A category $\mathcal{K}$ is triangulated if it is additive, has an additive automorphism $\Sigma : \mathcal{K} \to \mathcal{K}$ and has a class $\mathcal{D}$ of distinguished triangles $(a \to b \to c \to \Sigma a)$ satisfying

(1) If Δ and Δ' are isomorphic triangles, $\Delta \in \mathcal{D} \Rightarrow \Delta' \in \mathcal{D}$

For any object $a \in \mathcal{K}$, $(a \xrightarrow{id} a \to 0 \to \Sigma a) \in \mathcal{D}$

$(a \xrightarrow{f} b \to c \to \Sigma a) \in \mathcal{D} \iff (b \to c \to \Sigma a \xrightarrow{-\Sigma f} \Sigma b) \in \mathcal{D}$

(2) For every morphism $a \xrightarrow{f} b$, there exists some triangle $\Delta = (a \xrightarrow{f} b \to c \to \Sigma a) \in \mathcal{D}$

(3) If
$$\begin{tikzcd}
\Delta = (a \arrow[r] \arrow[d, "f"'] & b \arrow[r] \arrow[d, "g"] & c \arrow[r] \arrow[d, dotted] & \Sigma a) \in \mathcal{D} \arrow[d, "\Sigma f"] \\
\Delta' = (a' \arrow[r] & b' \arrow[r] & c' \arrow[r] & \Sigma a') \in \mathcal{D}
\end{tikzcd}$$
then $\exists$ $c \xrightarrow{h} c'$ making the whole thing $\circlearrowright$

Rq From (1) + (2) + (3), can prove that $(a \to b \to c \to \Sigma a) \Rightarrow$ the object c is unique up to iso, denoted by cone(f)

(4) If $a \xrightarrow{f} a' \xrightarrow{f'} a''$ are composable, there exists a dist triangle:
$$\big(\text{cone}(f) \longrightarrow \text{cone}(f' \circ f) \longrightarrow \text{cone}(f') \longrightarrow \Sigma\,\text{cone}(f)\big) \in \mathcal{D}$$
\+ some commut relationships

A functor F b/w triangulated categories is exact if it commutes w/ suspension up to iso & preserves distinguished triangles

ex The derived category of an abelian category is triangulated
The homotopy category of a stable ∞-cat is triangulated

A tensor triangulated (tt) category $(\mathcal{K}, \otimes, 1)$ is a triangulated category with a symm mon structure such that $\otimes$ is exact in both variables
\+ some compatibility

A tt functor is an exact functor that is also monoidal $\big(F(1) = 1,\ F(A \otimes B) \simeq F(A) \otimes F(B)\big)$

Rk We will only consider essentially small tt categories

An ideal $\mathcal{J}$ of a tt category $\mathcal{K}$ is a full subcat containing 0 satisfying: (1) if $\Delta = (a \to b \to c \to \Sigma a) \in \mathcal{D}$ and two of a, b, c lie in $\mathcal{J}$, then so does the third

(2) $\mathcal{J}$ is thick: if $a \oplus b$ lie in $\mathcal{J}$, then $a, b \in \mathcal{J}$

(3) If $a \in \mathcal{J}$ and $b \in \mathcal{K}$, then $a \otimes b \in \mathcal{J}$

A tt ideal $\mathcal{J}$ is prime if it is proper (ie $\mathcal{J} \neq \mathcal{K}$) and $a \otimes b \in \mathcal{J} \Rightarrow a \in \mathcal{J}$ or $b \in \mathcal{J}$

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The spectrum $\mathrm{Spc}(\mathcal{K})$ of a tt category $(\mathcal{K}, \otimes, \mathbb{1})$ is the top space whose underlying set is $\{P : \text{prime tt ideal of } \mathcal{K}\}$
+ basis for the topology : $\{\mathrm{supp}(a) = \{P \mid a \notin P\} \mid a \in \mathcal{K}\}$ closed subsets

Rk $\mathrm{supp}(0) = \emptyset$, $\mathrm{supp}(a) \cup \mathrm{supp}(b) = \mathrm{supp}(a \oplus b)$

lemma Let $\mathcal{J}$ be a tt-ideal of a tt-category $\mathcal{K}$ and let S be a multiplicative system (ie $a, b \in S \Rightarrow a \otimes b \in S$, $\mathbb{1} \in S$) such that $\mathcal{J} \cap S = \emptyset$
Then there exists a prime tt ideal P st $\mathcal{J} \subset P$, $P \cap S = \emptyset$

sketch of proof define the family of tt-ideals $\mathcal{I}$ st
$\mathcal{I} \cap S = \emptyset$; $\mathcal{J} \subset \mathcal{I}$; if $a \in \mathcal{K}$, $b \in S$, $a \otimes b \in \mathcal{I}$, then $a \in \mathcal{I}$
this family is not empty : $\mathcal{I}_0 = \{a \in \mathcal{K} \mid \exists c \in S \text{ st } a \otimes c \in \mathcal{J}\}$
↳ use Zorn's lemma to find P

cor $\mathrm{Spc}\, \mathcal{K} \neq \emptyset$ for $\mathcal{K} \neq 0$
pf take $S = \{\mathbb{1}\}$, $\mathcal{J} = \langle 0 \rangle$

lemma The assignment $a \mapsto \mathrm{supp}(a)$ satisfies :
(1) $\mathrm{supp}(0) = \emptyset$, $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}\, \mathcal{K}$
(2) $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$
(3) $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$ (b/c $a \to a \to 0 \to \Sigma a$ $\overset{\text{rot}}{\rightsquigarrow}$ $a \to 0 \to \Sigma a \to \Sigma a$)
(4) $\mathrm{supp}(a) \subset \mathrm{supp}(b) \cup \mathrm{supp}(c)$ if there is $(a \to b \to c \to \Sigma a) \in \mathcal{D}$
(5) $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$

prop Let $P \in \mathrm{Spc}\, \mathcal{K}$, then $\overline{\{P\}} = \{Q \in \mathrm{Spc}\, \mathcal{K} \mid Q \subset P\}$
pf $\overline{\{P\}} = \bigcap_{\substack{a \text{ st} \\ P \in \mathrm{supp}(a)}} \mathrm{supp}(a) = \bigcap_{a \notin P} \{Q \mid a \notin Q\} = \{Q \subset P\}$

def A support datum of a tt category $\mathcal{K}$ is a pair (X, σ) where X is a top space and $\sigma : \mathrm{ob}(\mathcal{K}) \longrightarrow \{\text{closed subsets of } \mathcal{K}\}$ such that σ satisfies properties (1)-(5) from the previous lemma

A morphism $f : (X, \sigma) \to (Y, \tau)$ is a continuous map st $\sigma(a) = f^{-1}(\tau(a))$ $\forall a \in \mathrm{ob}\, \mathcal{K}$

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Thm (Universal ppty of the spectrum) The pair $(\mathrm{Spc}\, \mathcal{K}, \mathrm{supp})$ is the final support datum of a tt category $\mathcal{K}$. For any support datum (X, σ), the morphism $(X, \sigma) \xrightarrow{f} (\mathrm{Spc}\, \mathcal{K}, \mathrm{supp})$ is given by $f(x) = \{a \in \mathcal{K} \mid x \notin \sigma(a)\}$

Pf Since (X, σ) is a support datum, $f(x)$ is a prime tt-ideal for all x

$\forall x, a$, $f(x) \in \mathrm{supp}(a) \Leftrightarrow a \notin f(x) \Leftrightarrow x \in \sigma(a)$

This implies that $f^{-1}(\mathrm{supp}(a)) = \sigma(a)$

Suppose that f' is another morphism $X \to \mathrm{Spc}\, \mathcal{K}$, then $f^{-1}(\mathrm{supp}\, a) = (f')^{-1}(\mathrm{supp}\, a)$ $\forall a$

$\Rightarrow$ $f(x) \in \mathrm{supp}(a) \Leftrightarrow f'(x) \in \mathrm{supp}(a)$

$\Rightarrow$ $\overline{\{f(x)\}} = \overline{\{f'(x)\}} \Rightarrow f(x) = f'(x)$

prop Spc defines a contravar functor from ess$^{\text{ly}}$ small tt cat to Top

For a tt functor $F : \mathcal{K} \to \mathcal{L}$, $\mathrm{Spc}\, F : \mathrm{Spc}\, \mathcal{L} \longrightarrow \mathrm{Spc}\, \mathcal{K}$

$$Q \longmapsto F^{-1}(Q)$$

Moreover $(\mathrm{Spc}\, F)^{-1}(\mathrm{supp}_{\mathcal{K}}(a)) = \mathrm{supp}_{\mathcal{L}}(F(a))$

Pf of the last formula: $(\mathrm{Spc}\, F)(Q) \in \mathrm{supp}_{\mathcal{K}}(a)$

$\Leftrightarrow a \notin (\mathrm{Spc}\, F)(Q) = F^{-1}(Q)$

$\Leftrightarrow F(a) \notin Q$

$\Leftrightarrow Q \in \mathrm{supp}_{\mathcal{L}}(F(a))$