configuration spaces and operads

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December 11th, 2018 @ Stockholm Topology Seminar
Configuration spaces

$M$: $n$-manifold

$$\text{Conf}_r(M) := \{(x_1, \ldots, x_r) \in M^k \mid \forall i \neq j, \ x_i \neq x_j\}$$

- Braid groups
- Loop spaces
- Moduli spaces of curves
- Particles in movement [physics]
- Motion planning [robotics]
Question

Does the homotopy type of $M$ determine the homotopy type of $\text{Conf}_r(M)$? How to compute homotopy invariants of $\text{Conf}_r(M)$?

Non-compact manifolds

False: $\text{Conf}_2(\mathbb{R}) \not\sim \text{Conf}_2(\{0\})$ even though $\mathbb{R} \sim \{0\}$.

Closed manifolds

Longoi–Salvatore (2005): counter-example (lens spaces)... but not simply connected.

Simply connected closed manifolds

Homotopy invariance is still open.

We can also localize: $M \sim_{\mathbb{Q}} N \implies \text{Conf}_r(M) \sim_{\mathbb{Q}} \text{Conf}_r(N)$?
**Configurations in a Euclidean spaces**

Presentation of $H^*(\text{Conf}_k(\mathbb{R}^n))$ [Arnold, Cohen]

- Generators: $\omega_{ij}$ of degree $n - 1$ (for $1 \leq i \neq j \leq r$)
- Relations:
  \[
  \omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0
  \]

**Theorem (Arnold 1969)**

**Formality:** $H^*(\text{Conf}_k(\mathbb{C})) \sim_{\mathbb{C}} \Omega_{dR}^*(\text{Conf}_k(\mathbb{C})), \omega_{ij} \mapsto d \log(z_i - z_j)$.

**Theorem (Kontsevich 1999, Lambrechts–Volić 2014)**

$H^*(\text{Conf}_k(\mathbb{R}^n)) \sim_{\mathbb{R}} \Omega_{dR}^*(\text{Conf}_k(\mathbb{R}^n))$ pour tout $k \geq 0$ et tout $n \geq 2$.

**Corollary**

The cohomology of $\text{Conf}_k(\mathbb{R}^n)$ determines its rational homotopy type.
Kontsevich’s graph complexes

Arnold relations: \[ R_{123} = \omega_{12}\omega_{23} + \omega_{23}\omega_{31} + \omega_{31}\omega_{12} \]

We can represent elements of \( H^*\left( \text{Conf}_r(\mathbb{R}^n) \right) \) by linear combinations of graphs with \( r \) vertices, modulo the \( R_{ijk} \)

\( \rightsquigarrow \) add “internal” vertices and a differential which contracts edges incident to these new vertices:

Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 1)

We get a quasi-free CDGA \( \text{Graphs}_n(r) \) and a quasi-isomorphism \( \text{Graphs}_n(r) \cong H^*\left( \text{Conf}_r(\mathbb{R}^n) \right) \).
Kontsevich’s integrals

The relations $R_{ijk}$ are only satisfied up to homotopy in $\Omega^*(\text{Conf}_r(\mathbb{R}^n))$. How to systematically find representatives to get $\text{Graphs}_n(k) \sim \Omega^*(\text{Conf}_k(\mathbb{R}^n))$?

Let $\varphi \in \Omega^{n-1}(\text{Conf}_2(\mathbb{R}^n))$ be the volume form. For $\Gamma \in \text{Graphs}_n(r)$ with $i$ internal vertices:

$$\omega(\Gamma) := \int_{\text{Conf}_{k+i}(\mathbb{R}^n) \to \text{Conf}_k(\mathbb{R}^n)} \bigwedge_{(ij) \in E\Gamma} \varphi_{ij}.$$

Theorem (Kontsevich 1999, Lambrechts–Volić 2014 – Part 2)

We get a quasi-isomorphism $\omega : \text{Graphs}_n(k) \sim \Omega(\text{Conf}_k(\mathbb{R}^n))$.

⚠ I’m cheating! We have to compactify $\text{Conf}_k(\mathbb{R}^n)$ to make sure $\int$ converges and to apply the Stokes formula correctly.
Compactification

Problem: $\text{Conf}_k$ is not compact.

Fulton–MacPherson compactification $\text{Conf}_k(M) \sim \text{FM}_M(k)$

\[ M \text{ closed manifold} \implies \text{semi-algebraic stratified manifold } \dim = nk \]
Animation n°2
Animation n°2
Animation n°3
We have to “normalize” $\text{Conf}_k(\mathbb{R}^n)$ to mitigate the non-compacity of $\mathbb{R}^n$:

$$\text{Conf}_k(\mathbb{R}^n) \sim \text{Conf}_k(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_{>0}) \sim \text{FM}_n(k)$$

$\Rightarrow$ semi-algebraic stratified manifold $\dim = nk - n - 1$
We see a new structure on $\text{FM}_n$: an operad! We can “insert” an infinitesimal configuration in another one:

$$1 \overset{\circ_2}{\circ} 2 \quad 1 \quad 2 \quad = \quad 1$$

$\text{FM}_n(k) \times \text{FM}_n(l) \xrightarrow{\circ_i} \text{FM}_n(k + l - 1), \quad 1 \leq i \leq k$

**Remark**

Weakly equivalent to the “little disks operad”.
Complete theorem

By functoriality, $H^*(\text{FM}_n) = H^*(\text{Conf}\cdot(\mathbb{R}^n))$ and $\Omega^*(\text{FM}_n)$ are Hopf cooperads. We check that $\text{Graphs}_n$ is one too, and:

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

The operad $\text{FM}_n$ is formal over $\mathbb{R}$:

$$\Omega^*(\text{FM}_n) \looparrowleft_{\omega} \text{Graphs}_n \looparrowright H^*(\text{FM}_n).$$

Formality has important applications, e.g. Deligne conjecture, deformation quantization of Poisson manifolds, etc.

Remark

$H_*(\text{FM}_n)$ is the operad governing Poisson $n$-algebras for $n \geq 2$. 

(Oriented) closed manifolds satisfy Poincaré duality:
\[ H^k(M) \otimes H^{n-k}(M) \to \mathbb{R}, \alpha \otimes \beta \mapsto \int_M \alpha \beta \] is non-degenerate.

**Poincaré duality CDGA** \((A, d, \varepsilon)\):

- \( (A, d) \): connected finite-type CDGA
- \( \varepsilon : A^n \to k \) s.t. \( \varepsilon \circ d = 0 \)
- \( A^k \otimes A^{n-k} \to k, a \otimes b \mapsto \varepsilon(ab) \) is non-degen \( \forall k \).

**Theorem (Lambrechts–Stanley 2008)**

Any simply connected closed manifold admits a Poincaré duality model \( A \sim \Omega^*(M) \).
The Lambrechts–Stanley model

$M$: oriented closed manifold
$A \sim \Omega(M)$: Poincaré duality model of $M$

$G_A(r)$: (conjectural) model of $\text{Conf}_r(M) = M^\times k \setminus \bigcup_{i \neq j} \Delta_{ij}$

- “Generators”: $A^\otimes r$ and the $\omega_{ij}$ from $\text{Conf}_k(\mathbb{R}^n)$
- Arnold relations + symmetry
- $d\omega_{ij}$ kills the dual of $[\Delta_{ij}]$. 

Examples:

- $G_A(0) = \mathbb{R}$ is a model of $\text{Conf}_0(M) = \{\emptyset\}$ ✓
- $G_A(1) = A$ is a model of $\text{Conf}_1(M) = M$ ✓
- $G_A(2) \sim A^\otimes 2/(\Delta_A)$ should be a model of $\text{Conf}_2(M) = M^2 \setminus \Delta$?
- $r \geq 3$: more complicated.
**Brief history of $G_A$**

1969 [Arnold, Cohen] $H^*(\text{Conf}_k(\mathbb{R}^n)) = G_{H^*(D^n)}(k)$

1978 [Cohen–Taylor] spectral sequence starting at $G_{H^*(M)}$

~1994 For smooth projective complex manifolds (Kähler):

- [Kříž] $G_{H^*(M)}(k)$ is a model of $\text{Conf}_k(M)$;

2004 [Lambrechts–Stanley] model for $r = 2$ if $\pi_{\leq 2}(M) = 0$


2008 [Lambrechts–Stanley] $H^i(G_A(k)) \cong_{\Sigma_k\text{-Vect}} H^i(\text{Conf}_k(M))$

2015 [Cordova Bulens] model for $r = 2$ if $\dim M = 2m$
By generalizing the proof of Kontsevich & Lambrechts–Volić:

**Theorem (1.)**

Let $M$ be a closed simply connected smooth manifold. Let $A$ be any Poincaré duality model of $M$. Then $G_A(k)$ is a real model of $\text{Conf}_r(M)$.

**Corollaries**

$M \sim_R N \implies \text{Conf}_r(M) \sim_R \text{Conf}_r(N)$ for all $k$.

We can “compute everything” over $\mathbb{R}$ for $\text{Conf}_r(M)$.

**Remark**

$\dim M \leq 3$: only spheres (Poincaré conjecture) and we know that $G_A$ is a model, but adapting the proof is problematic!
$M$ parallelized $\implies FM_M = \{FM_M(k)\}_{k \geq 0}$ is a right $FM_n$-module:

We can rewrite:

$$G_A(k) = (A^{\otimes k} \otimes H^*(FM_n(k))/\text{relations, } d)$$

A bit of abstract nonsense:

**Proposition**

$\chi(M) = 0 \implies G_A = \{G_A(k)\}_{k \geq 0}$ is a Hopf right $H^*(FM_n)$-comodule.
Theorem (l. 2016)

$M$: closed simply connected smooth manifold, $\dim M \geq 4$

\[
\begin{align*}
G_A & \xleftarrow{\sim} \text{Graphs}_R \longrightarrow \Omega^*_\text{PA}(FM_M) \\
\bigcirc^{\dagger} & \quad \bigcirc^{\dagger} & \quad \bigcirc^{\ddagger} \\
H^*(FM_n) & \xleftarrow{\sim} \text{Graphs}_n \longrightarrow \Omega^*_\text{PA}(FM_n)
\end{align*}
\]

$\dagger$ if $\chi(M) = 0$

$\ddagger$ if $M$ is parallelized.

Conclusion

Not only do we have a model of each $\text{Conf}_r(M)$, but for their richer structure if we look at them all at once.
Consider the space of embeddings: $\text{Emb}(M, N) = \{ f : M \hookrightarrow N \}$.

Goodwillie–Weiss manifold calculus [Boavida–Weiss, Turchin]: for parallelized manifolds of codimension $\geq 3$,

$$\text{Emb}(M, N) \simeq \text{Mor}^h_{\text{Conf}_\cdot(\mathbb{R}^n)}(\text{Conf}_\cdot(M), \text{Conf}_\cdot(N)).$$

Since the LS model is small and explicit, hope to do computations with these spaces.

**Remark**

Requires something like $\text{Mor}^h_{\text{Conf}_\cdot(\mathbb{R}^n)}(\text{Conf}_\cdot(M), \text{Conf}_\cdot(N)) \simeq_{\mathbb{R}} \text{Mor}^h_{\text{Conf}_\cdot(\mathbb{R}^n)^\mathbb{R}}(\text{Conf}_\cdot(M)^\mathbb{R}, \text{Conf}_\cdot(N)^\mathbb{R})$. 
Schematically, factorization homology = homology where $\otimes$ replaces $\oplus$. Can be seen as “quantum observables” on $M$. For an $E_n$-algebra $\mathcal{A}$,

$$\int_M \mathcal{A} = \hocolim (D^n \sqcup_k \hookrightarrow M \mathcal{A} \otimes^h k).$$

Alternate description: $\int_M \mathcal{A} \sim \text{Conf}_\bullet (M) \otimes^h_{\text{Conf}_\bullet (\mathbb{R}^n)} \mathcal{A}$ [Francis].

**Theorem (I. 2018, se also Markarian 2017, Döppenschmidt 2018)**

$M$ closed simply connected smooth manifold ($\dim \geq 4$),

$$\mathcal{A} = \mathcal{O}_{\text{poly}} (T^* \mathbb{R}^d [1 - n]) \implies \int_M \mathcal{A} \sim \mathbb{R} \mathbb{R}.$$
**Generalization 1: Manifolds with Boundary**


For manifolds with boundary: homotopy invariance of \( \text{Conf}_r(-) \), generalization of the Lambrechts–Stanley model (and more); under good conditions, including \( \dim M \geq \ldots \)

Allows to compute \( \text{Conf}_r \) by “induction”:

Roughly: we use 2-colored labeled graphs.
**Generalization 2: oriented manifolds**

\(M\): oriented \(n\)-manifold \(\leadsto\) framed configuration space

\[\text{Conf}^\text{fr}_r(M) := \{ (x \in \text{Conf}_r(M), B_1, \ldots, B_r) \mid B_i: \text{orth. basis of } T_{x_i}M \}\].

Natural action of the framed little disks operad on \(\{\text{Conf}^\text{fr}_\bullet(M)\}\).


Real model of this module based on graph complexes (little hope of analogue of Lambrechts–Stanley model...)

Should allow us to compute e.g. embedding spaces of non-parallelized manifolds. (Not enough, though: need partially framed configurations for the larger manifold \(N\).)
WIP: compute configuration spaces of complements $N \setminus M$ where $\dim N - \dim M \geq 2$.

**Motivation: Ayala–Francis–Tanaka conjecture**

Knot complement: should be related(?) to Khovanov homology.

There exists an operad $VSC_{mn}$ which models the local situation $\mathbb{R}^n \setminus \mathbb{R}^m$:

$$\in VSC_{13}(2, 2)$$

**Theorem (I. 2018)**

The operad $VSC_{mn}$ is formal over $\mathbb{R}$. 
Thank you for your attention!

These slides: https://idrissi.eu