \oplus \mathfrak{i})/(R + R')$:

- $E \hookrightarrow E \oplus \mathfrak{i}$ induces $P \hookrightarrow uP$
- $quP \cong P \oplus \mathfrak{i}$
- $R'$ has only quadratic-constant terms

**Examples**

$u\text{Ass}, u\text{Com}, c\text{Lie}, u\text{e}_n$...

**Algebra with QLC relations** $A = uP(V)/I$:

- $I$ is generated by $S := I \cap (\mathfrak{i} \oplus V \oplus E(V))$
- $S \cap (\mathfrak{i} \oplus V) = 0$ ("$V$ is minimal")

The second condition is difficult to check!
Curved KD for algebras over binary unital operads

\[ uP = \text{FOp}(E \oplus \mathcal{Y})/(R + R') \]: unital version of quadratic \( P = \text{FOp}(E)/(R) \)

\[ A = uP(V)/(S) \]: algebra w/ QLC relations \( S \subset E(V) \oplus V \oplus \mathcal{Y} \)

Koszul dual: curved \( P^i \)-coalgebra \( A^i = (qA^i, d_{A^i}, \theta_{A^i}) \)

- quadratic \( \sim \) \( qA = P(V)/(qS) \): “quadratization” of \( A \);
- linear \( \sim \) \( d_{A^i} \): coderivation;
- constant \( \sim \) \( \theta : qA^i \rightarrow \mathbb{R}^\mathcal{Y} \) (+ relations)

Generalization of bar/cobar adjunction:

\[ \Omega_\kappa : \{\text{curved } P^i \text{-coalgebras}\} \leftrightarrows \{\text{semi.aug. } uP \text{-algebras}\} : B_\kappa \]

Theorem (I. ’18)

If \( qA \) is Koszul then \( \Omega_\kappa A^i \sim A \) is a resolution.
**APPLICATION 1: FACTORIZATION HOMOLOGY**

*M*: framed $n$-manifold, *A*: $uE_n$-algebra (∃ version for unframed manifolds.)

<table>
<thead>
<tr>
<th>Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compute $\int_M A = \text{hocolim}_{(D^n) \sqcup k \hookrightarrow M} A \otimes k$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (Francis 2015)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_M A \simeq E_M \circ \xrightarrow{\text{L}} uE_n A = \text{hcoeq}(E_M \circ uE_n \circ A \Rightarrow E_M \circ A)$, where:</td>
</tr>
</tbody>
</table>

$$uE_n(k) = \text{Emb}^{\text{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k \times}, \mathbb{R}^n); \quad E_M(k) = \text{Emb}^{\text{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k \times}, M).$$

| Upshot: data is separated in three + resolution |


If we work over $\mathbb{R}$ and we just want chains:

$$C_*(\int_M A) \simeq C_*(E_M) \circ_{C_*(uE_n)} C_*(A).$$

Theorem (Kontsevich ’99; Tamarkin ’03 ($n = 2$); Lambrechts–Volić ’14; Petersen ’14 ($n = 2$); Fresse–Willwacher ’15)

The operad $C_*(uE_n)$ is formal: $C_*(uE_n) \simeq ue_n := H_*(uE_n) = \text{Com} \circ \text{Lie}_n$.

Theorem (I.)

$M$ closed, simply connected, smooth, $\dim M \geq 4 \implies$

Lambrechts–Stanley model of $C_*(E_M)$ as a right $C_*(uE_n)$-module:

$$LS_M = C^{CE}_*(\mathcal{M}^{n-*} \otimes \text{Lie}_n[1-n]) + \text{action of Com}.$$ 

Upshot: $C_*(\int_M A) \simeq LS_M \circ_{ue_n} \tilde{A}$

$\implies$ we need to resolve $A$ as a $ue_n$-algebra.
**Weyl Algebra** $O_{\text{poly}}(T^*\mathbb{R}^d[1-n])$

$$A = O_{\text{poly}}(T^*\mathbb{R}^d[1-n]) = S(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)$$

Action of $ue_n$: free symmetric algebra and $\{x_i, \xi_j\} = \delta_{ij}1$

$\implies$ quadratic-(linear-)-constant presentation

Quadratization $qA = S(x_i, \xi_j)$ free symmetric algebra + zero bracket

Koszul dual: $A^i = (qA^i, d, \theta)$

- $qA^i = S^c(\bar{x}_i, \bar{\xi}_j)$ cofree symmetric coalgebra + trivial cobracket
- $d = 0$
- curvature: $\theta(\bar{x}_i \wedge \bar{\xi}_j) = -\delta_{ij}$.

$\implies$ "small" resolution $Q_A := \Omega_\kappa A^i = (\text{SLS}^c(\bar{x}_i, \bar{\xi}_j), d) \sim A$

(If we had applied curved KD at the level of operads instead:

$\Omega_\kappa B_\kappa A \supset (\text{SL} S^c L^c S(x_i, \xi_j), d)$, + resolution of the unit...)}
We can also compute

$$\int_M \mathcal{O}_{\text{poly}} (T^*\mathbb{R}^d[1 - n]) \simeq \text{LS}_M \circ_{\text{ue}_n} (\text{SLS}^c(\bar{x}_i, \bar{\xi}_j), d)$$

**Theorem (I. ’18, see also Markarian ’17, Döppenschmitt ’18)**

$$\int_M \mathcal{O}_{\text{poly}} (T^*\mathbb{R}^d[1 - n]) \simeq \mathcal{C}^E (\mathcal{M}^{n-*} \otimes \mathbb{R} \langle 1, x_i, \xi_j \rangle) \simeq \mathbb{R}.$$ 

Intuition: quantum observable with values in $A \rightsquigarrow$ “expectation” lives in $\int_M A$, should be a number.
Operad $P + P$-algebra $A \quad \implies \quad$ notion of $A$-modules

**Examples**

- $P = \text{Ass} \to (A, A)$ bimodules; $P = \text{Com} \to A$-modules; $P = \text{Lie} \to$ representations of the Lie algebra.

∃ an associative algebra $U_P(A)$ s.t. left $U_P(A)$-modules = $A$-modules

**Proposition**

For $A = \mathcal{O}_\text{poly}(T^*\mathbb{R}^d[1 - n])$, the derived enveloping algebra $U_{\text{uen}}(A)$ is q.iso to the underived one.
Thank you for your attention!

These slides: https://idrissi.eu