Formality of a higher-codimensional Swiss-Cheese operad

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We study configurations of points in the complement of a linear subspace inside a Euclidean space, $\mathbb{R}^n \setminus \mathbb{R}^m$ with $n - m \geq 2$. We define a higher-codimensional Swiss-Cheese operad VSC_{mn} associated to such configurations, a variant of the classical Swiss-Cheese operad. The operad VSC_{mn} is weakly equivalent to the operad of locally constant factorization algebras on the stratified space $\{\mathbb{R}^m \subset \mathbb{R}^n\}$. We prove that this operad is formal over $\mathbb{R}$.

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Introduction

The theory of operads deals with types of algebras, e.g. associative algebras, commutative algebras, Lie algebras, etc. The little disks operads D_n (for $n \geq 1$) are central to this theory. Their applications include deformation quantization of Poisson manifolds, the Deligne conjecture, embedding spaces of manifolds, factorization homology, and configuration spaces. Briefly, an element of $\mathsf{D}_n(k)$ is an embedding of k little n-disks with disjoint interiors in the unit n-disk D^n. The operadic structure on $\mathsf{D}_n = \{\mathsf{D}_n(k)\}_{k \geq 0}$ is obtained by considering the composition of such embeddings. Elements of D_n represent operations acting on n-fold loop spaces, and the operadic structure reflects the composition of such operations. Up to homotopy, grouplike D_n-algebras are exactly n-fold loop spaces [BV68; May72]. A fundamental property of D_n is its "formality", i.e. its cohomology $H^*(\mathsf{D}_n; \mathbb{Q})$ encodes its rational homotopy type [Kon99; Tam03; LV14; Pet14; FW15].

Voronov's Swiss-Cheese operads SC_n [Vor99] (for $n \geq 2$) are two-colored operads which encode, in some sense, central actions of D_n-algebras on D_{n-1}-algebras. Elements of $\mathsf{SC}_n(k, l)$ are roughly speaking given by configurations of k half-disks and l full disks in the unit upper half-disk. Under good assumptions, SC_n-algebras are exactly relative iterated loop spaces [Duc14; Que15; HLS16; Vie18] (see Section 2.2). Unlike the little disks operads, the Swiss-Cheese operads are not formal [Liv15; Wil17], a fact related to the nonformality of the natural inclusion $\mathsf{D}_{n-1} \subset \mathsf{D}_n$ [TW18].

In this paper, we introduce variants of the Swiss-Cheese operads VSC_{mn} for $n - 2 \geq m \geq 1$. These operads encode the action of a D_n-algebra on a D_m-algebra by a central derivation (see Section 3.2). Elements of $\mathsf{VSC}_{mn}(k, l)$ are given by configurations of two kinds of little n-disks: l aerial disks which are forbidden from touching $D^m \subset D^n$, and k terrestrial disks which are centered on D^m. Our main result is:

Theorem A (See Theorem 5.27). *For $n - 2 \geq m \geq 1$, the higher-codimensional Swiss-Cheese operad VSC_{mn} is formal over $\mathbb{R}$, i.e. $H^*(\mathsf{VSC}_{mn}; \mathbb{R})$ (viewed as a cooperad in the category of commutative dg-algebras) encodes the real homotopy type of VSC_{mn}.*

It follows from results of Livernet [Liv15] that $\mathsf{VSC}_{(n-1)n}$ is not formal (Remark 3.27).

Motivation The motivation for this paper comes from the study of configuration spaces. Indeed, the little disks operads are intimately linked to configuration spaces of Euclidean spaces. If we take an element of $\mathsf{D}_n(k)$ and we keep only the centers of the disks, then we obtain an element of the ordered configuration space $\mathrm{Conf}_{\mathbb{R}^n}(k)$. We thus obtain homotopy equivalences $\mathsf{D}_n(k) \simeq \mathrm{Conf}_{\mathbb{R}^n}(k)$ for all k, $n \geq 0$, although they do not preserve the operadic structures.

This observation was the starting point of a series of papers on configuration spaces of manifolds. Campos–Willwacher [CW16] and we [Idr16] provided combinatorial models for the real homotopy types of configuration spaces of simply connected closed smooth manifolds. With Campos, Lambrechts, and Willwacher [CILW18], we provided similar models for configuration spaces of compact, simply connected smooth manifolds with simply connected boundary of dimension ≥ 4. With Campos, Ducoulombier, and Willwacher [CDIW18], we studied framed configuration spaces of orientable closed smooth manifolds, i.e. configurations of points equipped with trivializations of the tangent spaces.

In each case, we used models for the little disks operads or their variants. Indeed, a closed manifold is locally homeomorphic to $\mathbb{R}^n$. As we saw above, configuration spaces of $\mathbb{R}^n$ are linked to the little disks operad D_n. The formality of D_n, and more precisely its proof by Kontsevich and Lambrechts–Volić, was essential in [Idr16; CW16]. A manifold with boundary is locally homeomorphic to the upper half-space $\mathbb{H}^n$. By analogy, configuration spaces of $\mathbb{H}^n$ are linked to the Swiss-Cheese operad SC_n. While SC_n is not formal, Willwacher [Wil15] defined a model for the real homotopy type of SC_n. We used this model extensively in [CILW18]. For framed configuration spaces [CDIW18], we used the model for the framed little disks operad due to Khoroshkin–Willwacher [KW17].

The overarching goal of the present paper is to provide a stepping stone to study configuration spaces of more general manifolds. More precisely, let N be a closed manifold and $M \subset N$ be an embedded submanifold of codimension ≥ 2. Our goal is to study the configuration spaces of the complement $N \setminus M$, for example the complement of a knot $S^3 \setminus K$. Such a pair (N, M) is locally homeomorphic to the stratified space $(\mathbb{R}^n, \mathbb{R}^m)$. Using the analogy above, configuration spaces of $(\mathbb{R}^n, \mathbb{R}^m)$ are linked to the operad VSC_{mn}. Hence we hope that the proof of the formality of VSC_{mn} can be adapted in order to define models for the real homotopy type of $\mathrm{Conf}_{N\setminus M}(k)$.

Let us note that Willwacher studied another generalization of the Swiss-Cheese operad, called the "extended Swiss-Cheese operad" ESC_{mn} [Wil17]. He proved that this operad is formal for $n - m \geq 2$. The difference between ESC_{mn} and VSC_{mn} can be seen at the level of configuration spaces. In VSC_{mn}, the aerial disks in are forbidden from touching the "ground" D^m, whereas this is allowed in ESC_{mn}. We refer to Remark 2.11 for a precise statement. The formality of ESC_{mn} is equivalent to the formality of $\mathsf{D}_m \subset \mathsf{D}_n$. It does not seem easy to adapt the argument for VSC_{mn}, as it is obtained by removing a subspace from ESC_{mn}, an operation which is usually difficult to deal with in rational homotopy theory.

Proof The proof of our theorem is inspired by Kontsevich's proof of the formality of the little disks operad and its improvement by Lambrechts–Volić [Kon99; LV14]. For technical reasons, it is necessary to consider the Fulton–MacPherson compactification $\mathsf{FM}_n(k)$ of $\mathrm{Conf}_k(\mathbb{R}^n)$ [AS94; FM94; Sin04]. The collection FM_n assembles to form a topological operad which has the same homotopy type as D_n. The goal is to find a zigzag, in the category of cooperads in commutative differential-graded algebras, between the cohomology of FM_n and the forms on FM_n. Kontsevich builds a cooperad graphs_n, spanned by graphs with two kinds of vertices (external and internal), with a differential designed to kill the relations in $H^*(\mathsf{FM}_n)$. There is a quotient quasi-isomorphism $\mathsf{graphs}_n \to H^*(\mathsf{FM}_n)$, and a quasi-isomorphism $\omega : \mathsf{graphs}_n \to \Omega(\mathsf{FM}_n)$ given by integrals.

Using the same pattern, we first define VFM_{mn}, a variant of VSC_{mn} inspired by the Fulton–MacPherson compactification. We build an intermediate cooperad of graphs, $\mathsf{vgraphs}_{mn}$, which serves as a bridge between the cohomology $H^*(\mathsf{VFM}_{mn})$ and the forms $\Omega^*(\mathsf{VFM}_{mn})$. The definition of $\mathsf{vgraphs}_{mn}$ is inspired Willwacher's model for the Swiss-Cheese operad [Wil15] and the graph complex used in [KW17, Section 8]. The map $\mathsf{vgraphs}_{mn} \to \Omega^*(\mathsf{VFM}_{mn})$ is defined by integrals. Unfortunately, we cannot find a direct map $\mathsf{vgraphs}_{mn} \to H^*(\mathsf{VFM}_{mn})$, as the differential of $\mathsf{vgraphs}_{mn}$ depends on non-explicit integrals. However, using vanishing results on the cohomology of some graph complex, we are able to simplify $\mathsf{vgraphs}_{mn}$ up to homotopy, and then map it to $H^*(\mathsf{VFM}_{mn})$.

Outline In Section 1, we write down some necessary background on operads, the little disks operads, and the theory of piecewise algebraic forms. In Section 2, we define the Fulton–MacPherson compactification VFM_{mn} and we compare it with VSC_{mn} and with the operad $\mathsf{Disk}^{\mathrm{fr}}_{m\subset n}$ of locally constant factorization algebras on $\{\mathbb{R}^m \subset \mathbb{R}^n\}$ [AFT17]. We give examples of VSC_{mn}-algebras based on relative iterated loop spaces. In Section 3, we compute the cohomology of the operad VSC_{mn}. We also give a presentation by generators and relation of its homology $\mathsf{vsc}_{mn} = H_*(\mathsf{VSC}_{mn})$. In Section 4, we start by reviewing Kontsevich's proof of the formality of the little disks operad, and we define the cooperad $\mathsf{vgraphs}_{mn}$ that will be used to adapt that proof to VFM_{mn}. We moreover construct the map from $\mathsf{vgraphs}_{mn}$ to forms on VFM_{mn} using integrals. Finally, in Section 5, we complete the proof of the main theorem. We first show that certain integrals used in the definition of $\mathsf{vgraphs}_{mn}$ can be simplified. Then we construct the map into $H^*(\mathsf{VFM}_{mn})$ and we show that all our maps are quasi-isomorphisms.

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1 Prerequisites

Throughout this paper, we work with cohomologically graded dg-modules over the base field $\mathbb{R}$ (except in Section 3.1 where we work over $\mathbb{Z}$).

1.1 Operads

We work extensively with operads and cooperads. General references include [LV12] and [Fre17, Part I(a)]. We assume basic proficiency with the theory.

For some applications, especially when working with cooperads, it is easier to label the inputs of an operation by elements of an arbitrary finite set rather than $\{1, \dots, k\}$. Briefly, we fix a base symmetric monoidal category $\mathcal{C}$ (e.g. dg-modules). Let Bij be the category of finite sets and bijections A symmetric collection is a functor $\mathsf{Bij} \to \mathcal{C}$.

For $k \geq 0$, we let $\underline{k} = \{1, \dots, k\}$ (in particular $\underline{0} = \varnothing$). A symmetric collection M can equivalently be seen as a sequence $\{\mathsf{M}(n) \coloneqq \mathsf{M}(\underline{n})\}_{n \geq 0}$ with each $\mathsf{M}(n)$ equipped with an action of the symmetric group $\Sigma_n = \mathrm{Aut}_{\mathsf{Bij}}(\underline{n})$.

For a pair of finite sets $W \subset U$, we define the quotient $U/W = (U \setminus W) \sqcup \{*\}$. For example $U/\varnothing = U \sqcup \{*\}$. An operad P is a symmetric collection equipped with composition maps $\circ_W : \mathsf{P}(U/W) \otimes \mathsf{P}(W) \to \mathsf{P}(U)$, for each pair $W \subset U$, satisfying the usual axioms. Dually, a cooperad C is a symmetric collection equipped with cocomposition maps $\circ_W^\vee : \mathsf{C}(U) \to \mathsf{C}(U/W) \otimes \mathsf{C}(W)$. Following [Fre17], we call “Hopf cooperad” a cooperad in the category of commutative differential-graded algebras (CDGAs).

We also deal with some special particular two-colored operads called “relative operads” [Vor99] or “Swiss-Cheese type operads” [Wil16]. A compact definition is the following: given an operad P, a relative P-operad is an operad in the category of right P-modules. If we unpack this definition, a relative P-operad is a “bisymmetric collection”, i.e. a functor $\mathsf{Q} : \mathsf{Bij} \times \mathsf{Bij} \to \mathcal{C}$, equipped with two kinds of composition maps:

$$\begin{aligned} \circ_T &: \mathsf{Q}(U, V/T) \otimes \mathsf{P}(T) \to \mathsf{Q}(U, V) && \text{for } V \subset T, \\ \circ_{W,T} &: \mathsf{Q}(U/W, V) \otimes \mathsf{Q}(W, T) \to \mathsf{Q}(U, V \sqcup T) && \text{for } W \subset U, \end{aligned}$$

as well as a unit, satisfying associativity, equivariance, and unitality axioms. We will often write “the operad Q”, and P will be implicit from the context. A relative C-cooperad is defined dually. We also apply the adjective “Hopf” to denote such cooperads in the category of CDGAs.

1.2 Little disks and variants

Fix some $n \geq 0$. Let us define the little n-disks operad D_n. Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed disk. The space $\mathsf{D}_n(r)$ is the space of maps $c : (D^n)^{\sqcup r} \to D^n$ satisfying the following properties: (i) each $c_i : D^n \to D^n$ is an embedding given by a composition of a translation and a positive rescaling; (ii) the interiors of two different little disks are disjoint, i.e. $c_i(\mathring{D}^n) \cap c_j(\mathring{D}^n) = \varnothing$ for $i \neq j$. Using composition of embeddings, the collection $\mathsf{D}_n = \{\mathsf{D}_n(r)\}_{r \geq 0}$ assembles to form a topological operad.

Let us also describe two operads with the same homotopy type as D_n. The first comes from the link between the little disks operads and configuration spaces. We will need it for technical reasons. Given some space X, its rth (ordered) configuration space is:

$$\mathrm{Conf}_X(r) \coloneqq \{(x_1, \dots, x_k) \in X^k \mid \forall i \neq j,\, x_i \neq x_j\}. \tag{1.1}$$

More generally, for a finite (discrete) set U, $\mathrm{Conf}_X(U)$ is the set of injections $U \hookrightarrow X$.

Given an element of $\mathsf{D}_n(r)$, we can forget the radii and keep the center of each disk. This map defines a homotopy equivalence $\mathsf{D}_n(r) \to \mathrm{Conf}_{\mathbb{R}^n}(r)$. However, it is not possible to see the operad structure of D_n on the configuration spaces directly. To remedy this deficiency, we can use the Fulton–MacPherson compactification $\mathsf{FM}_n(r)$ of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ [FM94; AS94] (see also [Sin04] and [LV14, Chapter 5]). First, note that there is a natural action of the group $\mathbb{R}^n \rtimes \mathbb{R}_{>0}$ of translations and positive rescaling on

$\mathrm{Conf}_{R^n}(r)$. We can thus consider the quotient $\underline{\mathrm{Conf}}_n(r) \coloneqq \mathrm{Conf}_{\mathbb{R}^n}(r)/\mathbb{R}^n \rtimes \mathbb{R}_{>0}$, and the quotient map is a homotopy equivalence. We can then define an embedding:

$$(\theta_{ij}, \delta_{ijk}) : \underline{\mathrm{Conf}}_n(r) \to (S^{n-1})^{\binom{r}{2}} \times [0, +\infty]^{\binom{r}{3}}. \tag{1.2}$$

Given some $x \in \mathrm{Conf}_{\mathbb{R}^n}(r)$, we have $\theta_{ij}(x) \coloneqq (x_i - x_j)/\|x_i - x_j\|$ which records the direction from i to j, while $\delta_{ijk}(x) = \|x_i - x_j\|/\|x_i - x_k\|$ records the relative distance. Then the Fulton–MacPherson compactification $\mathsf{FM}_n(r)$ is the closure of the image of the embedding above. It is a stratified manifold, of dimension $nr - n - 1$ if $r \geq 2$ and reduced to a point otherwise. Its interior is $\underline{\mathrm{Conf}}_n(r)$, and the inclusion is a deformation retract. For example, we have $\mathsf{FM}_n(0) = \mathsf{FM}_n(1) = \{*\}$ and $\mathsf{FM}_n(2) = S^{n-1}$. There is a natural operad structure on the collection FM_n, defined by explicit formulas in $(S^{n-1})^{\binom{r}{2}} \times [0, +\infty]^{\binom{r}{3}}$, see [LV14, Section 5.2]. The operads FM_n and D_n have the same homotopy type, see [Mar99] and [Sal01, Proposition 4.9].

The second variant of the little disks operads serves as motivation for the paper. The little disks operad D_n has the same homotopy type as the operad $\mathsf{Disk}_n^{\mathrm{fr}}$ of locally constant framed factorization algebras on $\mathbb{R}^n$ which we now define, see e.g. [Lur17; AFT17; CG17]. Let us consider $\mathbb{R}^n$ with its canonical parallelization. The space $\mathsf{Disk}_n^{\mathrm{fr}}(k)$ is given by the space of embeddings of k copies of $\mathbb{R}^n$ into $\mathbb{R}^n$ which preserve the parallelization. The collection $\mathsf{Disk}_n^{\mathrm{fr}}$ forms a topological operad by considering composition of embeddings. The proof that $\mathsf{Disk}_n^{\mathrm{fr}} \simeq \mathsf{D}_n$ is almost identical to the proof of [Mar99]. The advantage of this definition is that we can take any stratified (parallelized) space instead of $\mathbb{R}^n$. The rest of the paper is devoted to the operad associated to the stratified space $\{\mathbb{R}^m \subset \mathbb{R}^n\}$.

Remark 1.3. There are some issues with units (i.e. the arity zero component $\mathsf{Disk}_n^{\mathrm{fr}}(0)$) in the ∞-categorical setting, especially the connection with the Ran space [Lur17, Section 5.5]. While what we wrote above is true in our strict topological setting, the name "operad governing locally constant framed factorization algebras on $\mathbb{R}^n$" might be a bit abusive if we allow $k = 0$. This is of no consequence in this paper.

1.3 Semi-algebraic sets and PA forms

For technical reasons, we will need to use the technology of semi-algebraic (SA) sets and piecewise algebraic (PA) forms. Initially developed by [KS00, Appendix 8], the theory was brought to its current form in [HLTV11].

An SA set is a subset of $\mathbb{R}^N$ (for some N) which is obtained as finite unions of finite intersections of sets defined by polynomial equalities and inequalities. One can also define SA manifolds and SA stratified sets [HLTV11, Section 2]. There also exists a notion of SA bundles [HLTV11, Definition 8.1].

Example 1.4. The Fulton–MacPherson compactification $\mathsf{FM}_n(k)$ is an SA stratified manifold [LV14, Proposition 5.1.2]. The projection $p_{ij} : \mathsf{FM}_n(k) \to \mathsf{FM}_n(2)$ which forgets all points but two is an SA bundle [LV14, Theorem 5.3.2].

Given an SA set X, a minimal form on X is a form of the type $f_0 df_1 \wedge \ldots \wedge df_k$, where the $f_i : X \to \mathbb{R}$ are SA maps [HLTV11, Section 5.2]. A PA form is roughly speaking one that is obtained as obtained by integrating a minimal form along the fibers of an SA bundle (or more generally, a strongly continuous family of chains) [HLTV11, Section 5]. The complex $\Omega^*_{\mathrm{PA}}(X)$ of all PA forms on X is a CDGA. If X is compact, then this CDGA is quasi-isomorphic to the CDGA of PL forms on X with real coefficients, $A^*_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$. In other words, it is a model for the real homotopy type of X [HLTV11, Theorem 6.1].

The reason we need to consider such PA forms is the existence of the map which integrates along the fibers of a PA bundle, which satisfies a good Stokes' formula [HLTV11, Section 8]. This is necessary for our purposes, because the projection map p_{ij} above is a PA bundle but not a submersion [LV14, Example 5.9.1].

Definition 1.5. An operad P in compact SA sets is formal (over $\mathbb{R}$) if there exists a zigzag of quasi-isomorphisms of Hopf cooperads $H^*(\mathsf{P}; \mathbb{R}) \leftarrow \cdot \rightarrow \Omega^*_{\mathrm{PA}}(\mathsf{P})$.

Another common definition of "formality" merely requires the dg-operads $C_*(\mathsf{P})$ and $H_*(\mathsf{P})$ to be quasi-isomorphic. The notion defined above is stronger: if P is formal, we can forget cup products and dualize to recover formality of chains.

Note that $\Omega^*_{\mathrm{PA}}(\mathsf{P})$ is not actually a Hopf cooperad, because the Künneth quasi-isomorphisms go in the wrong direction. However, if C is a Hopf cooperad, then it is possible to define a morphism $\mathsf{C} \to \Omega^*_{\mathrm{PA}}(\mathsf{P})$ as a collection of maps $\mathsf{C}(U) \to \Omega^*_{\mathrm{PA}}(\mathsf{P}(U))$ making the obvious diagrams commute [LV14, Chapter 3]. The results of Fresse [Fre17, §II.12] can be adapted to Ω^*_{PA} to show that if P is cofibrant in the category of Hopf cooperads satisfying $\mathsf{P}(0) = \{*\}$ (which is true of FM_n), then a Hopf cooperad which is quasi-isomorphic to $\Omega^*_{\mathrm{PA}}(\mathsf{P})$ encodes the real homotopy type of P. The constructions can be extended to colored operads.

2 Definition of VFM_{mn} and comparison

From now on and until the end of the paper, we fix integers $n > m \geq 1$. Apart from some remarks, we will always assume that $n - m \geq 2$. We identify $\mathbb{R}^m$ as the subspace of $\mathbb{R}^n$ given by $\mathbb{R}^m \times \{0\}^{n-m}$.

2.1 The compactification and its boundary

We define the colored configuration spaces by:

$$\mathrm{Conf}_{mn}(U, V) := \mathrm{Conf}_{\mathbb{R}^m}(U) \times \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(V) \subset \mathrm{Conf}_{\mathbb{R}^n}(U \sqcup V). \tag{2.1}$$

Roughly speaking, $\mathrm{Conf}_{mn}(U, V)$ is the set of configurations of two kinds of points in $\mathbb{R}^n$: U "terrestrial" points in $\mathbb{R}^m$, and V "aerial" points in $\mathbb{R}^n \setminus \mathbb{R}^m$. We will reuse the terminology "aerial/terrestrial" throughout the paper.

There is a natural action of the group $\mathbb{R}^m \rtimes \mathbb{R}_{>0}$ of translations and positive rescalings on $\mathrm{Conf}_{mn}(U, V)$. We define the quotient:

$$\underline{\mathrm{Conf}}_{mn}(U, V) := \mathrm{Conf}_{mn}(U, V) / (\mathbb{R}^m \rtimes \mathbb{R}_{>0}). \tag{2.2}$$

Since $\mathbb{R}^m \rtimes \mathbb{R}_{>0}$ is contractible, the quotient map is a weak homotopy equivalence. If $2\#U + \#V \geq 2$, then the action is free and properly discontinuous, thus $\underline{\mathrm{Conf}}_{mn}(U,V)$ is a manifold of dimension $m\#U + n\#V - m - 1$. However, if $\#U = 0$ and $\#V \leq 1$, then $\underline{\mathrm{Conf}}_{mn}(U,V)$ is merely a point as the action is clearly transitive. We define an embedding

$$(\theta_{ij}, \delta_{ijk}, \alpha_v) : \underline{\mathrm{Conf}}_{mn}(U,V) \hookrightarrow (S^{n-1})^{\binom{U\sqcup V}{2}} \times [0,+\infty]^{\binom{U\sqcup V}{3}} \times (S^{n-m-1})^V, \tag{2.3}$$

$$\theta_{ij}(x) \coloneqq (x_i - x_j)/\|x_i - x_j\|, \tag{2.4}$$

$$\delta_{ijk}(x) \coloneqq \|x_i - x_j\|/\|x_i - x_k\|, \tag{2.5}$$

$$\alpha_v(x) \coloneqq p_{(\mathbb{R}^m)^\perp}(x_v)/\|p_{(\mathbb{R}^m)^\perp}(v)\|, \tag{2.6}$$

where $p_{(\mathbb{R}^m)^\perp}$ is the orthogonal projection on $(\mathbb{R}^m)^\perp = \{0\}^m \times \mathbb{R}^{n-m} \subset \mathbb{R}^n$.

Definition 2.7. The space $\mathsf{VFM}_{mn}(U,V)$ is the closure of the image of the embedding $(\theta_{ij}, \delta_{ijk}, \vartheta_v)$ inside $(S^{n-1})^{\binom{U\sqcup V}{2}} \times [0,+\infty]^{\binom{U\sqcup V}{3}} \times (S^{n-m-1})^V$.

Example 2.8. We have $\mathsf{VFM}_{mn}(U,\varnothing) = \mathsf{FM}_m(U)$ and $\mathsf{VFM}_{mn}(0,1) = S^{n-m-1}$.

Proposition 2.9. *The space $\mathsf{VFM}_{mn}(U,V)$ is a compact semi-algebraic stratified manifold. Its dimension is $m\#U + n\#V - m - 1$ if $2\#V + \#U \geq 2$, and zero otherwise. The projections $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U,V)$ which forget some points are SA bundles.*

Proof. We can adapt the proofs of [LV14, Propositions 5.1.2, Theorem 5.3.2] in a straightforward way. ☐

Proposition 2.10. *The collection VFM_{mn} assembles to form a relative FM_n-operad in compact SA sets.*

Proof. We can define operadic structure maps on VFM_{mn} by straightforward formulas in the coordinates $(S^{n-1})^{\cdots} \times [0,+\infty]^{\cdots} \times (S^{n-m-1})^{\cdots}$. For example, for $T \subset V$, $x \in \mathsf{VFM}_{mn}(U, V/T)$, and $y \in \mathsf{FM}_n(T)$, we have $\theta_{ij}(x \circ_T y) = \theta_{ij}(y)$ if $i, j \in T$, and $\theta_{ij}(x \circ_T y) = \theta_{[i][j]}(x)$ otherwise. The other formulas are defined on a similar case by case basis. Compare with [LV14, Section 5.2] for the case of FM_n. ☐

Remark 2.11. The operad VFM_{mn} is not homotopy equivalent to the operad ESC_{mn} considered by Willwacher [Wil17]. Recall that $\mathsf{ESC}_{mn}(U,V) \coloneqq \mathsf{D}_n(U \sqcup V) \times_{\mathsf{D}_n(U)} \mathsf{D}_m(U)$, where $\mathsf{D}_n(U \sqcup V) \to \mathsf{D}_n(U)$ is the projection that forgets disks and $\mathsf{D}_m(U) \to \mathsf{D}_n(U)$ is the usual embedding. The crucial difference is that do not allow "aerial" points to be on $\mathbb{R}^m$. For example, $\mathsf{VFM}_{mn}(0,1) = S^{n-m-1} \not\simeq \mathsf{ESC}_{mn}(0,1) \simeq *$, and $\mathsf{VFM}_{mn}(1,1) \simeq S^{n-m-1} \not\simeq \mathsf{ESC}_{mn}(1,1) \simeq S^{n-1}$.

Let us describe the boundary of $\mathsf{VFM}_{mn}(U,V)$ and the fiberwise boundary of the projection maps.

Proposition 2.12. *We have a decomposition:*

$$\partial \mathsf{VFM}_{mn}(U,V) = \bigcup_{T \in \mathcal{BF}'(V)} \operatorname{im} \circ_T \cup \bigcup_{(W,T) \in \mathcal{BF}''(U;V)} \operatorname{im} \circ_{W,T},$$

where the boundary faces are indexed by:

$$\mathcal{BF}'(V) \coloneqq \{T \subset V \mid \#T \geq 2\},$$
$$\mathcal{BF}''(U;V) \coloneqq \{(W \subset U, T \subset V) \mid \#(W \cup T) < \#(U \cup V) \text{ and } 2 \cdot \#T + \#W \geq 2\}.$$

Each of these boundary faces is open in the boundary, and the intersection of two distinct faces is of positive codimension.

Proof. Adapting the proofs of [LV14, Proposition 5.4.1] is straightforward. □

If $p : E \to B$ is an SA bundle, then its fiberwise boundary is $E^\partial = \bigcup_{x \in B} \partial p^{-1}(x)$, see [HLTV11, Definition 8.1]. If p is an SA bundle of rank k, then $p^\partial : E^\partial \to B$ is an SA bundle of rank $k-1$. It is instructive to compute the fiberwise boundary of $\mathrm{pr}_1 : [0,1]^2 \to [0,1]$. We see that E^∂ is neither ∂E nor $\bigcup_{x \in B} p^{-1}(x) \cap \partial E$. The fiberwise Stokes formula for PA forms reads $d(p_* \alpha) = p_*(d\alpha) \pm p^\partial_* \alpha$ [HLTV11, Proposition 8.12].

Proposition 2.13. *The fiberwise boundary of the canonical projection* $p_{U,V} : \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) \to \mathsf{VFM}_{mn}(U,V)$ *is given by:*

$$\mathsf{VFM}^\partial_{mn}(U,V) = \bigcup_{T \in \mathcal{BF}'(V,J)} \operatorname{im} \circ_T \cup \bigcup_{(W,T) \in \mathcal{BF}''(U,I;V,J)} \operatorname{im} \circ_{W,T} \subset \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J),$$

where the subsets $\mathcal{BF}'(V,J) \subset \mathcal{BF}'(V \sqcup J)$ *and* $\mathcal{BF}''(U,I;V,J) \subset \mathcal{BF}''(U \sqcup I, V \sqcup J)$ *are respectively defined by the conditions* $\#(T \cap J) \leq 1$ *and by* $((U \subset W$ *and* $V \subset T)$ *or* $(V \cap T = \varnothing$ *and* $\#(U \cap W) \leq 1))$.

Proof. We can adapt the proof of [LV14, Proposition 5.7.1] immediately. □

2.2 Comparison with other operads

In this section, we compare VFM_{mn} with two operads: the operad $\mathsf{Disk}^{\mathrm{fr}}_{m \subset n}$ of locally constant factorization algebras on $\{\mathbb{R}^m \subset \mathbb{R}^n\}$, and an operad VSC_{mn} that is analogous to the Swiss-Cheese operad SC_n.

2.2.1 Comparison with $\mathsf{Disk}^{\mathrm{fr}}_{m \subset n}$

Let us first compare VFM_{mn} with $\mathsf{Disk}^{\mathrm{fr}}_{m \subset n}$. We refer to [AFT17, Section 4.3] for analogous definitions in the ∞-categorical setting.

Definition 2.14. As a space, $\mathsf{Disk}^{\mathrm{fr}}_{m \subset n}(U,V)$ is given by framed embeddings $\gamma = (\gamma_i)_{i \in U \sqcup V} : (\mathbb{R}^n)^{\sqcup U} \sqcup (\mathbb{R}^n)^{\sqcup V} \to \mathbb{R}^n$ such that (i) for $u \in U$, $\gamma_u(\mathbb{R}^m) \subset \mathbb{R}^m$, and (ii) for $v \in V$, $\gamma_v(\mathbb{R}^n) \subset \mathbb{R}^n \setminus \mathbb{R}^m$. It is a relative operad (Section 1.1) over the operad $\mathsf{Disk}^{\mathrm{fr}}_n$ (Section 1.2) using composition of embeddings.

Proposition 2.15. *There exists a zigzag a weak homotopy equivalences of operads between* $(\mathsf{Disk}^{\mathrm{fr}}_{m\subset n}, \mathsf{Disk}^{\mathrm{fr}}_n)$ *and* $(\mathsf{VFM}_{mn}, \mathsf{FM}_n)$.

Proof. Briefly, recall that for a topological operad P, the Boardman–Vogt resolution $W\mathsf{P} \xrightarrow{\sim} \mathsf{P}$ is a canonical resolution of P. The elements of $W\mathsf{P}(k)$ are rooted trees with k leaves, such that each internal vertex is labeled by an element of P (of arity equal to the number of incoming edges), and each internal edge is labeled by a time parameter $t \in [0,1]$. If an edge is labeled by $t = 0$, then the tree is identified with the quotient obtained by contracting the edge and applying the operadic structure of P on the labels of the corresponding vertices. The operadic structure of $W\mathsf{P}$ is obtained by grafting trees, assigning the time parameter $t = 1$ to the new edge. The map $W\mathsf{P} \to \mathsf{P}$ simply contracts all the edges, and applies the operadic structure of P to the vertices.

A weak equivalence $W\mathsf{Disk}^{\mathrm{fr}}_n \xrightarrow{\sim} \mathsf{FM}_n$ (or its analogue for D_n) was constructed in [Mar99, Section 3] and [Sal01, Proposition 3.9]. It is obtained by taking an element of $W\mathsf{Disk}^{\mathrm{fr}}_n$, rescaling the disks by the time parameter associated to the edge, and keeping the centers of the disks (applying the operadic composition if the time parameter goes to zero).

We want to construct a morphism $W\mathsf{Disk}^{\mathrm{fr}}_{m\subset n} \to \mathsf{VFM}_{mn}$, compatible with the previous one. Using the same method as in [Mar99; Sal01], there exists one that extends the map $\mathsf{Disk}^{\mathrm{fr}}_{m\subset n} \to \mathrm{Conf}_{mn}$ which keeps only the centers of the disks.

It remains to check that the induced map $W\mathsf{Disk}^{\mathrm{fr}}_{m\subset n} \to \mathsf{VFM}_{mn}$ is a homotopy equivalence in each arity. It is sufficient to check that $\mathrm{Conf}_{mn} \to \mathsf{VFM}_{mn}$ is a weak equivalence and to conclude by the 2-out-of-3 property. This is done by considering an explicit deformation retract, see [Sal01, Proposition 2.5]. □

2.2.2 Comparison with VSC_{mn} and a conjecture

We can also compare VFM_{mn} with the higher-codimensional Swiss-Cheese operad VSC_{mn}. Let $D^m = D^n \cap \mathbb{R}^m$ for convenience.

Definition 2.16. The space $\mathsf{VSC}_{mn}(U,V)$ is the space of maps $c : (D^n)^{U\sqcup V} \hookrightarrow D^n$ satisfying: (i) for all i, $c_i : D^n \hookrightarrow D^n$ is an embedding obtained by composing a translation and a positive rescaling; (ii) for $i \neq j$, we have $c_i(\mathring{D}^n) \cap c_j(\mathring{D}^n) = \varnothing$; (iii) for $u \in U$, we have $c_u(D^m) \subset D^m$; (iv) for $v \in V$, we have $c_v(D^n) \cap D^m = \varnothing$. Using composition of embeddings, VSC_{mn} is a relative D_n-operad.

Remark 2.17. When $n = m+1$, this operad is not the usual Swiss-Cheese operad SC_n. In fact, SC_n is a suboperad of $\mathsf{VSC}_{(n-1)n}$, by considering the connected components where all the aerial disks are in the upper half-disk.

Proposition 2.18. *There exists a zigzag of weak homotopy equivalences of operads* $(\mathsf{VSC}_{mn}, \mathsf{D}_n) \simeq (\mathsf{VFM}_{mn}, \mathsf{FM}_n)$.

Proof. There is an embedding of operads $(\mathsf{VSC}_{mn}, \mathsf{D}_n) \subset (\mathsf{Disk}^{\mathrm{fr}}_{m\subset n}, \mathsf{Disk}^{\mathrm{fr}}_n)$. We also have obvious factorizations:

$$\mathsf{D}_n(U) \xrightarrow{\sim} \mathrm{Conf}_{\mathbb{R}^n}(U) \xleftarrow{\sim} \mathsf{Disk}^{\mathrm{fr}}_n(U), \quad \mathsf{D}_n(U) \hookrightarrow \mathsf{Disk}^{\mathrm{fr}}_n(U),$$

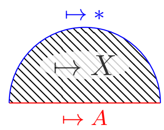
$$\mathsf{VSC}_{mn}(U,V) \xrightarrow{\sim} \mathrm{Conf}_{mn}(U,V) \xleftarrow{\sim} \mathsf{Disk}^{\mathrm{fr}}_{m\subset n}(U,V), \quad \mathsf{VSC}_{mn}(U,V) \hookrightarrow \mathsf{Disk}^{\mathrm{fr}}_{m\subset n}(U,V).$$

Hence we can conclude by the 2-out-of-3 property and Proposition 2.15. □

Let us also give examples of VSC_{mn}-algebras. For motivation, let us first recall algebras over D_n and SC_n. For a pointed space $* \in X$, the iterated loop space $\Omega^n X$ is the space of maps $\gamma : D^n \to X$ such that $\gamma(\partial D^n) = *$. The space $\Omega^n X$ is an algebra over D_n. Conversely, the recognition principle states that any "grouplike" D_n-algebra is weakly equivalent to an iterated loop space [BV68; May72].

For a pair of pointed topological spaces $* \in A \subset X$, there is a associated relative iterated loop space $\Omega^n(X, A) = \mathrm{hofib}(\Omega^{n-1}A \to \Omega^{n-1}X)$. Concretely, consider the upper half-disk $D^n_h = D^n \cap \mathbb{H}^n$. Its boundary ∂D^n_h is obtained by gluing the lower disk $\partial_- D^n_h = D^n \cap \partial\mathbb{H}^n \cong D^{n-1}$ to the upper hemisphere $\partial_+ D^n_h = \partial D^n \cap \mathbb{H}^n \cong D^{n-1}$ along the circle $\partial D^n \cap \partial \mathbb{H}^n \cong S^{n-2}$.

The relative iterated loop space $\Omega^n(X, A)$ is then the space of maps $\gamma : D^n_h \to X$ such that $\gamma(\partial_- D^n_h) \subset A$ and $\gamma(\partial_+ D^n_h) = *$. For example, we have that $\Omega^1(X, A) = \{\gamma : [0, 1] \to X \mid \gamma(0) \in A, \gamma(1) = *\}$. A sketch for $n = 2$ is on the side. The pair $(\Omega^n(X, A), \Omega^n X)$ is an algebra over the operad SC_n. Conversely, a relative recognition principle states that any algebra over that operad satisfying good properties is weakly equivalent to such a pair [Duc14; Que15; HLS16; Vie18].



By analogy, we define the (n, m)-relative iterated loop space:

$$\Omega^{n,m}(X, A) \coloneqq \{\gamma : D^n \to X \mid \gamma(D^m) \subset A \text{ and } \gamma(\partial D^n) = *\}. \tag{2.19}$$

Then the pair $(\Omega^{n,m}(X, A), \Omega^n X)$ is an algebra over the operad VSC_{mn}. We conjecture that an analogous relative recognition principle holds: any VSC_{mn}-algebra satisfying appropriate conditions should be weakly equivalent to such a pair.

3 (Co)homology of VSC_{mn}

In this section, we compute the integral cohomology of $\mathsf{VSC}_{mn}(U, V)$ (Definition 2.16). We then give a presentation of the operad $H_*(\mathsf{VSC}_{mn})$ by generators and relations. Unless specified otherwise, the ring of coefficients is $\mathbb{Z}$ in this section.

3.1 The cohomology as a Hopf cooperad

We will first compute the cohomology of $\mathrm{Conf}_W(l)$ with $W \coloneqq \mathbb{R}^n \setminus \mathbb{R}^m$. The computation is inspired by the methods of [Sin13]. We prove that it free as an abelian group, thus we will be able to apply Künneth's formula to get the cohomology of $\mathsf{VSC}_{mn}(k, l) \simeq \mathrm{Conf}_{\mathbb{R}^m}(k) \times \mathrm{Conf}_W(l)$ as a tensor product. Then we study what maps are induced on cohomology by the operad structure of VSC_{mn}.

Let us consider the Fadell–Neuwirth fibration, which forgets (for example) the last point of a configuration. Note that W with l points removed is homotopy equivalent to $S^{n-m-1} \vee \bigvee^l S^{n-1}$. Therefore we get fibrations:

$$S^{n-m-1} \vee \bigvee\nolimits^l S^{n-1} \hookrightarrow \mathrm{Conf}_W(l+1) \xrightarrow{p_{l+1}} \mathrm{Conf}_W(l) \tag{3.1}$$

Definition 3.2. The Poincaré polynomial of a space X is $\mathscr{P}(X) \coloneqq \sum_{i\geq 0}(\operatorname{rk} H^i(X)) \cdot t^i$. For P, $Q \in \mathbb{N}[[t]]$, we say that $P \preceq Q$ if all the coefficients of $Q - P$ are nonnegative.

Proposition 3.3. *The Poincaré polynomial of* $\mathrm{Conf}_W(l)$ *satisfies:*

$$\mathscr{P}(\mathrm{Conf}_W(l)) \preceq \prod_{i=0}^{l-1}(1 + t^{n-m-1} + it^{n-1}). \tag{3.4}$$

Moreover, if the equality is reached and the homology of $\mathrm{Conf}_W(l-1)$ *is free as a* $\mathbb{Z}$*-module, then the homology of the total space* $\mathrm{Conf}_W(l)$ *is free too.*

Proof. We use the fibrations of Equation (3.1), the Serre spectral sequence, and induction to deduce the proposition. □

Remark 3.5. We used $n - m \geq 2$ to be able to use the Serre spectral sequence without problems. If $n - m - 1 \leq 1$, then the fiber of p_{k+1} is not simply connected, so we should in principle be more careful. However, we can reuse the ideas of the proof of [Coh76, Lemma 6.3] to show that the coefficient system is trivial, thus the proposition still holds.

First, let us recall the usual presentation of $H^*(\mathrm{Conf}_{\mathbb{R}^n}(l))$ [Arn69; Coh76]. It is a quotient of a free graded symmetric algebra, on generators ω_{ij} of degree $n-1$:

$$\mathsf{e}_n^\vee(k) \coloneqq H^*(\mathrm{Conf}_{\mathbb{R}^n}(k)) = \frac{S(\omega_{ij})_{1\leq i\neq j\leq k}}{(\omega_{ji} - (-1)^n\omega_{ij},\, \omega_{ij}^2,\, \omega_{ab}\omega_{bc} + \omega_{bc}\omega_{ca} + \omega_{ca}\omega_{ab})}. \tag{3.6}$$

Definition 3.7. We define a algebra, with generators η_i of degree $n - m - 1$:

$$\mathsf{vsc}_{mn}^\vee(0,l) \coloneqq \frac{\mathsf{e}_n^\vee(l) \otimes S(\eta_i)_{1\leq i\leq l}}{(\eta_i^2, \eta_i\omega_{ij} - \eta_j\omega_{ij})}. \tag{3.8}$$

Remark 3.9. This is very similar to the Lambrechts–Stanley model G_A (see [Idr16]) applied to $A = H^*(D^n \setminus D^m)$ with vanishing diagonal class.

Lemma 3.10. *The dg-module* $\mathsf{vsc}_{mn}^\vee(0,l)$ *has the same Poincaré polynomial as the RHS of Inequality* (3.4)*.* □

Proposition 3.11. *We have a well-defined algebra map* $\mathsf{vsc}_{mn}^\vee(0,l) \to H^*(\mathrm{Conf}_W(l))$*, given on generators by* $\omega_{ij} \mapsto \theta_{ij}^*(\mathrm{vol}_{n-1})$ *and* $\eta_i \mapsto \alpha_i^*(\mathrm{vol}_{n-m-1})$ *(with* θ_{ij}*,* α_i *as in Section 2).*

We need an intermediate lemma before proving this proposition.

Lemma 3.12. *The abelian group* $H_{n-m-1}(\mathrm{Conf}_W(l))$ *is free of rank* l*.*

Proof. We use the long exact sequence in homotopy of the Fadell–Neuwirth fibration and the fact that this fibration admits a section. By induction, this shows that $\mathrm{Conf}_W(l)$ is $(n-m-2)$-connected and that $\pi_{n-m-1}(\mathrm{Conf}_W(l))$ is free abelian of rank l. We then conclude by Hurewicz's theorem. □

Proof of Proposition 3.11. The only thing we need to check is that we have the relation $\eta_i\omega_{ij} = \eta_j\omega_{ij}$ in $H^*(\mathrm{Conf}_W(l))$. It is sufficient to check this on $\mathrm{Conf}_W(2)$, as the three classes involved are pulled back from this space. Let us compute the cohomological Serre spectral sequence for the Fadell–Neuwirth fibration with $k = 2$. We find the following E_2 page (the second column is in degree $n - m - 1$ while the second and third rows are in respective degrees $n - m - 1$ and $n - 1$):

2	ω_{12}	$\eta_1\omega_{12}$
1	η_2	$\eta_1\eta_2$
0	1	η_1
	0	1

Checking the degrees, there is only one possibly nonzero differential, only if $n = 2(m + 1)$, which would be $d^{n-m-1}\omega_{12} \propto \eta_1\eta_2$. However, we have a sequence of embeddings $\mathrm{Conf}_{\mathbb{R}^n}(2) \to \mathrm{Conf}_W(2) \to \mathrm{Conf}_{\mathbb{R}^n}(2)$, where the second map is simply the inclusion, and the first map is obtained by choosing any embedding of $\mathbb{R}^n$ into W isotopic to the identity. The composite of these two maps is homotopic to the identity of $\mathrm{Conf}_{\mathbb{R}^n}(2)$. They induce morphisms of the associated Serre spectral sequences. We note that the Serre spectral sequence for $\mathrm{Conf}_{\mathbb{R}^n}(2)$ is trivial, concentrated on one column. By definition, ω_{12} is the pullback of the fundamental class of $\mathrm{Conf}_{\mathbb{R}^n}(2) \simeq S^{n-1}$. But if we pull back once more to $\mathrm{Conf}_{\mathbb{R}^n}(2)$, we get back the generator of $H^{n-1}(\mathrm{Conf}_{\mathbb{R}^n}(2))$, which survives to the end. If $d^{n-m-1}\omega_{12}$ were nonzero in the spectral sequence of $\mathrm{Conf}_W(2)$, then it would not survive until the end. We would thus get $\omega_{12} \mapsto 0 \mapsto \omega_{12} \neq 0 \in E_\infty^{n-1,0}$, which is absurd.

It follows that the spectral sequence collapses. Hence $H^{(n-1)+(n-m-1)}(\mathrm{Conf}_W(2))$ is free of rank 1, generated by $\eta_1\omega_{12}$ (by the multiplicativity of the Serre spectral sequence and Lemma 3.12). Therefore $\eta_2\omega_{12} = \lambda\eta_1\omega_{12}$ for some $\lambda \in \mathbb{Z}$. By a symmetric argument, there also exists a constant $\mu \in \mathbb{Z}$ such that $\eta_1\omega_{12} = \mu\eta_2\omega_{12}$. It follows that $\lambda\mu = 1$, thus $\lambda = \mu = \pm 1$. Choosing the orientations correctly, we obtain $\eta_1\omega_{12} = \eta_2\omega_{12}$. □

Proposition 3.13. *The map* $\mathsf{vsc}^\vee_{mn}(0, l) \to H^*(\mathrm{Conf}_W(l))$ *is an isomorphism.*

Proof. Our proof is inspired by the proof of [Sin13, Theorem 4.9]. Let us check that it is an isomorphism over each field $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$. Then by induction and using the Serre spectral sequence of Proposition 3.11, we show that the homology of $\mathrm{Conf}_W(l)$ is free as a $\mathbb{Z}$-module. This will then imply that the map of the proposition is an isomorphism over $\mathbb{Z}$ using the universal coefficients theorem.

Given Inequality (3.4), the universal coefficients theorem, and the fact that we are working over a field, it is sufficient to show that the map tensored with $\Bbbk$ is injective. We may create classes in $H_*(\mathrm{Conf}_W(l); \Bbbk)$ using trees where vertices are possibly decorated by a loop (to represent the homology class corresponding to η_i). The Jacobi relation is satisfied, as these homology classes come from the subspace $\mathrm{Conf}_{\mathbb{R}^n}(l)$ (with $\mathbb{R}^n$ being e.g. the upper half-space). Similarly, the classes from $\mathsf{vsc}^\vee_{mn}(l)$ correspond to graphs modded out by the Arnold relation, with each connected component possibly decorated by η. The duality pairing between homology and cohomology classes correspond to the pairing between graphs and trees of [Sin13]. As this pairing is nondegenerate [Sin13, Theorem 4.7], we therefore get that $\mathsf{vsc}^\vee_{mn}(l) \to H^*(\mathrm{Conf}_W(l))$ is injective. $\square$

Definition 3.14. More generally, we define, for integers $k, l \geq 0$:

$$\mathsf{vsc}^\vee_{mn}(k,l) \coloneqq \mathsf{e}^\vee_m(k) \otimes \mathsf{vsc}^\vee_{mn}(0,l). \tag{3.15}$$

For cosmetic reasons, for $m \geq 2$ we will write $\tilde{\omega}_{ij}$ for the generators of $\mathsf{e}^\vee_m(k)$, to distinguish them from the generators of $\mathsf{vsc}^\vee_{mn}(0,l)$. Recall that for $m = 1$ then $\mathsf{e}^\vee_1(k)$ is simply the group algebra $\mathbb{R}[\Sigma_k]$. The CDGA $\mathsf{vsc}^\vee_{mn}(k,l)$ is equipped with the obvious action of the product of the symmetric groups $\Sigma_k \times \Sigma_l$. Therefore, we can view $\mathsf{vsc}^\vee_{mn}$ as a bisymmetric collection.

Proposition 3.16. *There is an isomorphism of algebras*

$$\mathsf{vsc}^\vee_{mn}(k,l) \cong H^*(\mathsf{VSC}_{mn}(k,l)) = H^*(\mathrm{Conf}_{\mathbb{R}^m}(k)) \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n\setminus\mathbb{R}^m}(l)). \tag{3.17}$$

Proof. The homology groups of $\mathrm{Conf}_{\mathbb{R}^m}(k)$ and $\mathrm{Conf}_{\mathbb{R}^n\setminus\mathbb{R}^m}(l)$ are free and finitely generated. Hence we may conclude by the Künneth formula and the homotopy equivalence $\mathsf{VSC}_{mn}(k,l) \simeq \mathrm{Conf}_{\mathbb{R}^m}(k) \times \mathrm{Conf}_{\mathbb{R}^n\setminus\mathbb{R}^m}(l)$. $\square$

Next we turn to the cooperad structure of $\mathsf{vsc}^\vee_{mn}$. For reference, the cooperad structure of $\mathsf{e}^\vee_n$ is given as follows. For a pair $W \subset U$ and some $u \in U$, we let $[u] \in U/W$ be its class in the quotient. The cooperad structure is then given, for $n \geq 2$, by:

$$\begin{aligned} \circ^\vee_W : \mathsf{e}^\vee_n(U) &\to \mathsf{e}^\vee_n(U/W) \otimes \mathsf{e}^\vee_n(W), \\ \omega_{uv} &\mapsto \begin{cases} 1 \otimes \omega_{uv} & \text{if } u,\, v \in W; \\ \omega_{[u][v]} \otimes 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.18}$$

Proposition 3.19. *The CDGAs $\mathsf{vsc}^\vee_{mn}(U,V)$ assemble into a relative Hopf cooperad over $\mathsf{e}^\vee_n$, with structure maps $\mathsf{vsc}^\vee_{mn}(U,V) \xrightarrow{\circ^\vee_T} \mathsf{vsc}^\vee_{mn}(U,V/T) \otimes \mathsf{e}^\vee_n(T)$ and $\mathsf{vsc}^\vee_{mn}(U,V \sqcup T) \xrightarrow{\circ^\vee_{W,T}} \mathsf{vsc}^\vee_{mn}(U/W,V) \otimes \mathsf{vsc}^\vee_{mn}(W,T)$ given by:*

$$\circ^\vee_T(\tilde{\omega}_{uu'}) = \tilde{\omega}_{uu'} \otimes 1. \qquad \circ^\vee_{W,T}(\tilde{\omega}_{uu'}) = \begin{cases} 1 \otimes \tilde{\omega}_{uu'}, & \text{if } u, u' \in W; \\ \tilde{\omega}_{[u][u']} \otimes 1, & \text{otherwise.} \end{cases}$$

$$\circ^\vee_T(\omega_{vv'}) = \begin{cases} 1 \otimes \omega_{vv'}, & \text{if } v, v' \in T; \\ \omega_{[v][v']} \otimes 1 & \text{otherwise.} \end{cases} \qquad \circ^\vee_{W,T}(\omega_{vv'}) = \begin{cases} 1 \otimes \omega_{vv'}, & \text{if } v, v' \in T; \\ \omega_{vv'} \otimes 1 & \text{if } v, v' \in V; \\ 0 & \text{otherwise.} \end{cases}$$

$$\circ^\vee_T(\eta_v) = \eta_{[v]} \otimes 1. \qquad \circ^\vee_{W,T}(\eta_v) = \begin{cases} \eta_v \otimes 1 & \text{if } v \in V; \\ 1 \otimes \eta_v & \text{if } v \in T. \end{cases}$$

For $m = 1$ we instead extend the maps $\circ^\vee_T(1) = 1 \otimes 1$ and $\circ^\vee_{W,T}(1) = 1 \otimes 1$ equivariantly to $\mathsf{e}^\vee_1(U) = \mathbb{R}[\Sigma_U]$ (and with the same behavior as above on the generators $\omega_{vv'}$ and η_v).

Proof. We just need to check the Hopf cooperad axioms (coassociativity, counit, compatibility with the product), which are all straightforward. □

Proposition 3.20. *The maps* $(\mathsf{vsc}^\vee_{mn}, \mathsf{e}^\vee_n) \to (H^*(\mathsf{VSC}_{mn}), H^*(\mathsf{D}_n))$ *define an isomorphism of relative Hopf cooperads.*

Proof. Our proof is similar to the proof of [Sin13, Theorem 6.3]. We already know that these maps are isomorphisms in each arity, so we just need to check that they are compatible with the cooperad structures on both sides.

Let us first check that $\circ^\vee_T : \mathsf{vsc}^\vee_{mn}(U,V) \to \mathsf{vsc}^\vee_{mn}(U,V/T) \otimes \mathsf{e}^\vee_n(T)$ models $\circ_T : \mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \to \mathsf{D}_{mn}(U,V)$. To this end, we consider the maps $\theta_{uu'} : \mathsf{VSC}_{mn}(U,V) \to S^{n-1}$, $\theta_{vv'} : \mathsf{VSC}_{mn}(U,V) \to S^{m-1}$, and $\alpha_v : \mathsf{VSC}_{mn}(U,V) \to S^{n-m-1}$ defined in Section 2. We would like to check that, when they are composed with the insertion maps $\circ_T$, we obtain the behavior prescribed in Proposition 3.19.

For the maps $\theta_{uu'}$ and $\theta_{vv'}$, this is a computation identical to the one necessary to check that $\mathsf{e}^\vee_n$ (resp. $\mathsf{sc}^\vee_m$) is the cohomology, as a cooperad, of D_n (resp. SC_m). Next we want to determine the homotopy class of the map $\alpha_v(-\circ_T-) : \mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \to S^{n-m-1}$ for some $v \in V$.

- If $v \notin T$, then the composite map $\alpha_v(- \circ_T -)$ factors as

  $$\mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \xrightarrow{\mathrm{pr}_1} \mathsf{D}_{mn}(U,V/T) \xrightarrow{\alpha_v} S^{n-m-1}. \tag{3.21}$$

  We therefore find that

  $$\circ^\vee_T(\eta_v) = (\alpha_v(- \circ_T -))^*(\mathrm{vol}_{n-m-1}) = \mathrm{pr}_1^*(\alpha_v^*(\mathrm{vol}_{n-m-1})) = \eta_v \otimes 1. \tag{3.22}$$

- If $v \in T$, let us consider the homotopy which precomposes the embedding indexed by $[v] \in V/T$ in $\mathsf{D}_{mn}(U,V/T)$ by the map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto tx$. At the limit $t = 0$, we find the composite:

  $$\mathsf{VSC}_{mn}(U,V/T) \times \mathsf{D}_n(T) \xrightarrow{\mathrm{pr}_1} \mathsf{VSC}_{mn}(U,V/T) \xrightarrow{\alpha_{[v]}} S^{n-m-1}. \tag{3.23}$$

  The homotopy class of the map is constant as t changes, thus $\circ^\vee_T(\eta_v) = \eta_{[v]} \otimes 1$.

Now let us consider the insertion map $\mathsf{VSC}_{mn}(U/W,V) \times \mathsf{VSC}_{mn}(W,T) \to \mathsf{D}_{mn}(U, V \sqcup T)$. We again need to check that the (homotopy classes of the) composite maps agree with the ones prescribed in Proposition 3.19. For the maps $\theta_{uu'}(- \circ_{W,T} -)$, this is again a computation similar to the case of D_n. For the maps $\eta_v(- \circ_{W,T} -)$, it is easy to see that $\eta_v(- \circ_{W,T} -)$ factors by the projection on one of the two factors in the product. For the maps $\theta_{vv'}(- \circ_{W,T} -)$:

- If v, v' are both in V (resp. both in T), then we find that $\theta_{vv'}(- \circ_{W,T} -)$ factors by the projection onto the factor $\mathsf{VSC}_{mn}(U/W,V)$ (resp. $\mathsf{VSC}_{mn}(W,T)$) followed by the appropriate map $\theta_{vv'}$, and thus that $\circ^\vee_{W,T}(\omega_{vv'})$ equals $\omega_{vv'} \otimes 1$ (resp. $1 \otimes \omega_{vv'}$).

- Otherwise, let's assume that $v \in V$ and $v' \in T$ (the other case is symmetrical). If we contract the appropriate disks by a homotopy which linearly decreases the radius as above, then we obtain at the limit a constant map, therefore $\circ^\vee_{W,T}(\omega_{vv'}) = 0$. □

In the sequel, we will write $\mathsf{vsc}^\vee_{mn}$ or $H^*(\mathsf{VSC}_{mn}) = H^*(\mathsf{VFM}_{mn})$ indiscriminately.

Remark 3.24. We can compare $\mathsf{vsc}^\vee_{mn}$ with the cohomology of ESC_{mn} (see Remark 2.11). Recall that $\mathsf{ESC}_{mn}(U, V)$ is the fiber product $\mathsf{D}_m(U) \times_{\mathsf{D}_n(U)} \mathsf{D}_n(U \sqcup V)$. The cohomology of ESC_{mn} is the pushout $H^*(\mathsf{ESC}_{mn}(U, V)) = \mathsf{e}^\vee_m(U) \otimes_{\mathsf{e}^\vee_n(U)} \mathsf{e}^\vee_n(U \sqcup V)$, see [Wil17, Proposition 4.1]. Here, the morphism $\mathsf{e}^\vee_n(U) \to \mathsf{e}^\vee_n(U \sqcup V)$ is the obvious inclusion, while the morphism $\mathsf{e}^\vee_n(U) \to \mathsf{e}^\vee_m(U)$ sends all the generators $\omega_{uu'}$ to zero. Thus $H^*(\mathsf{ESC}_{mn}(U, V)) = \mathsf{e}^\vee_m(U) \otimes \mathsf{e}^\vee_n(U \sqcup V)/\mathsf{e}^\vee_n(U)$. The cooperadic structure maps are by formulas similar to Proposition 3.19 (forgetting the η_v). We also have an obvious inclusion $\mathsf{VSC}_{mn} \subset \mathsf{ESC}_{mn}$, which induces on cohomology the composite $\mathsf{e}^\vee_m(U) \otimes \mathsf{e}^\vee_n(U \sqcup V)/\mathsf{e}^\vee_n(U) \twoheadrightarrow \mathsf{e}^\vee_m(U) \otimes \mathsf{e}^\vee_n(V) \hookrightarrow \mathsf{vsc}^\vee_{mn}(U, V)$.

3.2 Generators and relations for the homology

Although it is not strictly necessary for our purposes, we also describe a presentation of the homology $\mathsf{vsc}_{mn} \coloneqq H_*(\mathsf{VFM}_{mn})$ by generators and relations. It is more convenient to describe this presentation by saying what the algebras over the operad are. First, let us recall the homology of the little disks operad, $\mathsf{e}_n \coloneqq H_*(\mathsf{FM}_n)$:

Theorem 3.25 (Cohen [Coh76])**.** *An algebra over e_1 is a unital associative algebra. For $n \geq 2$, an algebra over e_n is a unital Poisson n-algebra, i.e. a unital commutative algebra equipped with a Lie bracket of degree $1 - n$ such that the bracket is a biderivation for the product and the unit is a central element for the bracket.*

Using the description of the cohomology from Proposition 3.20, we find:

Proposition 3.26. *For $n - m \geq 2$, an algebra over vsc_{mn} is the data (A, B, f, δ) consisting of an e_m-algebra A, an e_n-algebra B, a central morphism of algebras $f : B \to A$, and a central derivation $\delta : B[n - m - 1] \to A$.*

Central means that f and α land in the center $Z(A) = \{a \in A \mid \forall b \in A, [a, b] = 0\}$, where the bracket is either the graded commutator ($m = 1$) or the shifted Lie bracket ($m \geq 2$). The map α is a derivation with respect to f, i.e. $\delta(xy) = \delta(x)f(y) \pm f(x)\delta(y)$.

Proof. The proof is an exercise in dualizing the description from Proposition 3.19. The product, bracket, and unit of B are duals to $1 \in \mathsf{e}^\vee_n(2)$, $\omega_{12} \in \mathsf{e}^\vee_n(2)$, and $1 \in \mathsf{e}^\vee_n(0)$. The product and unit of A are duals to $1 \in \mathsf{vsc}^\vee_{mn}(2, 0) = \mathsf{e}^\vee_m(2)$ and $1 \in \mathsf{vsc}^\vee_{mn}(0, 0) = \mathsf{e}^\vee_m(0)$; its bracket is dual to $\omega_{12} \in \mathsf{vsc}^\vee_{mn}(2, 0) = \mathsf{e}^\vee_m(2)$ for $m \geq 2$. The relations between the operations acting exclusively on A (resp. B), i.e. the products, brackets, and units, follow from Theorem 3.25 above.

The morphism f is dual to $1 \in \mathsf{vsc}^\vee_{mn}(0, 1)$. The derivation δ is dual to $\eta_1 \in \mathsf{vsc}^\vee_{mn}(0, 1)$. The fact that f is a morphism follows from $\circ^\vee_{\{x,y\}}(1) = 1 \otimes 1$ in $\mathsf{vsc}^\vee_{mn}(\varnothing, \{x, y\})$. The fact that δ is a derivation follows from $\circ^\vee_{\{x,y\}}(\eta_x) = \circ^\vee_{\{x,y\}}(\eta_y) = \eta_* \otimes 1$ (so dually we have

$\eta_*^\vee \circ 1^\vee = \eta_x^\vee + \eta_y^\vee$). The equations $[x, f(y)] = [x, \delta(y)] = 0$ follow from degrees reasons in $\mathsf{vsc}_{mn}^\vee(\{x\}, \{y\})$ (or, if $m = 1$, from an explicit homotopy which shows the centrality).

Finally, the element $\omega_{xy} \in \mathsf{vsc}_{mn}^\vee(\varnothing, \{x, y\})$ is dual to $f([x, y])$ and does not satisfy any new relation. Similarly $\eta_x \omega_{xy} = \eta_y \omega_{xy}$ is dual to $\delta([x, y])$ and does not satisfy any new relation. □

We can rephrase Proposition 3.26 more compactly. Let A be an e_m-algebra. We let $A[\varepsilon] := A \otimes \mathbb{R}[\varepsilon]/(\varepsilon^2)$ be the algebra obtained by adjoining a new square-zero variable ε of degree $n - m - 1$. If $m \geq 2$, then there is a Poisson bracket on $A[\varepsilon]$ given by $[x + \varepsilon y, x' + \varepsilon y'] = [x, x'] + \varepsilon([x, y'] \pm [x', y])$. Then an vsc_{mn}-algebra is the data of an e_m-algebra A, an e_n-algebra B, and a central morphism $f + \varepsilon\delta : B \to A[\varepsilon]$.

Compare this result with the ∞-categorical counterparts from [AFT17, Section 4.3]. An algebra over the ∞-categorical version $\mathscr{D}isk_{m\subset n}^{\mathrm{fr}}$ consist of a $\mathscr{D}isk_m^{\mathrm{fr}}$-algebra A, a $\mathscr{D}isk_n^{\mathrm{fr}}$-algebra B, and a morphism of $\mathscr{D}isk_{m+1}^{\mathrm{fr}}$-algebras $\alpha : \int_{S^{n-m-1}} B \to \mathrm{HC}^*_{\mathsf{D}_m}(A)$, where $\int_{S^{n-m-1}} B$ is the "factorization homology" of S^{n-m-1} with coefficients in B, and $\mathrm{HC}^*_{\mathsf{D}_m}$ refers to Hochschild cochains. We view this as a "up to homotopy" version of an vsc_{mn}-algebra, the morphism $f + \varepsilon\delta$ being the part $\int_{S^{n-m-1}} B \to \mathrm{HC}^0_{\mathsf{D}_m}(A)$ and the higher terms being homotopies. It would be interesting to make this observation precise.

Remark 3.27. An algebra over the homology $\mathsf{sc}_n := H_*(\mathsf{SC}_n)$ of the Swiss-Cheese operad is the data of (A, B, f) as in the proposition, see [Liv15, Section 4.3]. However, our computation for $H^*(\mathsf{VSC}_{(n-1)n})$ in Section 3.1 above does not apply (e.g. the class $\omega_{12} \in H^*(\mathsf{VSC}_{(n-1)n}(0, 2))$ vanishes on some connected components). We can compute: $\mathsf{vsc}_{(n-1)n}$-algebras are given by quadruples (A, B, f, g) where (A, B, f) are as above and $g : B \to A$ is another central morphism (i.e. A is a unitary B-bimodule). There is an embedding $\mathsf{SC}_n \subset \mathsf{VSC}_{(n-1)n}$. On homology, an $\mathsf{vsc}_{(n-1)n}$-algebra (A, B, f, g) viewed as an sc_n-algebra is simply (A, B, f), i.e. we forget the right action. Livernet proved that SC_n is not formal by exhibiting a nontrivial operadic Massey product $\langle \mu_B, f, \lambda_A \rangle$, where μ_B represents the product of B and λ_A represents the Lie bracket of A [Liv15, Theorem 4.3]. This shows that the operad of chains $C_*(\mathsf{VFM}_{(n-1)n}; \mathbb{Q})$ cannot be formal either, because we obtain a nontrivial Massey product there too.

4 Graph complexes

In this section, we define a two-colored Hopf cooperad, whose operations in the second color are given by Kontsevich's cooperad graphs_n [Kon99], and whose operations in the first color will be called $\mathsf{vgraphs}_{mn}$. Our tool of choice to define $\mathsf{vgraphs}_{mn}$ will be "operadic twisting" [Wil14; DW15], just like in [Wil15]. To give an idea of how $\mathsf{vgraphs}_{mn}$ is built, we first recall the steps in the definition of graphs_n.

4.1 Recollections: the cooperad graphs_n

In this section, we recall the definition of Kontsevich's graph cooperad graphs_n. We assume that $n \geq 2$ in the whole section.

Remark 4.1. Unlike some earlier works, we use the notation graphs_n for the cooperad rather than its dual operad. Its linear dual $\mathsf{graphs}_n^\vee$ is an operad which is quasi-isomorphic to the homology of the little disks operad. We make this choice because in this paper, the dual operad $\mathsf{graphs}_n^\vee$ never appears (and the cooperad is in some sense more fundamental, as the operad is given by its dual, while there is no straightforward way to define the cooperad out of the operad). Moreover, Kontsevich's graph complex where the differential splits vertices will also be denoted with a dual sign, $\mathrm{GC}_n^\vee$ (dual of the complex where the differential contracts edges). See e.g. [Idr16; CILW18] for matching notations.

4.1.1 Untwisted

The first step is to define an untwisted Hopf cooperad Gra_n, given in each arity by the following CDGA, with generators e_{uv} of degree $n-1$:

$$\mathsf{Gra}_n(U) := S(e_{uv})_{u \neq v \in U} / (e_{vu} = (-1)^n e_{uv}). \tag{4.2}$$

We have a graphical interpretation of $\mathsf{Gra}_n(U)$. The CDGA $\mathsf{Gra}_n(U)$ is spanned by graphs on the set of vertices U. The monomial $e_{u_1v_1} \dots e_{u_rv_r}$ corresponds to the graph with edges $\overrightarrow{u_1v_1}, \dots, \overrightarrow{u_rv_r}$. The identification $e_{vu} = \pm e_{uv}$ allows us to view the graphs as undirected, although we need directions to define the signs precisely for odd n. Moreover, if n is even then $\deg e_{uv}$ is odd, thus we need to order the edges to get precise signs. Note that we explicitly allow tadpoles (e_{uu}) and double edges (e_{uv}^2). However, for even n, $e_{uv}^2 = 0$ because $\deg e_{uv}$ is odd; and for odd n, $e_{uu} = (-1)^n e_{uu} = -e_{uu}$ thus $e_{uu} = 0$.

The product is given by gluing graphs along their vertices The cooperadic structure is given by subgraph contraction. Explicitly, the map $\circ_W^\vee : \mathsf{Gra}_n(U) \to \mathsf{Gra}_n(U/W) \otimes \mathsf{Gra}_n(W)$ is given on generators by $\circ_W^\vee(e_{uv}) = 1 \otimes e_{uv}$ if $u, v \in W$, and by $\circ_W^\vee(e_{uv}) = e_{[u][v]} \otimes 1$ otherwise. One may then produce a first zigzag of Hopf cooperads, defined on generators by:

$$\begin{array}{ccccc} H^*(\mathsf{FM}_n) = \mathsf{e}_n^\vee & \leftarrow & \mathsf{Gra}_n & \xrightarrow{\omega'} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n), \\ \omega_{uv} & \leftarrow\!\shortmid & e_{uv} & \mapsto & p_{uv}^*(\varphi_n), \end{array} \tag{4.3}$$

where $p_{uv} : \mathsf{FM}_n(U) \to \mathsf{FM}_n(2)$ is the projection and φ_n is the "propagator":

$$\varphi_n := \mathrm{cst} \cdot \sum_{i=1}^{n} \pm x_i dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_n \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{FM}_n(2)) = \Omega^{n-1}_{\mathrm{PA}}(S^{n-1}). \tag{4.4}$$

Given a graph $\Gamma \in \mathsf{Gra}_n(U)$, one may define its *coefficient*:

$$\mu(\Gamma) := \int_{\mathsf{FM}_n(U)} \omega'(\Gamma). \tag{4.5}$$

This element μ has a very simple description: it vanishes on all graphs, except for the one with exactly two vertices and an edge between the two [LV14, Lemma 9.4.3]. In other words, in the dual basis, one has:

$$\mu = \textcircled{1}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{2} \in \mathsf{Gra}_n^\vee(2) \subset \prod_{i \geq 0} \mathsf{Gra}_n^\vee(i). \tag{4.6}$$

4.1.2 Twist

The second step is to *twist* the cooperad Gra_n. The general reference for this construction is [DW15], and the application to graph complexes is spelled out in [Wil14].

Let Lie_n be the operad governing Lie algebras shifted by $n-1$. The element μ above defines a morphism from Lie_n to $\mathsf{Gra}_n^\vee$: we send the generating bracket to the graph appearing in Equation (4.6). The general framework of [DW15] produces a twisted cooperad $\mathrm{Tw}\,\mathsf{Gra}_n$ from Gra_n and this morphism. Note that despite the fact that μ is suppressed from the notation, $\mathrm{Tw}\,\mathsf{Gra}_n$ depends on it. The terminology comes from the fact that a $(\mathrm{Tw}\,\mathsf{Gra}_n)$-coalgebra is a Gra_n-coalgebra with a differential twisted by a Maurer–Cartan element.

Let us now describe this twisted cooperad. As a graded module, we have:

$$\mathrm{Tw}\,\mathsf{Gra}_n(r) \coloneqq \Big(\bigoplus_{i\geq 0}\big(\mathsf{Gra}_n(r+i)\otimes \Bbbk[n]^{\otimes i}\big)_{\Sigma_i}, d_\mu\Big). \tag{4.7}$$

The cooperadic cocomposition is inherited from Gra_n. The differential uses the action of μ on Gra_n.

Remark 4.8. The cooperad Gra_n is a Λ-cooperad [Fre17], i.e. for an injection $i : U \hookrightarrow V$ there is an induced map $i_* : \mathsf{Gra}_n(U) \to \mathsf{Gra}_n(V)$ which adds isolated vertices (and a coaugmentation $\varepsilon : \mathbb{R} \to \mathsf{Gra}_n(U)$ given by the empty graph). It follows easily that $\mathrm{Tw}\,\mathsf{Gra}_n$ is a Λ-cooperad too, with a similar graphical description for the Λ-structure. The Hopf structure and the Λ-structure together induce a Hopf structure on $\mathrm{Tw}\,\mathsf{Gra}_n$. More precisely, in order to multiply a graph $\Gamma \in \mathsf{Gra}_n(U \sqcup I) \subset \mathrm{Tw}\,\mathsf{Gra}_n(U)$ with a graph $\Gamma' \in \mathsf{Gra}_n(U \sqcup J) \subset \mathrm{Tw}\,\mathsf{Gra}_n(U)$, one should first use the Λ-structure to send both graphs in $\mathsf{Gra}_n(U \sqcup I \sqcup J)$, then use the product of Gra_n.

Let us now give a graphical interpretation of this definition. The CDGA $\mathrm{Tw}\,\mathsf{Gra}_n(U)$ is spanned by graphs with two kinds of vertices: external vertices, which are in bijection with U, and internal vertices, which are indistinguishable (in pictures they will be drawn in black). Given a graph Γ, its differential $d\Gamma = \sum_e \pm\Gamma/e$ is obtained as a sum over all the edges $e \in E_\Gamma$ connected to internal vertex of the graphs obtained by contracting these edges. Note that in this differential, edges connected to univalent vertices are not contracted, roughly speaking because the contraction appears twice in $d\Gamma$ and cancels out, see [Wil14, Appendix I.3]. The product of two graphs is the graph obtained by gluing them along their external vertices. The cooperadic structure is given by subgraph contraction (summing over all choices of whether internal vertices are in the subgraph or not).

One checks that the zigzag of Equation (4.3) extends to a zigzag:

$$\mathsf{e}_n^\vee \leftarrow \mathrm{Tw}\,\mathsf{Gra}_n \xrightarrow{\omega} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n). \tag{4.9}$$

The left-pointing map sends all graphs with internal vertices to zero. The right-pointing map is given by the following integrals. Given a graph $\Gamma \in \mathsf{Gra}_n(U \sqcup I) \subset \mathrm{Tw}\,\mathsf{Gra}_n(U)$, the form $\omega(\Gamma)$ is the integral of $\omega'(\Gamma)$ along the fiber of the PA bundle $p_U : \mathsf{FM}_n(U \sqcup I) \to$

$\mathsf{FM}_n(U)$ which forgets points in the configuration:

$$\omega(\Gamma) := (p_U)_*(\omega'(\Gamma)) = \int_{\mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)} \omega'(\Gamma). \tag{4.10}$$

4.1.3 Interlude: Graph complex

We record the following definition for future use.

Definition 4.11. The full graph complex fGC_n is defined to be $\mathrm{Tw}\,\mathsf{Gra}_n(\varnothing)[-n]$. It is spanned by graphs with only internal vertices, and the differential is given by edge contraction. The degree of $\gamma \in \mathrm{fGC}_n$ is $\deg\gamma = (n-1)\#E_\gamma - n\#V_\gamma + n$.

Remark 4.12. This degree shift by n morally comes from the fact that we mod out $\mathrm{Conf}_{\mathbb{R}^n}(k)$ by $\mathbb{R}^n \rtimes \mathbb{R}_{>0}$ in the definition of FM_n.

The module fGC_n is a shifted CDGA, with a product given by disjoint union of graphs. The differential is given by edge contraction. The space $\mathrm{Tw}\,\mathsf{Gra}_n(U)$ is a module over the shifted CDGA fGC_n, by taking disjoint unions. One can moreover define the subcomplex $\mathrm{GC}_n \subset \mathrm{fGC}_n$ of connected graphs, and there is an isomorphism of (shifted) CDGAs:

$$\mathrm{fGC}_n = S(\mathrm{GC}_n[n])[-n]. \tag{4.13}$$

The module GC_n is a (pre)Lie coalgebra. Its cobracket Δ is given by subgraph contraction (i.e. it is inherited from the cooperad structure on Gra_n). Dually, $\mathrm{GC}_n^\vee$ is a (pre)Lie algebra, with a bracket given by graph insertion. The differential of GC_n is given by $(\mu \otimes 1 + 1 \otimes \mu)\Delta$, where $\mu \in \mathrm{GC}_n^\vee$ is the Maurer–Cartan element defined in Equation (4.6). Dually, the differential on $\mathrm{GC}_n^\vee$ is $[\mu, -]$.

4.1.4 Reduction

The next step is to mod out graphs with *internal components*, i.e. connected components with only internal vertices, to obtain a new Hopf cooperad Graphs_n. Formally, we consider the tensor product

$$\mathsf{Graphs}_n(U) := \mathrm{Tw}\,\mathsf{Gra}_n(U) \otimes_{\mathrm{fGC}_n} \mathbb{R}, \tag{4.14}$$

where the fGC_n-module structure on $\mathbb{R}$ is simply given by the augmentation, i.e. the action of any nonempty graph vanishes. In other words, in Graphs_n, a graph with a connected component consisting entirely of internal vertices is set equal to zero. The map ω of Equation (4.10) factors through the quotient defining Graphs_n thanks to [LV14, Lemma 9.3.7], and the quotient map $\mathrm{Tw}\,\mathsf{Gra}_n \to \mathsf{e}_n^\vee$ clearly does.

We can moreover reduce further the operad. We consider the quotient graphs_n where we have killed graphs containing: internal vertices that are univalent or bivalent; double edges; or tadpoles. It follows again from the lemmas of [LV14, Section 9.3] that ω factors through this quotient (and the map $\mathsf{Graphs}_n \to \mathsf{e}_n^\vee$ clearly does).

Theorem 4.15 ([Kon99; LV14])**.** *This defines a zigzag of quasi-isomorphisms of Hopf cooperads* $\mathsf{e}_n^\vee \xleftarrow{\sim} \mathsf{graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$*. Thus the operad* FM_n *is formal over* $\mathbb{R}$*.*

4.2 The cooperad $\mathsf{VGraphs}_{mn}$

Let us now define $\mathsf{VGraphs}_{mn}$, using the same methodology that was used to define Graphs_n. Note that we must take special care of the case $m = 1$. We define the further reduced cooperad $\mathsf{vgraphs}_{mn}$ in the next section.

4.2.1 Untwisted

The first step is the definition of the untwisted graph cooperad.

Definition 4.16. For $m \geq 2$, the untwisted graph cooperad is a relative VGra_{mn}-cooperad given in each arity by:

$$\mathsf{VGra}_{mn}(U, V) := \frac{S(\tilde{e}_{uu'})_{u,u' \in U} \otimes S(e_{ij})_{i,j \in U \sqcup V}}{(e_{ji} = (-1)^n e_{ij}, \tilde{e}_{uu'} = (-1)^m \tilde{e}_{u'u})},$$

where $\deg \tilde{e}_{uu'} = m - 1$ and $\deg e_{ij} = n - 1$.

Definition 4.17. For $m = 1$, we instead define (where $\Sigma_U = \mathsf{Bij}(U, \{1, \dots, \#U\})$):

$$\mathsf{VGra}_{1n}(U, V) = \frac{S(e_{ij})_{i,j \in U \sqcup V} \otimes \mathbb{R}[\Sigma_U]}{(e_{ji} = (-1)^n e_{ij}, \tilde{e}_{uu'} = (-1)^m \tilde{e}_{u'u})}.$$

Let us now give a graphical interpretation of $\mathsf{VGra}_{mn}(U, V)$. We will concentrate first on the case $m \geq 2$. As a vector space, it is spanned by graphs with two kinds of vertices: terrestrial (in bijection with U) and aerial (in bijection with V). We will draw the aerial vertices as circles, and the terrestrial vertices as semicircles, below the aerial ones. There are also two kind of edges: full edges (corresponding to the e_{ij}) between any two vertices, and dashed edges (corresponding to the $\tilde{e}_{uu'}$) between two terrestrial vertices Note that we allow tadpoles (edges between a vertex an itself) as well as double edges. See Figure 4.1 for an example. If $m = 1$, the interpretation is similar, except that there are no dashed edges, and in addition the set of terrestrial vertices is ordered.

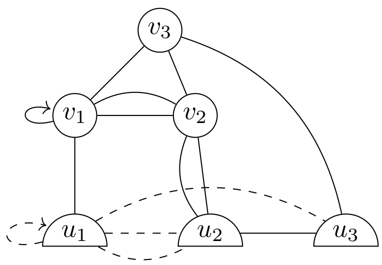


Figure 4.1: Example of a graph in VGra_{mn} (for $m \geq 2$)

The product glues graphs along their vertices, and the differential is zero. These CDGAs assemble to form a relative Hopf Gra_n-cooperad, using subgraph contraction. Let us now define a zigzag of Hopf cooperad maps

$$H^*(\mathsf{VFM}_{mn}) = \mathsf{vsc}^\vee_{mn} \leftarrow \mathsf{VGra}_{mn} \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}). \tag{4.18}$$

The left pointing map is defined by $\tilde{e}_{uu'} \mapsto \tilde{\omega}_{uu'}$, $e_{vv'} \mapsto \omega_{vv'}$ for $v, v' \in V$, and $e_{ij} \mapsto 0$ if $i \in U$ or $j \in U$. The right pointing map is defined using the following three "propagators":

- We have the identification $\mathsf{VFM}_{mn}(2,0) = \mathsf{FM}_m(2) \cong S^{m-1}$, for which we can use the propagator $\varphi_m \in \Omega^{m-1}_{\mathrm{PA}}(\mathsf{FM}_m(2))$ of Equation (4.4).
- We have the map $\theta_{12} : \mathsf{VFM}_{mn}(1,1) \to S^{n-1}$ which records the direction from the aerial point to the terrestrial point. We then define the propagator to be the pullback of the volume form of S^{n-1} along θ_{12}:
$$\psi^{\partial}_{mn} \coloneqq \theta_{12}^*(\mathrm{vol}_{S^{n-1}}) \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{VFM}_{mn}(1,1)). \tag{4.19}$$
- Similarly, there is the map $\theta_{12} : \mathsf{VFM}_{mn}(0,2) \to S^{n-1}$ which records the direction from the second point to the first point. Then the propagator is:
$$\psi_{mn} \coloneqq \theta_{12}^*(\mathrm{vol}_{S^{n-1}}) \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{VFM}_{mn}(0,2)). \tag{4.20}$$

Remark 4.21. By construction, these propagators are all minimal forms on VFM_{mn}, because the volume form on S^d is $\mathrm{vol}_{S^d} = \mathrm{cst} \cdot \sum_i (-1)^i x_i dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_d$. Hence they can be pushed forward (once) along PA bundles [HLTV11].

We may then define a morphism
$$\omega' : \mathsf{VGra}_{mn}(U,V) \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}(U,V)) \tag{4.22}$$
as follows, using the convention that u, $u' \in U$ and v, $v' \in V$:
$$\begin{aligned} \omega'(\tilde{e}_{uu'}) &\coloneqq p^*_{uu'}(\varphi_m); & \omega'(e_{vv'}) &\coloneqq p^*_{vv'}(\psi_{mn}); \\ \omega'(e_{uu'}) &\coloneqq 0; & \omega'(e_{vu}) &\coloneqq p^*_{vu}(\psi^{\partial}_{mn}). \end{aligned} \tag{4.23}$$
Here, $p_{\bullet\bullet}$ is the map which forget all but two points in a configuration.

Lemma 4.24. *This defines a zigzag of Hopf cooperads* $\mathsf{vsc}^{\vee}_{mn} \leftarrow \mathsf{VGra}_{mn} \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$.

Proof. This is immediate. □

Let us define some more notation. Given a graph $\Gamma \in \mathsf{VGra}_{mn}(U,V)$, we define $V_\Gamma = V^t_\Gamma \cup V^a_\Gamma = U \cup V$ to be its set of vertices, partitioned into terrestrial and aerial ones. Similarly, $E_\Gamma = E^f_\Gamma \cup E^d_\Gamma$ is its set of edges, split into full edges and dashed edges. The graph Γ induces a map
$$\Phi_\Gamma : \mathsf{VFM}_{mn}(U,V) \to (S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma}, \tag{4.25}$$
obtained using the maps θ_{ij} from Section 2. We also define
$$\mathrm{vol}_\Gamma \in \Omega^{\deg \Gamma}_{\mathrm{PA}}((S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma}) \tag{4.26}$$
to be the product of the volume forms. Then by definition, $\omega'(\Gamma) = \Phi^*_\Gamma(\mathrm{vol}_\Gamma)$.

Given a graph $\Gamma \in \mathsf{VGra}_{mn}(U,V)$, we may define its *coefficient* $c(\Gamma)$ by
$$c(\Gamma) \coloneqq \int_{\mathsf{VFM}_{mn}(U,V)} \omega'(\Gamma). \tag{4.27}$$

Remark 4.28. In the case $(n, m) = (2, 1)$, these are analogous to the coefficients in Kontsevich's universal formality morphism $T_{\mathrm{poly}} \to D_{\mathrm{poly}}$ from [Kon03]. The difference is that we allow points in the lower half-space.

4.2.2 Twisted

Together with μ from Equation (4.5), this collection of coefficients c defines a morphism of colored operads $(\mathsf{Lie}_n, \mathsf{Lie}_{mn}) \to (\mathsf{Gra}_n^\vee, \mathsf{VGra}_{mn}^\vee)$, where Lie_{mn} is the operad encoding the action of an n-shifted Lie algebra on an m-shifted Lie algebra. We can therefore apply the formalism of operadic twisting.

Definition 4.29. The twisted graph cooperad $\mathrm{Tw}\,\mathsf{VGra}_{mn}$ is the relative $(\mathrm{Tw}\,\mathsf{Gra}_n)$-cooperad obtained by twisting VGra_{mn} with respect to μ and c.

Since VGra_{mn} is a Hopf Λ-cooperad, we find again that $\mathrm{Tw}\,\mathsf{VGra}_{mn}$ inherits a Hopf Λ-cooperad structure, see Remark 4.8.

This twisted cooperad has a graphical description. The CDGA $\mathrm{Tw}\,\mathsf{VGra}_{mn}(U, V)$ is spanned by graphs with four types of vertices:

- external terrestrial vertices, in bijection with U (drawn as semicircles);
- external aerial vertices, in bijection with V (drawn as circles);
- internal terrestrial vertices, indistinguishable among themselves, of degree $-m$ (drawn as black semicircles);
- internal aerial vertices, indistinguishable among themselves, of degree $-n$ (drawn as black circles).

There are two kinds of edges: aerial (full) edges of degree $n - 1$, and terrestrial (dashed) edges of degree $m - 1$. If $m = 1$, then there are no dashed edges, and the whole set of terrestrial vertices is ordered.

Definition 4.30 (Quotient graph)**.** Let $\Gamma' \subset \Gamma$ be a subgraph, not necessarily full. We define the quotient graph Γ/Γ' as follows. The set of vertices of Γ/Γ' is the quotient set $V_\Gamma / V_{\Gamma'}$, identifying all the vertices of Γ' to produce a new vertex $[\Gamma']$, always terrestrial even if Γ' contains no terrestrial vertices. The set of edges of Γ/Γ' is the difference $E_\Gamma \setminus E_{\Gamma'}$, and we will informally view the edges of Γ' as having been "contracted". If $e \in E_\Gamma \setminus E_{\Gamma'}$ has an endpoint in Γ', then the corresponding endpoint of $e \in E_{\Gamma/\Gamma'}$ is the new terrestrial vertex $[\Gamma']$. In particular, if an edge is not in Γ' but both of its endpoints are in Γ', then in Γ/Γ' this edge becomes a tadpole on the vertex $[\Gamma']$.

Example 4.31. The quotient $\Gamma/\varnothing$ is Γ with a new isolated external terrestrial vertex.

The differential is given by the twisting procedure, with several summands coming from μ and c. Given a graph Γ, its differential $d\Gamma$ is given as a sum of:

- contractions of full edges between an aerial internal vertex and an aerial vertex of any kind (this uses the simple description of μ in Equation (4.6));

- contractions of subgraphs Γ' containing at most one external vertex, necessarily aerial, with result $c(\Gamma') \cdot \Gamma/\Gamma'$;
- forgetting of all vertices outside a subgraph Γ'' which contains all the external vertices, with result $c(\Gamma/\Gamma'') \cdot \Gamma''$.

Lemma 4.32. *The morphism ω' extends to a morphism of Hopf cooperads*

$$\omega : \mathrm{Tw}\,\mathsf{VGra}_{mn}(U, V) \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$$

by setting, for $\Gamma \in \mathsf{VGra}_{mn}(U \sqcup I, V \sqcup J) \subset \mathrm{Tw}\,\mathsf{VGra}_{mn}(U, V)$ and $\#U + 2\#V \geq 2$:

$$\omega(\Gamma) := (p_{U,V})_*(\omega'(\Gamma)) = \int_{\mathsf{VFM}_{mn}(U\sqcup I, V\sqcup J) \to \mathsf{VFM}_{mn}(U,V)} \omega'(\Gamma).$$

If $\#U + 2\#V \leq 1$, i.e. if there are no aerial vertices and at most one terrestrial vertex, then we set

$$\omega(\Gamma) = \begin{cases} 1, & \text{if } \Gamma \text{ is the unit graph;} \\ 0, & \text{otherwise.} \end{cases} \tag{4.33}$$

Proof. We use the double pushforward formula, Stokes' formula and the decomposition of the fiberwise boundary $\mathsf{VFM}_{mn}(U, V)$ from Section 2.1 (in a manner identical to the case of the little disks operad [LV14]). We must deal separately with the case $\#U + 2\#V \leq 1$ because, in this case, the dimension of the fibers of $p_{U,V}$ is smaller than expected (compare with [LV14, Section 9.1]). □

4.2.3 Interlude: Graph complex

Before turning to the final step of the construction, we will need the following definition:

Definition 4.34. The full Swiss-Cheese graph complex is given by:

$$\mathrm{fVGC}_{mn} = \mathrm{Tw}\,\mathsf{VGra}_{mn}(\varnothing, \varnothing)[-m]$$

It is free as a (shifted) CDGA, generated by the subcomplex VGC_{mn} of connected graphs.

The dual $\mathrm{VGC}^\vee_{mn}$ is at the same time a module over the Lie algebra $\mathrm{GC}^\vee_n$ and a (pre-)Lie algebra, in both cases using insertion of graphs (respectively at aerial and terrestrial vertices). The differential of $\mathrm{VGC}^\vee_{mn}$ is given by $[\mu + c, -]$ where c represents the coefficients from Equation (4.27). The fact that c defines a morphism of CDGAs $\mathrm{fVGC}_{mn} \to \mathbb{R}$ is equivalent to the fact that the element $\mu + c \in \mathrm{GC}^\vee_n \ltimes \mathrm{VGC}^\vee_{mn}$ satisfies the Maurer–Cartan equation: $[\mu, \mu] = 0$ and $[\mu, c] + \frac{1}{2}[c, c] = 0$.

4.2.4 Reduction

We now define $\mathsf{VGraphs}_{mn}$. We would like to define it by modding out graphs with internal components. However, just like in [Wil15], the case $m = 1$ is special.

Lemma 4.35. *Given a disconnected graph $\gamma \in \mathrm{fVGC}_{mn}$, the coefficient $c(\gamma)$ vanishes unless $m = 1$ and γ is the graph with two terrestrial vertices and no edges.*

Proof. Let us first assume that γ has no isolated terrestrial vertices. Then γ factors as a product $\gamma = \gamma_1 \cdot \gamma_2$, with corresponding sets of vertices (I_1, J_1) and (I_2, J_2). Thanks to our assumption, we have $\#I_\bullet + 2\#J_\bullet \geq 2$ for $\bullet \in \{1, 2\}$. The coefficient $c(\gamma) = \int_{\mathsf{VFM}_{mn}(I,J)} \omega'(\gamma)$ is defined as an integral, and the degree of $\omega'(\gamma)$ must be equal to the dimension of $\mathsf{VFM}_{mn}(I, J)$ for the integral to be nonzero. Let us assume that this is the case (otherwise we are done).

We have $\omega'(\gamma) = \Phi_\gamma^*(\mathrm{vol}_\gamma)$. The map Φ_γ factors through the product $\mathsf{VFM}_{mn}(I_1, J_1) \times \mathsf{VFM}_{mn}(I_2, J_2)$. But we know that:

$$\dim \mathsf{VFM}_{mn}(I, J) - \dim \mathsf{VFM}_{mn}(I_1, J_1) \times \mathsf{VFM}_{mn}(I_2, J_2) = m + 1 > 0, \tag{4.36}$$

therefore $\deg \mathrm{vol}_\gamma > \dim \mathsf{VFM}_{mn}(I_1, J_1) \times \mathsf{VFM}_{mn}(I_2, J_2)$. It follows that $\omega'(\gamma) = 0$ and therefore $c(\gamma) = 0$.

Let us now assume that γ has an isolated terrestrial vertex, say $i \in I$. As before, $\omega'(\gamma) = \Phi_\gamma^*(\mathrm{vol}_\gamma)$, and Φ_γ factors through $\mathsf{VFM}_{mn}(I \setminus \{i\}, J)$. If $(\#I - 1) + 2\#J \geq 2$, in other words if the graph is not the graph with two terrestrial vertices and no aerial ones, then

$$\dim \mathsf{VFM}_{mn}(I, J) - \dim \mathsf{VFM}_{mn}(I \setminus \{i\}, J) = m > 0 \tag{4.37}$$

and we may again conclude that $\omega'(\gamma) = 0 \implies c(\gamma) = 0$. However, if Γ is the graph with two terrestrial vertices and no aerial ones, i.e. $(\#I, \#J) = (2, 0)$, then the codimension is $m - 1$, because $\dim \mathsf{VFM}_{mn}(1, 0) = 0$ and not -1 as the general formula would give. Thus in the case $m > 1$, we may still conclude that $\omega'(\gamma) = 0$, because the codimension is positive. $\square$

Remark 4.38. In the case $m = 1$, we have that $\mathsf{VFM}_{1n}(2, 0) \cong \mathsf{FM}_1(2) \cong S^0$. Hence we find that for the following graph, $c(\gamma) = 1$:

$\gamma =$  $\in \mathrm{fVGC}_{1n}$.

Definition 4.39. For $m \geq 2$, we define the graph cooperad $\mathsf{VGraphs}_{mn}$ to be the relative Graphs_n-cooperad given by quotient of $\mathrm{Tw}\,\mathsf{VGra}_{mn}$ by the Hopf cooperadic ideal of graphs with internal components. For $m = 1$, we define $\mathsf{VGraphs}_{1n}$ to be the quotient of $\mathrm{Tw}\,\mathsf{VGra}_{1n}$ by the Hopf cooperadic ideal of "Lie-disconnected" graphs (see [Wil15] for the definition).

Remark 4.40. Willwacher [Wil15] defined a model $\mathsf{SGraphs}_n$ for the Swiss-Cheese operad. There are important differences between $\mathsf{VGraphs}_{(n-1)n}$ and $\mathsf{SGraphs}_n$. In $\mathsf{SGraphs}_n$, there are no "dashed" edges, the full edges are oriented, and their source is always aerial.

Proposition 4.41. *The morphism ω factors through the quotient and defines a morphism $\omega : \mathsf{VGraphs}_{mn} \to \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$.*

Proof. The proof is identical to the proof of [LV14, Lemma 9.3.7]. We must check that ω vanishes on any graph with a connected component consisting only of internal vertices. As ω is a morphism of CDGAs, it is enough to show that ω vanishes on graphs where all the edges are between internal vertices. Let $\gamma \in \mathrm{Tw}\,\mathsf{VGra}_{mn}(U, V)$ be such a graph, and let I and J be the sets of terrestrial (resp. aerial) internal vertices of γ. The proposition is obvious by definition if $\#U + 2\#V \leq 1$, so let us assume that we are not in this case.

Thanks to our hypothesis on γ, we can fill the diagram, where ρ is the product of two projections:

$$\begin{array}{ccc}
\mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) & \xrightarrow{\quad\Phi_\gamma\quad} & (S^{m-1})^{E^t_\Gamma} \times (S^{n-1})^{E^a_\Gamma} \\
\downarrow{\scriptstyle p_{U,V}} \quad \searrow{\scriptstyle \rho} & & \nearrow{\scriptstyle \exists \Phi'_\gamma} \\
\mathsf{VFM}_{mn}(U, V) \xleftarrow{\mathrm{pr}_1} & \mathsf{VFM}_{mn}(U, V) \times \mathsf{VFM}_{mn}(I, J) &
\end{array}.$$

By definition, $\omega'(\Gamma) = \Phi^*_\gamma(\mathrm{vol}_\Gamma)$. Thus we find that:

$$\omega(\Gamma) = (p_{U,V})_*(\omega'(\Gamma)) = (p_{U,V})_*(\rho^*(\Phi'^*_\gamma \mathrm{vol}_\Gamma)). \tag{4.42}$$

The dimension of the fiber of $p_{U,V}$ is $\dim \mathrm{fib}\, p_{U,V} = m\#I + n\#J$. We find that

$$\dim \mathrm{fib}\, \mathrm{pr}_1 = \dim \mathsf{VFM}_{mn}(I, J) \leq m\#I + n\#J - m < \dim \mathrm{fib}\, p_{U,V}, \tag{4.43}$$

therefore $\omega(\Gamma) = 0$ by [HLTV11, Proposition 5.1.2]. □

4.3 Vanishing lemmas and $\mathsf{vgraphs}_{mn}$

We now prove some vanishing lemmas about ω and c, which allows us to define the further reduced cooperad $\mathsf{vgraphs}_{mn}$. This will be useful to prove that the quotient map $\mathsf{VGra}_{mn} \to \mathsf{vsc}^\vee_{mn}$ extends to $\mathsf{vgraphs}_{mn}$ in Section 5.

Lemma 4.44. *The morphism ω' vanishes on graphs with loops or double edges.*

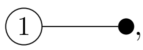
Proof. This follows from simple dimension arguments, cf. [LV14, Section 9.3]. □

Compare the following lemma with [LV14, Lemma 9.3.9].

Lemma 4.45. *The morphism ω vanishes on graphs containing univalent aerial internal vertices, as well as univalent terrestrial vertices connected to another terrestrial vertex.*

Proof. The lemma is trivial if the graph has no external aerial vertices and at most one external terrestrial vertex, so let us assume that we are not in this case.

Let us first deal with univalent aerial internal vertices connected to another aerial vertex. The following graph is of negative degree:

$\Gamma =$  , (4.46)

thus its image under ω must vanish. We can then apply the double pushforward formula for integral along fibers to see that the image of any graph containing Γ as a subgraph vanishes (see a similar argument spelled out in detail in the proof of [Idr16, Corollary 3.26], see also the proof of [LV14, Lemma 9.3.9]).

Let us now assume that Γ is a graph with a univalent internal vertex connected to a terrestrial vertex. If that univalent vertex is terrestrial and the edge is full, then the integral is zero by definition (see Equation (4.23)), so let us assume that we are not in that case, i.e. either the univalent is aerial, or the incident edge is dashed.

Let $(U \sqcup I, V \sqcup J)$ be the sets of vertices of Γ. Let i be the univalent internal vertex, and let u be the terrestrial vertex to which it is connected. We have a commutative diagram:

$$\begin{array}{ccccc} \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J) & \xrightarrow{\rho} & X & \xrightarrow{q} & \mathsf{VFM}_{mn}(U, V) \\ \downarrow{\scriptstyle \Phi_\Gamma} & \swarrow{\scriptstyle \exists \Phi'_\Gamma} & & & \\ (S^{m-1})^{E^d_\Gamma} \times (S^{n-1})^{E^f_\Gamma} & & & & \end{array}, \qquad p_{U,V} = q \circ \rho \tag{4.47}$$

where

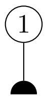
- either $X = \mathsf{VFM}_{mn}(\{u\}, \{i\}) \times \mathsf{VFM}_{mn}(U \sqcup I, V \sqcup J \setminus \{i\})$ if i is aerial,
- or $X = \mathsf{VFM}_{mn}(\{u, i\}, \varnothing) \times \mathsf{VFM}_{mn}(U \sqcup I \setminus \{i\}, V \sqcup J)$ if i is terrestrial.

The map ρ is in both cases the product of two projections, q is the projection, and the factorization Φ'_Γ exists due to the hypothesis that i is univalent and connected to u.

The dimension of the fiber of $p_{U,V}$ is $m\#I + n\#J$. The dimension of the fiber of q is, in both cases, given by $m\#I + n\#J - 1$, which is strictly smaller. Hence, by [HLTV11, Proposition 5.1.2], we conclude that

$$\omega(\Gamma) = (p_{U,V})_*(\Phi_\Gamma(\omega'(\Gamma))) = (q \circ \rho)_*(\rho^*(\Phi'_\Gamma(\omega'(\Gamma)))) = 0. \qquad \square$$

Remark 4.48. If a graph contains a univalent internal terrestrial vertex connected to an aerial vertex, then the argument fails. In fact, if we consider the following graph, then we find that $\omega(\Gamma) \in \Omega^{n-m-1}_{\mathrm{PA}}(\mathsf{VFM}_{mn}(0,1))$ represents the fundamental class of $\mathsf{VFM}_{mn}(0,1) \cong S^{n-m-1}$, i.e. the class η from Section 3.1:

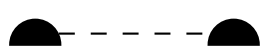
$\Gamma =$  $\in \mathsf{vgraphs}_{mn}(0,1)$, (4.49)

Indeed, by the definition of ψ^∂_{mn} in (4.19) and by [HLTV11, Proposition 8.10], we find that $\int_{\mathsf{VFM}_{mn}(1,1)} \psi^\partial_{mn} = \int_{S^{n-1}} \mathrm{vol}_{n-1} = 1$. Therefore $\int_{\mathsf{VFM}_{mn}(0,1)} \omega(\Gamma) = 1$ by [HLTV11, Proposition 8.13]. This also implies that if γ is obtained from the Γ above by making the external vertex internal, then we get $c(\gamma) = 1$.

Lemma 4.50. *The coefficient c vanishes on graphs with more than two vertices and which either contain a univalent aerial vertex, or which contain a univalent terrestrial vertex connected to another terrestrial vertex. It also vanishes on the graph with exactly two vertices, both aerial, and a (full) edge between the two.*

Proof. We can reuse the proof of Lemma 4.45 almost verbatim. There is one caveat: the graph γ with the univalent vertex removed must still satisfy the hypothesis $\#I + 2\#J \geq 2$. Otherwise, a degree shift occurs, to deal with the fact that $\dim \mathsf{VFM}_{mn}(I, J) = 0$ and not -1 or $-m-1$ as the general formula would give. This is the case unless γ has exactly two vertices with at most one aerial. If γ has exactly two aerial vertices and one edge, then $c(\gamma)$ vanishes for degree reasons ($\dim \mathsf{VFM}_{mn}(0,2) = 2n - m - 1 > n - 1$). $\square$

Remark 4.51. The restriction about the number of vertices in the lemma is necessary. Indeed, for the following graph we find $c(\gamma) = \int_{\mathsf{VFM}_{mn}(2,0)} \varphi_m = \int_{S^{m-1}} \mathrm{vol}_{S^{m-1}} = 1$:

$\gamma =$  $\in \mathrm{VGC}_{mn}$. (4.52)

Lemma 4.53. *The morphism ω and the coefficient c vanish on graphs with bivalent internal terrestrial vertices connected to two terrestrial vertices.*

Proof. We will do case by case, depending on what kind of edges are incident to the external vertex. Let us first consider the case of the graph

$\Gamma =$  (4.54)

with three terrestrial vertices and two edges as indicated. Using the identification $\mathsf{VFM}_{mn}(U, \varnothing) \cong \mathsf{FM}_m(U)$ and the fact that the terrestrial propagator is the same as the one used in the definition of the map $\mathsf{Graphs}_m \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_m)$, we deduce that $\omega(\Gamma) = 0$ from [LV14, Lemma 9.3.9] (see also [Kon94, Lemma 2.1] for the origin of this "trick"). Briefly, the proof uses a symmetry argument: there is an involution on $\mathsf{FM}_m(U)$ which leaves $\omega'(\Gamma)$ invariant but reverses the orientation, from which $\omega(\Gamma) = -\omega(\Gamma)$.

Next we consider the two graphs:

$\Gamma' =$  and $\Gamma'' =$  (4.55)

They are of respective degrees $n-2$ and $2n-m-2$. However the dimension of $\mathsf{VFM}_{mn}(2,0)$ is $m-1$, which is strictly smaller than both of these degrees. Hence $\omega(\Gamma') = \omega(\Gamma'') = 0$ by [HLTV11, Proposition 5.1.2].

For the general case (i.e. for any graph which contains Γ, Γ', or Γ'' as a subgraph), we use the theorems for pushforwards of PA forms like in Lemma 4.45. $\square$

Definition 4.56. The reduced graph cooperad $\mathsf{vgraphs}_{mn}$ is the relative graphs_n-cooperad given in each arity by the quotient of $\mathsf{VGraphs}_{mn}(U, V)$ by the submodule of graphs containing loops, double edges, univalent aerial internal vertices, univalent terrestrial vertices connected to another terrestrial vertex, or bivalent internal terrestrial vertices connected to two terrestrial vertices.

Proposition 4.57. *The CDGA $\mathsf{vgraphs}_{mn}(U, V)$ is well-defined, and ω factors through the quotient to define $\mathsf{vgraphs}_{mn}(U, V) \xrightarrow{\omega} \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}(U, V))$.*

Proof. This follows from Lemmas 4.44, 4.45, 4.50, and 4.53. $\square$

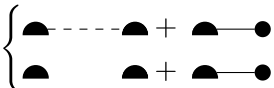
5 Proof of the formality

In this section we complete the proof of the formality of the operad VFM_{mn}. We first show that, up to homotopy, the differential of $\mathsf{vgraphs}_{mn}$ can be simplified. Then we construct a map from our graph complex to the cohomology of VFM_{mn}, and we prove that this map, as well as ω, are quasi-isomorphisms.

5.1 Change of Maurer–Cartan element and $\mathsf{vgraphs}^0_{mn}$

We would like to define a morphism $\mathsf{vgraphs}_{mn} \to \mathsf{vsc}^\vee_{mn}$. However, the Hopf cooperad $\mathsf{vgraphs}_{mn}$ depends on the Maurer–Cartan element $c \in \mathrm{VGC}^\vee_{mn}$ from Equation (4.27). We do not know the precise form of c. We just know its leading terms:

Proposition 5.1. *We have $c = c_0 + (\dots)$, where $(\dots)$ denotes terms where $\#\{\text{terr. vert.}\} + 2\#\{\text{aer. vert.}\} > 3$, and:*



$$c_0 := \begin{cases} \cdots + \cdots \in \mathrm{VGC}^\vee_{mn}, & \text{if } m \geq 2; \\ \cdots + \cdots \in \mathrm{fVGC}^\vee_{1n}, & \text{if } m = 1. \end{cases} \tag{5.2}$$

Proof. This follows from Remark 4.51, Remark 4.48, and Remark 4.38 for $m = 1$. We simply check the degrees of the few other graphs that satisfy $\#\{\text{terr. vert.}\} + 2\#\{\text{aer. vert.}\} \leq 2$ (discounting those with loops or double edges thanks to Lemma 4.44). $\square$

If we knew that $c = c_0$, then we would be able to build a map $\mathsf{vgraphs}_{mn} \to \mathsf{vsc}^\vee_{mn}$ easily (see Section 5.2). In this section, we show that c is gauge equivalent to c_0. For this we show that the cohomology of $\mathrm{VGC}^\vee_{mn}$ twisted by c_0 vanishes in the right degree. Obstruction theory then shows that c is gauge equivalent to c_0.

Fortunately, as we will see, the graph complex

$$\mathrm{VGC}^{\vee, c_0}_{mn} := (\mathrm{VGC}^\vee_{mn}, d + [c_0, -]) \tag{5.3}$$

is quasi-isomorphic to the better-known "hairy graph complex" HGC_{mn}, see e.g. [FW15, Section 2.2.6]. This allows us to use known vanishing results about HGC_{mn}.

Definition 5.4. For $k \geq 0$, let $\mathsf{Graphs}'_n(k)$ be the quotient of $\mathsf{Graphs}_n(k)$ by the ideal spanned by graphs which are disconnected or whose external vertices are not exactly univalent. The hairy graph complex is (with differential induced from Graphs'_n):

$$\mathrm{HGC}^\vee_{mn} := \prod_{k \geq 1} \big(\mathsf{Graphs}'_n(k)^\vee \otimes (\mathbb{R}[m])^{\otimes k}\big)^{\Sigma_k}[-m]. \tag{5.5}$$

As usual, this complex has a graphical interpretation. The complex HGC_{mn} can be seen as spanned by (infinite sums of) graphs whose external vertices are exactly univalent and indistinguishable. The differential is given by vertex splitting. Each external vertex, together with its only incident edge, can be seen as a "hair", which justifies the terminology.

Proposition 5.6 ([FW15, Proposition 2.2.7])**.** *When $n - m \geq 2$, the cohomology of the hairy graph complex* $\mathrm{HGC}_{mn}^{\vee}$ *vanishes in degrees* > -1*.*

Proof. Note that our definition of the hairy graph complex (denoted by HGC_{mn} without the dual in [FW15]) is slightly different, as we allow bivalent and univalent internal vertices. However, we can reuse their arguments to show that the inclusion of their complex into ours is a quasi-isomorphism (see also [Wil14, Proposition 3.4] for a similar argument). Briefly, we can filter both complexes by the number of internal vertices of valence ≥ 3. Both spectral sequences collapse starting on page E^2, and the inclusion induces an isomorphism on this page. We can then use [FW15, Proposition 2.2.7] to show the vanishing of the homology in degrees > -1 (note that in the reference, homologically graded complexes are used, so we just use the natural correspondence that reverse degrees). ☐

There is a natural preLie product on $\mathrm{HGC}_{mn}^{\vee}$, induced by the operad structure of $\mathsf{Graphs}_n^{\vee}$. Roughly speaking, $\Gamma \circ \Gamma'$ is obtained by inserting Γ' in an external vertex of Γ and reconnecting the incident edge to a vertex of Γ', in all possible ways. Moreover, there is a natural action of the Lie algebra $\mathrm{GC}_n^{\vee}$ (see Definition 4.11 and the discussion that follows) on $\mathrm{HGC}_{mn}^{\vee}$. Given $\Gamma \in \mathrm{HGC}_{mn}^{\vee}$ and $\gamma \in \mathrm{GC}_n^{\vee}$, the action $\Gamma \cdot \gamma$ is given by inserting γ at a vertex of Γ in all possible ways.

Lemma 5.7. *There is an inclusion of dg-modules* $\mathrm{HGC}_{mn}^{\vee} \subset \mathrm{VGC}_{mn}^{\vee,c_0}$ *obtained by considering all external vertices as terrestrial, with no dashed edges. This inclusion is compatible with the (pre)Lie algebra structure and the action of the Lie algebra* $\mathrm{GC}_n^{\vee}$ *on both sides.*

Proof. Simple inspection shows that the inclusion is well-defined, and that it is compatible with the differential and with the algebraic structures. ☐

Proposition 5.8. *The inclusion* $\mathrm{HGC}_{mn}^{\vee} \subset \mathrm{VGC}_{mn}^{\vee,c_0}$ *is a quasi-isomorphism.*

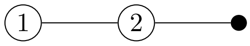
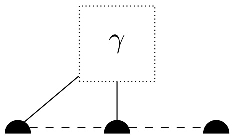
Proof. The proof is similar to the proof of [FW15, Theorem 2.3.3], which is in turn based on [Wil14, Proposition 4.3]. Indeed, the graph complex $\mathrm{VGC}_{mn}^{\vee,c_0}$ is very close to the deformation complex of the inclusion $\mathsf{Graphs}_m^{\vee} \to \mathsf{Graphs}_n^{\vee}$, denoted by E_{mn}^{**} in [FW15] (it is not necessary to know about deformation complexes to follow the arguments).

We first filter both complexes by the number of full edges, which is the only kind of edges in $\mathrm{HGC}_{mn}^{\vee}$. The differential of $\mathrm{HGC}_{mn}^{\vee}$ always increases this number strictly by 1. Let us write $c_0 = c_0' + c_0''$, where c_0' is the part with two terrestrial vertices, and c_0'' with one vertex of each kind (see Equation (5.2)). The differential of $\mathrm{VGC}_{mn}^{\vee,c_0}$ increases the filtration number by 1 (for the action of $\mu + c_0''$) or keeps it constant (for the action of c_0'). Hence on the associated spectral sequences, the differential of $E^0\mathrm{HGC}_{mn}^{\vee}$ vanishes, while the differential of $E^0\mathrm{VGC}_{mn}^{\vee,c_0}$ is just the bracket $[c_0', -]$.

We now check that the inclusion induces a quasi-isomorphism on these E^0 pages, from which the proposition follows. Let us first assume that $m \geq 2$. Given a graph $\Gamma \in \mathrm{VGC}_{mn}^{\vee}$, we can define its “character” $[\Gamma] \in \mathrm{HGC}_{mn}^{\vee}$ by considering the same graph with all terrestrial vertices and dashed edges removed, and making the dangling edges into hairs (see [Wil14, Lemma 4.4] for an analogous definition). Note that the differential $[c_0', -]$ does not change the character of a graph. Hence the complex $E^0\mathrm{VGC}_{mn}^{\vee,c_0}$ splits:

$$E^0\mathrm{VGC}_{mn}^{\vee,c_0} = \prod_{\gamma\in\mathrm{HGC}_{mn}^{\vee}} C_\gamma, \tag{5.9}$$

where $C_\gamma = \{\Gamma \in E^0\mathrm{VGC}_{mn}^{\vee,c_0} \mid [\Gamma] = \gamma\}$. Let $\gamma \in \mathrm{HGC}_{mn}^{\vee}$ be a graph with k hairs. The complex C_γ is isomorphic to $\mathsf{Graphs}_m^\vee(k)$ (with some degree shift depending on γ) as follows. Given some $\Gamma \in \mathsf{Graphs}_m^\vee(k)$, we view it as a graph with dashed edges, and we glue the external vertices of Γ to the endpoints of the hairs of γ to obtain an element of C_γ. For example, if γ is a graph with two hairs, then:

$\Gamma =$  $\in \mathsf{Graphs}_m^\vee(2) \mapsto$  $\in C_\gamma.$ (5.10)

The differential $[c_0', -]$ replicates the differential of $\mathsf{Graphs}_m^\vee(k)$ (i.e. vertex splitting), thus this is an isomorphism of dg-modules.

The homology of $\mathsf{Graphs}_m^\vee$ is the m-Poisson operad (see Theorem 4.15). Checking the degrees and the induced differential $[\mu + c_0'', -]$, we can identify the page $E^1\mathrm{VGC}_{mn}^{\vee,c_0}$ with (a shift of) the deformation complex E_{mn}^{**} from the m-Poisson operad to $\mathsf{Graphs}_n^\vee$ considered in [FW15, Theorem 2.3.3]. Since the differential of $\mathrm{HGC}_{mn}^\vee$ raises the number of edges by exactly 1, the page $E^1\mathrm{HGC}_{mn}^\vee$ is just $\mathrm{HGC}_{mn}^\vee$. We can then conclude using the result of [FW15, Theorem 2.3.3].

For $m = 1$, the proof is similar. The difference is that instead of getting an isomorphism $\mathsf{Graphs}_m^\vee(k) \cong C_\gamma$ (where γ has k hairs), we instead directly get $\mathsf{e}_1^\vee(k) = \mathbb{R}[\Sigma_k] \cong C_\gamma$. This isomorphism is given by sending $\sigma \in \Sigma_k$ to the graph γ viewed as an element of C_γ, i.e. with the external vertices replaced by terrestrial internal vertices, and with σ specifying the order of the resulting terrestrial vertices. We then get that $E^1\mathrm{VGC}_{1n}^{\vee,c_0}$ is the deformation complex of $\mathsf{e}_1 \to \mathsf{Graphs}_n^\vee(k)$, whose homology is the hairy graph complex. The induced morphism on the E^2 pages is the identity and we can conclude. □

Corollary 5.11. *For $n - m \geq 2$, the cohomology of $\mathrm{VGC}_{mn}^{\vee,c_0}$ vanishes in degrees > -1. Therefore the Maurer–Cartan element $c - c_0 \in \mathrm{VGC}_{mn}^{\vee,c_0}$ is gauge equivalent to zero; equivalently, c and c_0 are gauge equivalent.*

Proof. The first claim follows by combining Propositions 5.6 and 5.8. The second claim follows by applying the Goldman–Millson theorem to the inclusion of the truncation $\tau_{<0}\mathrm{VGC}_{mn}^{\vee,c_0} \subset \mathrm{VGC}_{mn}^{\vee,c_0}$. □

Definition 5.12. Let $\mathsf{vgraphs}_{mn}^0$ be the variant of $\mathsf{vgraphs}_{mn}$ where we use c_0 instead of c to twist the Hopf cooperad VGra_{mn} in the step of Definition 4.29.

Corollary 5.13. *The Hopf cooperads $\mathsf{vgraphs}_{mn}$ and $\mathsf{vgraphs}_{mn}^0$ are quasi-isomorphic.*

Proof. This follows from the same general arguments of [CILW18, Section 5.4]. Let us briefly describe them. Let $S(t, dt)$ be the algebra of polynomial forms on the interval $[0, 1]$, with $\deg t = 0$ and $\deg dt = 1$. Let $\mathrm{VGC}^{\vee;\sim}_{mn}$ be the Lie algebra with differential $[\mu, -]$, i.e. we are only allowed to split aerial vertices. Both c and c_0 are Maurer–Cartan elements, i.e. they satisfy $[\mu, c] + \frac{1}{2}[c, c] = [\mu, c_0] + \frac{1}{2}[\mu, c_0] = 0$. The Lie algebra $\mathrm{VGC}^{\vee}_{mn}$ is the twist of $\mathrm{VGC}^{\vee;\sim}_{mn}$ with respect to the Maurer–Cartan element c.

The gauge equivalence between c and c_0 can be seen as a Maurer Cartan element $c_t \in \mathrm{VGC}^{\vee;\sim}_{mn} \otimes S(t, dt)$ whose restriction at $t = 1$ (resp. $t = 0$) is c (resp. c_0). We can use this element c_t to produce a differential on $\mathsf{vgraphs}_{mn} \otimes S(t, dt)$ such that restriction at $t = 1$ (resp. $t = 0$) gives $\mathsf{vgraphs}_{mn}$ (resp. $\mathsf{vgraphs}^0_{mn}$). In other words, we have a zigzag:

$$\mathsf{vgraphs}_{mn} \xleftarrow{\mathrm{ev}_{t=1}} \mathsf{vgraphs}_{mn} \otimes S(t, dt) \xrightarrow{\mathrm{ev}_{t=0}} \mathsf{vgraphs}^0_{mn}. \tag{5.14}$$

The evaluation maps $\mathrm{ev}_{t=0}, \mathrm{ev}_{t=1} : S(t, dt) \to \mathbb{R}$ are quasi-isomorphisms of CDGAs. This implies that the two maps in the zigzag above are quasi-isomorphisms too. □

5.2 Connecting the graphs to the cohomology

The goal of this section is to describe a quasi-isomorphism of Hopf cooperads $\pi : \mathsf{vgraphs}^0_{mn} \to \mathsf{vsc}^{\vee}_{mn}$, where $\mathsf{vsc}^{\vee}_{mn} = H^*(\mathsf{VFM}_{mn})$ was obtained in Section 3.1.

For simplicity, we will describe this map on generators. The CDGA $\mathsf{vgraphs}^0_{mn}(U, V)$ is free as an algebra. Its generators are the "internally connected" graphs, i.e. the graphs which stay connected when all the external vertices are removed (keeping dangling edges). For example, if Γ has no internal vertices, then it is internally connected iff it has exactly one edge (an empty graph is not connected).

If Γ is such an internally connected graph, then $\pi(\Gamma) \in \mathsf{vsc}^{\vee}_{mn}(U, V)$ is given by:

- If $\Gamma = 1$ is the empty graph then $\pi(\Gamma) = 1$. If $m = 1$ then this is extended Σ_U-equivariantly to $\mathsf{vsc}^{\vee}_{mn}(U, \varnothing)$.
- If $\Gamma = e_{vv'}$ has no internal vertices and one full edge between v, $v' \in V$, then $\pi(\Gamma) = \omega_{vv'}$.
- If $\Gamma = \tilde{e}_{uu'}$ (for $m \geq 2$) has no internal vertices and one dashed edge between u, $u' \in U$, then $\pi(\Gamma) = \tilde{\omega}_{uu'}$.
- If Γ is the graph from Remark 4.48, i.e. it has exactly one internal vertex, which is terrestrial, and exactly one edge connecting that internal vertex to some aerial vertex u, then $\pi(\Gamma) = \eta_u$.
- In all other cases, $\pi(\Gamma) = 0$.

Proposition 5.15. *The map π is a quasi-isomorphism of Hopf cooperads $\mathsf{vgraphs}^0_{mn} \to \mathsf{vsc}^{\vee}_{mn}$.*

The proof of this proposition is split in a series of lemmas, which occupies the rest of this section (until the conclusion, Theorem 5.27).

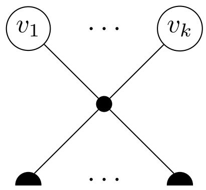
Lemma 5.16. *The map π is a well-defined algebra map and is equivariant with the symmetric group actions.*

Proof. Since we defined π on the generators of a free algebra (forgetting about the differential), it is well-defined. It is moreover clearly equivariant. □

Lemma 5.17. *The map π commutes with the differentials, i.e. $\pi d = 0$.*

Proof. Since π is an algebra map and the differential is a derivation, it is sufficient to check this on generators. Let Γ be an internally connected graph.

- If Γ has no internal vertices, then $d\Gamma = 0$ thus $\pi d\Gamma = 0$.
- If Γ has exactly one internal vertex, then $\pi d\Gamma = 0$ follows from the Arnold relations in $\mathsf{e}_n^\vee$ (if the vertex is aerial) or in $\mathsf{e}_m^\vee$ (if the vertex is terrestrial) and the fact that full edges incident to terrestrial vertices are mapped to zero.
- Finally let us deal with the case where Γ has at least two internal vertices. For a summand in $d\Gamma$ to be nonzero, it must be the case that, after contracting one edge, all remaining edges are between external vertices or between an external aerial vertex and a univalent terrestrial internal one. This implies first that there can be at most one aerial internal vertex. Moreover, since contracting an edge cannot reduce the valence of the remaining vertices (because contracting dead ends is forbidden), there can only be one internal vertex of valence greater than one, necessarily aerial. Using the internal connectedness of Γ, this special vertex must be connected to all the univalent terrestrial vertices by a full edge. In other words, the graph Γ must be of this type (plus some disconnected external vertices):



(5.18)

Then the Arnold relations in $\mathsf{e}_n^\vee$, the symmetry relation $\eta_v \omega_{vv'} = \eta_{v'} \omega_{vv'}$, and the relation $\eta_v^2 = 0$ (if there is more than one terrestrial vertex) show that $\pi d\Gamma = 0$. □

Lemma 5.19. *The map π commutes with the cooperad structure maps.*

Proof. Once again it is sufficient to check this on generators, i.e. internally connected graphs. The verifications require a large number of cases but do not present any particular difficulty. □

5.3 Proof that π is a quasi-isomorphism

The last step for Proposition 5.15 is proving that π is a quasi-isomorphism. We split this proof in several sub-lemmas which occupy the rest of this section until the conclusion, Theorem 5.27. We must deal separately with the cases $m \geq 2$ and $m = 1$.

5.3.1 Case $m \geq 2$

Let us give a rough outline of our strategy. It is clear that π is surjective on cohomology, so we just need to check that $\mathsf{vgraphs}^0_{mn}(k,l)$ has the same Betti numbers as $\mathsf{VFM}_{mn}(k,l)$. In a first step, we prove the case $l = 0$, using an inductive argument inspired by the proof of [LV14, Theorem 8.1]. Then, we prove that we can reduce to the case of "split" graphs, where external aerial vertices and external terrestrial vertices are not in the same connected components. This mirrors the fact that as a space, $\mathsf{VFM}_{mn}(k,l) \simeq \mathrm{Conf}_{\mathbb{R}^m}(k) \times \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)$. Finally, we prove the case $k = 0$, again using an inductive argument. We conclude using the Künneth formula.

We make an observation that will be useful throughout the proof. A graph $\Gamma \in \mathsf{vgraphs}^0_{mn}(k,l)$ determines a partition of $\{1,\dots,k\} \sqcup \{1,\dots,l\}$, by looking at connected components of Γ. We can define the subcomplex of connected graphs:

$$\mathsf{vgraphs}^0_{mn}(k,l)_{\mathrm{cn}} \subset \mathsf{vgraphs}^0_{mn}(k,l). \tag{5.20}$$

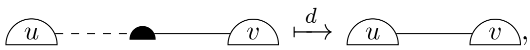
Then the complex $\mathsf{vgraphs}^0_{mn}(k,l)$ splits as a direct sum, over all partitions of $\{1,\dots,k\} \sqcup \{1,\dots,l\}$, of tensor products of complexes fo the type $\mathsf{vgraphs}^0_{mn}(-,-)_{\mathrm{cn}}$, one for each set in the partition (see [LV14, Equation (8.4)] for a similar statement about Graphs_n). For ease of notation, we will abbreviate the Betti numbers of this complex as:

$$\beta^j(k,l) := \dim H^j(\mathsf{vgraphs}^0_{mn}(k,l)_{\mathrm{cn}}). \tag{5.21}$$

Note that we will focus on the two cases $k = 0$ and $l = 0$, as these will be the relevant ones for the application of the Künneth formula.

Lemma 5.22. *The map* $\pi : \mathsf{vgraphs}^0_{mn}(k,0) \to \mathsf{vsc}^\vee_{mn}(k,0) = \mathsf{e}^\vee_m(k)$ *is a quasi-isomorphism for all* $k \geq 0$.

If $\mathsf{vgraphs}^0_{mn}(k,0)$ had no aerial (internal) vertices and no full edges, then it would be equal to $\mathsf{Graphs}_m(k)$, which is quasi-isomorphic to $\mathsf{e}^\vee_m(k)$ by [LV14, Theorem 8.1]. The following proof is the formalization of the intuitive fact that a full edge is killed by:



(5.23)

and that internal vertices, whether aerial or terrestrial, do not produce any homotopy class and are just here to kill the Arnold relations.

Proof of Lemma 5.22. The case $k = 0$ is obvious: since there are no external vertices and each connected component must contain an external vertex, it follows that the graph must be empty. Therefore we get $\mathsf{vgraphs}^0_{mn}(0,0) = \mathsf{e}_m(0) = \mathbb{R}$, and the map π is the identity.

Let us now assume that $\mathsf{vgraphs}^0_{mn}(k,0) \to \mathsf{e}^\vee_m(k)$ is a quasi-isomorphism for a given $k \geq 0$. We want to prove that the same statement is true for $k+1$. The quotient map is clearly surjective on cohomology ($\tilde{\omega}_{ij}$ is represented by the graph with a single dashed edge between external vertices i and j). Hence it is sufficient to prove that $\mathsf{vgraphs}^0_{mn}(k+1,0)$ and $\mathsf{e}^\vee_m(k+1)$ have the same Betti numbers.

Using the known recurrence relation satisfied by the Betti numbers of $\mathrm{Conf}_{\mathbb{R}^m}$ and the splitting of the complexes in terms of connected components (see [LV14, Section 9]), we can restrict our attention to connected graphs (see Equation (5.21)). It suffices to prove that the Betti numbers of the connected part satisfy the relation $\beta^i(k+1,0) = k \cdot \beta^{i-m+1}(k,0)$.

We split the complex $\mathsf{vgraphs}^0_{mn}(k+1,0)_{\mathrm{cn}}$ in several submodules.

- The submodule U_i ($1 \le i \le k$) spanned by graphs such that the $(k+1)$th external vertex is univalent and connected to the external vertex i by a dashed edge. For $k \ge 1$, we set $U := \bigoplus_{i=1}^k U_i$. In case $k = 0$, we instead set $U = \mathbb{R}$, the one-dimensional space spanned by the unit graph.
- the submodule V spanned by graphs such that the $(k+1)$th external vertex is univalent and connected by a dashed edge to an internal vertex;
- the submodule W spanned by graphs such that the $(k+1)$th external vertex is at least bivalent, or univalent and connected by a full edge to an internal vertex.

The submodule U is closed under the differential. We moreover have an isomorphism $(U_i, d) \cong \mathsf{vgraphs}^0_{mn}(k,0)_{\mathrm{cn}}[1-m]$ (for all $1 \le i \le k$), by removing the vertex $k+1$ and its incident edge. In case $k=0$, we clearly have $U = \mathbb{R} \cong \mathsf{vgraphs}^0_{mn}(0,0) = \mathsf{vgraphs}^0_{mn}(0,0)_{\mathrm{cn}}$

Let $\mathcal{Q} = \mathsf{vgraphs}^0_{mn}(k+1,0)_{\mathrm{cn}}/U$. As a graded module, we have $\mathcal{Q} = V \oplus W$, with some induced differential. Let us show that $\mathcal{Q}$ is acyclic. We put a filtration $F_\bullet\mathcal{Q}$ on $\mathcal{Q}$. A graph of V is in $F_p\mathcal{Q}$ if it has at most p edges, while a graph of W is in $F_p\mathcal{Q}$ if it has at most $p-1$ edges. On the $E^0\mathcal{Q}$ page of the associated spectral sequence, we see that the differential maps V isomorphically onto W. This shows that $E^1\mathcal{Q} = 0$, thus $\mathcal{Q}$ is acyclic.

Therefore the inclusion $U \subset \mathsf{vgraphs}^0_{mn}(k+1,0)_{\mathrm{cn}}$ is a quasi-isomorphism. If $k=0$ then we have $U = \mathbb{R}$, which coincides with $\mathsf{e}^\vee_m(1) = \mathbb{R}$. If $k \ge 1$, then, as we observed earlier, we have $U_i \cong \mathsf{vgraphs}^0_{mn}(k,0)_{\mathrm{cn}}[1-m]$. Hence we find that the Betti numbers satisfy $\beta^j(k+1,0) = \sum_{i=1}^k \dim H^j(U_i) = k \cdot \beta^{j-m+1}(k,0)$, as expected. ☐

Let us now turn to the second step of the proof. We prove that we can, in some sense, "split" our graph complex in two: external aerial and external terrestrial.

Lemma 5.24. *Let $k, l \ge 1$ and let $I_{k,l} \subset \mathsf{vgraphs}^0_{mn}(k,l)$ be the module spanned by graphs where one of the connected components contains an external aerial vertex and an external terrestrial vertex. Then I is an acyclic dg-ideal.*

Proof. Contracting edges does not affect connected components. Therefore the differential cannot change the partition of the set of external vertices induced by some graph Γ, and thus $I_{k,l}$ is closed under the differential. It is also clearly an ideal: gluing along external vertices can merge connected components but never split them.

Let us now prove that $I_{k,l}$ is acyclic. We can restrict our attention to connected graphs (the general case follows by the Künneth formula). We use a technique similar to the proof of Lemma 5.22, by induction on $k \ge 1$, for any fixed $l \ge 1$.

For $k=1$ we have an explicit homotopy showing that $I_{1,l}$ is acyclic. Namely, for a graph Γ, the graph $h(\Gamma)$ is obtained by replacing the external terrestrial vertex by an internal one, adding a new external terrestrial vertex, and connecting it to the old external vertex. We then check that $(dh + hd)(\Gamma) = \Gamma$ for all $\Gamma \in I_{1,l}$.

Let us now assume that the claim is true for a given $k \geq 1$, and let us prove the claim for $k + 1$. Just like in the proof of Lemma 5.22, we can split $I_{k+1,l}$ in three subcomplexes, depending on whether the last external terrestrial vertex is: univalent, connected by a dashed edge to an external vertex; univalent, connected by a dashed edge to an internal vertex; all other cases. The first subcomplex is isomorphic to k copies of the ideal $I_{k,l}$, thus it is acyclic by the induction hypothesis. On the quotient by this subcomplex, we can put a filtration just like in Lemma 5.22, so that on the E^0 page, the second complex is mapped isomorphically onto the third. This shows that the quotient is acyclic too, and therefore $I_{k+1,l}$ is acyclic. □

Lemma 5.25. *The map* $\pi : \mathsf{vgraphs}^0_{mn}(0, l) \to \mathsf{vsc}^\vee_{mn}(0, l)$ *is a quasi-isomorphism* $\forall l \geq 0$.

Proof. This final lemma is also proved by induction. Once again this map is clearly surjective on cohomology, so it suffices to prove that both complexes have the same Betti numbers. Using the results of Section 3.1, the Poincaré polynomial of $\mathsf{VFM}_{mn}(0, l) \simeq \mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)$ is given by $\mathscr{P}(\mathrm{Conf}_{\mathbb{R}^n \setminus \mathbb{R}^m}(l)) = \prod_{i=0}^{l-1}(1 + t^{n-m-1} + it^{n-1})$.

We can again work with the connected part of the graph complex $\mathsf{vgraphs}^0_{mn}(0, l)_{\mathrm{cn}}$. Note that the case $l = 0$ is covered by Lemma 5.22. The base case that we need to prove is $\beta^0(0, 1) = \beta^{n-m-1}(0, 1) = 1$, and $\beta^j(0, 1) = 0$ for other j. The recurrence relation is $\beta^j(0, l + 1) = l \cdot \beta^{j-n+1}(0, l)$ for all j and all $l \geq 1$.

For $l = 1$, we again have an explicit homotopy. Given a graph Γ, the graph $h(\Gamma)$ is obtained by replacing the external (aerial) vertex of Γ by an internal (aerial) one, adding back an external vertex, and connected the new external vertex to the old one by a (full) edge. We then check that if Γ is the unit graph or the graph from Remark 4.48, then $(dh + hd)(\Gamma) = 0$, while if Γ is another graph, then $(dh + hd)(\Gamma) = \Gamma$.

To prove the recurrence relation, we split $\mathsf{vgraphs}^0_{mn}(0, l + 1)_{\mathrm{cn}}$:

- The submodule U_i where the external vertex $(k + 1)$ is univalent, connected to the external vertex i. We also set $U = \bigoplus_{i=1}^k U_i$.
- The submodule V, where the external vertex $(k + 1)$ is at least bivalent.
- The submodule V' where the external vertex $(k + 1)$ is univalent, connected to an aerial internal vertex.
- The submodule W, where the external vertex $(k + 1)$ is univalent, connected to a terrestrial internal vertex; this terrestrial vertex is itself bivalent, and its other incident edge is dashed.
- The submodule W', where the external vertex $(k + 1)$ is univalent, connected to a terrestrial internal vertex; this terrestrial vertex is either at least trivalent, or bivalent and both incident edges are full.

Let $\mathcal{Q} = \mathsf{vgraphs}^0_{mn}(0, l + 1)/U$ be the quotient. We can set up a spectral sequence just like in the proof of Lemma 5.22, such that on the E^0 page, the differential takes V'

isomorphically onto V, and W' isomorphically onto W. This show that $\mathcal{Q}$ is acyclic; thus, the inclusion $U \subset \mathsf{vgraphs}^0_{mn}(0, l+1)$ is a quasi-isomorphism. We have an isomorphism $U_i \cong \mathsf{vgraphs}^0_{mn}(0, l)[1-m]$ being given by removing the last external vertex and its incident edge. It follows that the Betti numbers satisfy the expected recurrence relation, $\beta^j(0, l+1) = l \cdot \beta^{j-n+1}(0, l)$. □

5.3.2 Case $m = 1$

We deal separately with the case $m = 1$, because e_1 is the associative operad and not the Poisson operad. To summarize the differences, recall that: the graphs do not have dashed edges, and the terrestrial vertices are ordered (Definition 4.17); the notion of "disconnected" is replaced by "Lie-disconnected" (Definition 4.39); the differential $[c_0, -]$ merges Lie clusters instead of contracting dashed edges (Equation (5.2)).

Proposition 5.26. *The map $\pi : \mathsf{vgraphs}^0_{1n}(k, l) \to \mathsf{vsc}^\vee_{1n}(k, l)$ is a quasi-isomorphism for all $k, l \geq 0$.*

Proof. As in the case $m = 2$, the map π is clearly surjective on cohomology, so we just need to check that $\mathsf{vgraphs}^0_{1n}(k, l)$ has the correct Betti numbers.

The proofs of Lemmas 5.22, 5.24, and 5.25 can be adapted in a straightforward manner. We can follow the same proofs, replacing m with 1. The crucial difference will be in the splitting of the complex $\mathsf{vgraphs}^0_{1n}(k, l)_{\mathrm{cn}}$ or of $I_{k,l}$.

- In $\mathsf{vgraphs}^0_{1n}(k, 0)_{\mathrm{cn}}$ (for Lemma 5.22), we set U to be the submodule where the $(k+1)$th external vertex is isolated but not adjacent to a terrestrial internal vertex (terrestrial vertices are ordered for $m = 1$), V the submodule where the $(k+1)$th external vertex is isolated and adjacent to a terrestrial internal vertex, and W all other kinds of graphs.
- In $I_{k,l}$ (for Lemma 5.24), we use the same splitting as for $\mathsf{vgraphs}^0_{1n}(k, 0)_{\mathrm{cn}}$.
- In $\mathsf{vgraphs}^0_{1n}(0, l)$ (for Lemma 5.25), we keep the same U, V, and V' as in the proof of Lemma 5.25. We change the submodules W and W': in W, we require the last external vertex to be connected to a univalent internal terrestrial vertex, while in W' we put all other graphs.

With these changes, we obtain the correct recurrence relations on the Betti numbers. □

5.3.3 Conclusion

Theorem 5.27. *The operad VFM_{mn} is formal over $\mathbb{R}$ for $n - 2 \geq m \geq 1$.*

Proof. We have a zigzag:

$$\mathsf{vsc}^\vee_{mn} \xleftarrow{\pi} \mathsf{vgraphs}^0_{mn} \leftarrow \mathsf{vgraphs}_{mn} \otimes S(t, dt) \to \mathsf{vgraphs}_{mn} \xrightarrow{\omega} \Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn}), \quad (5.28)$$

where $\mathsf{vsc}^\vee_{mn}$ was defined in Section 3.1, $\mathsf{vgraphs}_{mn}$ in Section 4.3, $\mathsf{vgraphs}^0_{mn}$ in Definition 5.12, and $\mathsf{vgraphs}_{mn} \otimes S(t, dt)$ in the proof of Corollary 5.13. The map π is defined at the beginning of Section 5.2, and the map ω is defined in Proposition 4.57.

We proved in Proposition 3.20 that $\mathsf{vsc}_{mn}^{\vee} \cong H^*(\mathsf{VFM}_{mn})$ as Hopf cooperads. We moreover proved in Corollary 5.13 that the two maps involving the three variants of $\mathsf{vgraphs}_{mn}$ were quasi-isomorphisms of Hopf cooperads. In addition, we proved that π was a quasi-isomorphism of Hopf cooperads in Proposition 5.15 (for $m \geq 2$) and Proposition 5.26 (for $m = 1$). Therefore it just remains to check that ω is a quasi-isomorphism of Hopf cooperads to conclude.

By the previous results, we know that $\mathsf{vgraphs}_{mn}$ and $\Omega^*_{\mathrm{PA}}(\mathsf{VFM}_{mn})$ have the same cohomology, namely $\mathsf{vsc}_{mn}^{\vee}$. Thus it is sufficient to show that the map ω is surjective on cohomology. The generators of $\mathsf{e}_n^{\vee}$ are represented by full edges between aerial external vertices. Generators of $\mathsf{e}_m^{\vee}$ are represented by dashed edges between terrestrial external vertices. The classes η_v are represented by graphs of the type seen in Remark 4.48. □

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