

Analogies & differences: usual spectrum & Balmer spectrum

	\mathcal{K} : tt-category	R : comm ring
Underlying set	$\text{Spec } \mathcal{K}$	$\text{Spec } R$
$\{P \mid a \notin P\}$	prime ideals	prime ideal
$\overline{\{P\}}$	basis of closed sets	basis of open set
	$\{Q \mid Q \subset P\}$	$\{Q \mid P \subset Q\}$

prop Let R be any comm ring, then there is an order-reversing homeo

$$\text{Spec}(R) \xrightarrow{\sim} \text{Spec}(\mathcal{K}^b(R\text{-proj})) \quad \text{natural in } R$$

Where $\mathcal{K}^b(R\text{-proj})$ is the homotopy category of bounded chain complexes of fp proj R -modules

To describe the map: $\mathcal{K}^b(R\text{-proj}) \simeq D^{\text{perf}}(R\text{-mod})$

$$p \in \text{Spec } R \mapsto \text{ker}(D^{\text{perf}}(R\text{-mod}) \rightarrow D^{\text{perf}}(R_{p\text{-mod}})) = "p\text{-torsion modules}"$$

extension of scalars \hookrightarrow localization at p

prop The space $\text{Spec } \mathcal{K}$ is always a spectral space, ie, it is quasi-compact, admits a basis of quasi-compact subspaces, and every closed irreducible subset has a unique generic pt

Let J be a t-t ideal in \mathcal{K}

The radical $\sqrt{J} = \{a \in \mathcal{K} \mid \exists n \text{ st } a^{\otimes n} \in J\}$

An ideal is called **radical** if $J = \sqrt{J}$

lemma $\sqrt{J} = \bigcap_{J \subset P, P \text{ prime}} P$ proof: obvious

Rk Often, all tt-ideals are radical

ex If $a \in \mathcal{K}$ has a dual object $D(a)$, then a is a direct summand of $a \otimes a \otimes D(a)$ $\Rightarrow a \in \langle a \otimes a \rangle$

+ lemma TFAE:

- every ideal is radical
- $\forall a \in \mathcal{K}, a \in \langle a \otimes a \rangle$

+ Lemma TFAE:

- every ideal is radical
- $\forall a \in K, a \in \langle a \otimes a \rangle$

Th (Classification of tt-ideals) Let K be a tt-category

The assignment $Y \mapsto K_Y = \{a \in K \mid \text{supp}(a) \subset Y\}$ is an order-preserving bijection b/w $\{Y \in \text{Spec } K \mid Y = \bigcup_{i \in I} \text{supp}(a_i)\}$ and radical ideals for some family $(a_i)_{i \in I}$

The inverse map is given by $J \mapsto \text{supp}(J) = \bigcup_{a \in J} \text{supp}(a)$

Lemma For any $Y \in \text{Spec } K$, $K_Y = \bigcap_{P \notin Y} P$

proof $a \in K_Y \Leftrightarrow \text{supp}(a) \subset Y$. The second part is $\Leftrightarrow \forall P \notin Y, a \in P$

Thus, $K_Y \subset \bigcap_{P \notin Y} P$

proof of the theorem: Both maps are well def, we want to check that they are inverse to each other.

• $K_{\text{supp } J} = \bigcap_{P \notin \text{supp } J} P$ If $P \notin \text{supp } J$, then $\forall a \in J, P \notin \text{supp } a \Rightarrow a \in P$

$\Rightarrow K_{\text{supp } J} = \bigcap J = J$ (b/c J is radical)

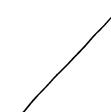
• $\text{supp}(K_Y) = Y$? if $P \in \text{supp } K_Y$, then $\exists a \in K_Y$ s.t. $P \in \text{supp } (a)$

$\Rightarrow \text{supp}(K_Y) \subset Y$

conversely, suppose $P \in Y = \bigcup_{i \in I} \text{supp}(a_i)$

$\Rightarrow \exists i$ s.t. $P \in \text{supp}(a_i) \subset Y$

$\Rightarrow a_i \in K_Y$ and $P \in \text{supp}(K_Y)$



notation: $U(a) = (\text{supp}(a))^c = \{P \mid a \in P\}$

$U(S) = \bigcup_{a \in S} U(a) = \{P \mid S \cap P \neq \emptyset\} \rightarrow$ any open is of this form

Thm Let K be a tt-cat, then:

i) $U(a)$ is quasi-compact

ii) If $U(S)$ is quasi-compact, then $\exists S \subset U(S), U(a)$

con $\text{Spec } K = U(0)$ is quasi-compact

Thm Let K be a tt-cat, \mathcal{Z} : non-empty closed subset of $\text{Spec } K$

| TFAE : 1) \mathcal{Z} is irreducible

2) $P = \{a \in K \mid \mathcal{Z} \cap U(a) = \emptyset\}$ is prime

| Moreover, if this holds then $\mathcal{Z} = \overline{\{P\}}$ (and P is unique)