Curved Koszul Duality for Algebras over Unital Operads

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A: $u\mathsf{E}_n$-algebra = locally constant (framed) factorization algebra on $\mathbb{R}^n$

Main goal: factorization homology

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Tool:

Theorem (Francis 2015)

$\int_M A \simeq \mathsf{E}_M \circ^{\mathbb{L}}_{u\mathsf{E}_n} A$, where:

$$u\mathsf{E}_n(k) = \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k\times}, \mathbb{R}); \quad \mathsf{E}_M(k) = \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n}_{k\times}, M).$$

If we work over $\mathbb{R}$ and we just want chains:

$$C_*(\textstyle\int_M A; \mathbb{R}) \simeq C_*(\mathsf{E}_M) \circ^{\mathbb{L}}_{C_*(u\mathsf{E}_n)} C_*(A).$$

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$\implies$ Koszul complex $K_A := (A \otimes A^{¡}, d_\kappa)$; A is Koszul if K_A is acyclic

Example

$F(E)$ and $S(E)$ are both Koszul.

Bar/cobar adjunction:

$$\Omega : \{\text{coaug.coalgebras}\} \leftrightarrows \{\text{aug.algebras}\} : B$$

where $BA = (F^c(\Sigma\bar{A}), d_B)$ and $\Omega C = (F(\Sigma^{-1}\bar{C}), d_\Omega)$.

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Much smaller resolution!

Examples

$A = F(E) \implies \Omega A^{\text{i}} = A$

$A = S(E) \implies \Omega A^{\text{i}} = F(\Lambda^c(E))$, to compare with $\Omega BA = F(F^c(S(E)))$.

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Goal: do this for more general types of algebras (e.g. Poisson algebras).

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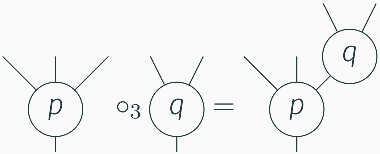

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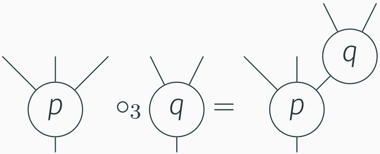

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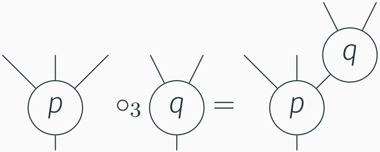

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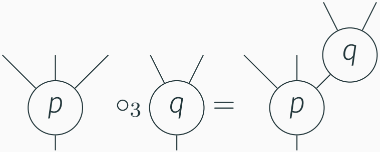

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$H_*(\mathsf{E}_n)$, $n \geq 2$ = Poisson n-algebras.

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$\mathtt{Ass}^! = \mathtt{Ass}$; $\mathtt{Com}^! = \mathtt{Lie}$, $\mathtt{Lie}^! = \mathtt{Com}$; $H_*(\mathsf{E}_n)^! = H_*(\mathsf{E}_n)\{-n\}$.

Formally similar definitions: bar/cobar adjunction

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Examples

$\mathtt{Ass}_\infty = A_\infty$-algebras, $\mathtt{Com}_\infty = C_\infty$-algebras, $\mathtt{Lie}_\infty = L_\infty$-algebras...

Curved KD for QLC operads [Hirsh–Millès]

Extension to operads with quadratic-linear-constant relations:

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If $qu\mathsf{P}$ is Koszul, then $u\mathsf{P}_\infty := \Omega(u\mathsf{P}^{\mathrm{i}}) \xrightarrow{\sim} u\mathsf{P}$: resolution of $u\mathsf{P}$

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$\mathsf{P} = \mathsf{Ass}$: recovers the classical Koszul duality of associative algebras.

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Application: $\mathrm{Poly}(T^*\mathbb{R}^d[1-n])$

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(If we had applied curved KD at the level of operads instead:
$\Omega_\kappa B_\kappa A \supset (\underbrace{SL}_{\text{cobar}}\underbrace{S^cL^c}_{\text{bar}}\underbrace{S(x_i, \xi_j)}_{A}, d)$, + resolution of the unit...)

Operad P + P-algebra $A \implies$ notion of A-modules

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$\mathsf{P} = \mathtt{Ass} \to (A, A)$ bimodules; $\mathsf{P} = \mathtt{Com} \to A$-modules; $\mathsf{P} = \mathtt{Lie} \to$ representations of the Lie algebra.

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$\exists$ an associative algebra $U_{\mathtt{P}}(A)$ s.t. left $U_{\mathtt{P}}(A)$-modules = A-modules

Application: derived enveloping algebra

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Proposition

For $A = \mathrm{Poly}(T^*\mathbb{R}^d[1-n])$, the derived enveloping algebra $U^{\mathbb{L}}_{H_*(u\mathsf{E}_n)}(A)$ is q.iso to the underived one + explicit description.

Application: $\int_M \mathrm{Poly}(T^*\mathbb{R}^d[1-n])$

We can also compute

$$\int_M \mathrm{Poly}(T^*\mathbb{R}^d[1-n]) \simeq \mathsf{LS}_M \circ_{H_*(u\mathsf{E}_n)} (SLS^c(\bar{x}_i, \bar{\xi}_j), d)$$

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A bit of homological algebra + explicit description of LS_M:

Theorem (I. '18, see also Markarian '17, Döppenschmitt '18)

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Intuition: quantum observable with values in $A \rightsquigarrow$ "expectation" lives in $\int_M A$, should be a number.

Thank you for your attention!

These slides, links to papers: `https://idrissi.eu`