Analogies & differences: usual spectrum & Balmer spectrum

| | $\mathcal{K}$: tt-category | $R$: comm ring |
|---|---|---|
| | $\operatorname{Spec} \mathcal{K}$ | $\operatorname{Spec} R$ |
| Underlying set | prime ideals | prime ideal |
| $\{P \mid a \notin P\}$ | basis of closed sets | basis of open set |
| $\overline{\{P\}}$ | $\{Q \mid Q \subset P\}$ | $\{Q \mid P \subset Q\}$ |

prop Let $R$ be any comm ring, then there is an order-reversing homeo
$$\operatorname{Spec}(R) \xrightarrow{\sim} \operatorname{Spec}(K^b(R\text{-proj})) \quad \text{natural in } R$$
Where $K^b(R\text{-proj})$ is the homotopy category of bounded chain complexes of f.gen proj $R$-modules

To describe the map: $K^b(R\text{-proj}) \simeq D^{perf}(R\text{-mod})$
$$p \in \operatorname{Spec} R \mapsto \ker\big(D^{perf}(R\text{-mod}) \xrightarrow{\text{extension of scalars}} D^{perf}(R_p\text{-mod})\big) = \text{"}p\text{-torsion modules"}$$
$R_p$: localization at $p$

prop The space $\operatorname{Spec} \mathcal{K}$ is always a spectral space, ie, it is quasi-compact, admits a basis of quasi-compact subspaces, and every closed irred subset has a unique generic pt

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Let $J$ be a t-t ideal in $\mathcal{K}$
The radical $\sqrt{J} = \{a \in \mathcal{K} \mid \exists n \text{ st } a^{\otimes n} \in J\}$
An ideal is called radical if $J = \sqrt{J}$

lemma $\sqrt{J} = \bigcap_{J \subset P,\, P \text{ prime}} P$ proof: obvious

Rk Often, all tt-ideals are radical

ex If $a \in \mathcal{K}$ has a dual object $D(a)$, then $a$ is a direct summand of $a \otimes a \otimes D(a) \Rightarrow a \in \langle a \otimes a \rangle$

\+ lemma TFAE:
- every ideal is radical
- $\forall a \in \mathcal{K}$, $a \in \langle a \otimes a \rangle$

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Th (Classification of tt-ideals) Let $\mathcal{K}$ be a tt-category
The assignment $Y \mapsto \mathcal{K}_Y = \{a \in \mathcal{K} \mid \operatorname{supp}(a) \subset Y\}$ is an order-preserving bijection b/w $\{Y \subset \operatorname{Spc} \mathcal{K} \mid Y = \bigcup_{i \in I} \operatorname{supp}(a_i)\}$ and radical ideals
for some family $(a_i)_{i \in I}$

The inverse map is given by $\mathcal{J} \mapsto \operatorname{supp}(\mathcal{J}) = \bigcup_{a \in \mathcal{J}} \operatorname{supp}(a)$

lemma For any $Y \subset \operatorname{Spc} \mathcal{K}$, $\mathcal{K}_Y = \bigcap_{P \notin Y} P$

proof $a \in \mathcal{K}_Y \Leftrightarrow \operatorname{supp}(a) \subset Y$. The second part is $\Leftrightarrow$ to $\forall P \notin Y$, $a \in P$
Thus, $\mathcal{K}_Y \subset \bigcap_{P \notin Y} P$

proof of the theorem: Both maps are well def, we want to check that they are inverse to each other.

• $\mathcal{K}_{\operatorname{supp} \mathcal{J}} = \bigcap_{P \notin \operatorname{supp} \mathcal{J}} P$ If $P \notin \operatorname{supp} X$, then $\forall a \in \mathcal{J}$, $P \notin \operatorname{supp} a \Leftrightarrow a \in P$

$\Rightarrow \mathcal{K}_{\operatorname{supp} \mathcal{J}} = \sqrt{\mathcal{J}} = \mathcal{J}$ (b/c $\mathcal{J}$ is radical)

• $\operatorname{supp}(\mathcal{K}_Y) = Y$? if $P \in \operatorname{supp} \mathcal{K}_Y$, then $\exists \underbrace{a \in \mathcal{K}_Y}_{\Rightarrow \operatorname{supp}(a) \subset Y}$ st $P \in \operatorname{supp}(a)$

$\Rightarrow \operatorname{supp}(\mathcal{K}_Y) \subset Y$

conversely, suppose $P \in Y = \bigcup_{i \in I} \operatorname{supp}(a_i)$
$\Rightarrow \exists i$ st $P \in \operatorname{supp}(a_i) \subset Y$
so $a_i \in \mathcal{K}_Y$ and $P \in \operatorname{supp}(\mathcal{K}_Y)$

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notation: $U(a) = (\operatorname{supp}(a))^c = \{P \mid a \in P\}$
$U(S) = \bigcup_{a \in S} U(a) = \{P \mid S \cap P \neq \emptyset\}$ → any open is of this form

Thm Let $\mathcal{K}$ be a tt-cat, then:
1) $U(a)$ is quasi-compact
2) If $U(S)$ is quasi-compact, then $\exists S$ st $U(S) = U(a)$

cor $\operatorname{Spc} \mathcal{K} = U(0)$ is quasi-compact

**Thm** Let $\mathcal{K}$ be a tt-cat, $Z$: non-empty closed subset of $\operatorname{Spc} \mathcal{K}$

TFAE: 1) $Z$ is irreducible

2) $P = \{ a \in \mathcal{K} \mid Z \cap U(a) = \emptyset \}$ is prime

Moreover, if this holds then $Z = \overline{\{P\}}$ (and $P$ is unique)