Things about spectra

Suppose $X = (X_0, X_1, \dots)$ are pointed top spaces st $X_i \simeq \Omega X_{i+1}$

$\Rightarrow X_0$ has an ∞-loop space spectrum

Notation: $\Omega^\infty X = X_0$, $\Omega^{\infty-1} X = X_1$, etc

Homotopy groups: $\pi_n X := \operatorname{colim}_k \pi_k (X_{n+k})$

Rk Makes sense for $n < 0$

General idea: set $X \rightsquigarrow$ free abelian group $\mathbb{Z}[X]$

part of an adjunction $\mathrm{Free}_*: \mathrm{Set}_* \rightleftarrows \mathrm{Ab}_*: \mathrm{Forget}_*$ of sym mon products

↳ if pointed, different monoidal struct: smash product

In topology: $\Sigma^\infty : \mathrm{Spaces}_* \rightleftarrows \mathrm{Spectra} : \Omega^\infty$

Σ^∞ is sym mon, Ω^∞ is oplax monoidal (wrt smash product)

Brown's representability theorem: correspondence b/w cohomology theories $\leftrightarrow$ spectra
$E^* : CW^{op} \to \mathrm{Ab}$

given a spectrum E, $E^k(X) = [X, E_k]$

notation: $E^k X = [X, \Omega^{\infty-n} E]$

can also define E-homology: $E_n X = \pi_n (E \wedge X)$

We want to understand thick subcategories of the category of spectra

$\longrightarrow$ Chromatic homotopy theory

Motivation: cohomology $\longleftrightarrow$ formal group laws

$\mathbb{C}P^\infty$: classifies complex line bundles

$\{\text{complex line bundles on } X\}/\simeq \;\longleftrightarrow\; \{X \to \mathbb{C}P^\infty\}/\simeq$

$\mathcal{L} \quad \longleftrightarrow \quad f \text{ st } f^* \mathcal{O}(1) = \mathcal{L}$

Chern class: $c_1(\mathcal{O}(1)) \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$

$\Rightarrow$ can define $c_1(\mathcal{L}) = f^* c_1(\mathcal{O}(1)) \in H^2(X; \mathbb{Z})$

If E is another spectrum, one can also define $c_1^E(\mathcal{L})$

Ordinary Chern classes behave well: $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$

↳ not necessarily true for c_1^E; instead, c_1^E follows a formal group law

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formal group law: $f(x,y) \in R[[x,y]]$

st $f(x,y) = x + y$ + higher order terms

We say that a cohom theory E is multiplicative if $\exists\, E \wedge E \to E$ assoc & unital (up to homotopy)

moreover called complex orientable if $E^2(\mathbb{C}P^\infty) \to E^2(S^2)$ is surjective

complex-oriented = choice of generator in $E^2(S^2)$

Thm If E is a complex oriented cohom theory, then we get a formal group law over the ring $\bigoplus_m \pi_{2m}(E) = \pi_* E$

Chern class $C_1^E(\mathcal{L} \otimes \mathcal{L}') = c_1^E(\mathcal{L}) + c_1^E(\mathcal{L}') + \dots$

Universal formal gp law:

prop There is a comm ring L and a FGL $f(u,v) \in L[[u,v]]$ st any FGL over a ring R can be obtained from $f(u,v)$ under some $L \to R$

L = "Lazard ring"

One specific example of complex oriented cohomology

MU = spectrum for which $L \cong \bigoplus_m \pi_* MU$

Where does it come from?

$BU(n)$ = classif sp for $\mathbb{C}^n$ bdl of rank n

$\Rightarrow MU(n) := \Sigma_+^{\infty - 2n}\left(BU(n)/BU(n-1)\right)$

MU is called the complex bordism spectrum: $\pi_n MU$ is the group of bordism classes of n-dim mfd w/ almost $\mathbb{C}$ structure

Bousfield localization

$\mathcal{C}$: full subcategory of Sp closed under shifts & hocolim st $\exists$ small subcat $\mathcal{C}_0 \subseteq \mathcal{C}$ which generates $\mathcal{C}$ under hocolim

A spectrum X is $\mathcal{C}$-local if every map $Y \to X$ is nullhomotopic when $Y \in \mathcal{C}$

↝ category $\mathcal{C}^\perp$ of $\mathcal{C}$-local spectra

Let's build a localization functor:

the inclusion $\mathcal{C} \hookrightarrow Sp$ has a right adjoint which gives $G : Sp \to Sp$ (factoring through $\mathcal{C}$)

+ natural maps $G(X) \to X$

Let $L(X)$ = cofiber $(G(X) \to X)$ ... cofiber seq $G(X) \to X \to L(X)$

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let $L(X) = \text{cofiber}(G(X) \to X)$, cofiber seq $G(X) \to X \to L(X)$

For a spectrum E, we say that X is E-acyclic if $X \wedge E = 0$

$$\rightsquigarrow \underbrace{G_E(X)}_{E\text{-acyclic}} \to X \to \underbrace{L_E(X)}_{E\text{-local}}$$

E-local, ie every $Y \to X$ is $\simeq 0$ when Y is E-acyclic

L_E = Bousfield localization wrt E

For a prime number p, p-completion = Bousfield localization wrt the Moore spectrum $\mathbb{M}_p = \text{cofib}(\mathbb{S} \xrightarrow{\times p} \mathbb{S})$

Morava K-theory:

spectrum st $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$, $\deg v_n = 2(p^n - 1)$

we could have five lectures about this

Classification of thick subcategories of Sp

$\mathcal{C}_0$ = p-local finite spectra

$\mathcal{C}_n \subseteq \mathcal{C}_0$: full subcat of $K(n-1)$-acyclic spectra

$\mathcal{C}_\infty$ = contractible spectra

There is a sequence $\mathcal{C}_\infty \subseteq \cdots \subseteq \mathcal{C}_{n+1} \subseteq \mathcal{C}_n \subseteq \cdots \subseteq \mathcal{C}_0$

Thm (Hopkins-Smith) If $\mathcal{C}$ is a thick subcat of finite p-local spectra, then $\mathcal{C} = \mathcal{C}_n$ for some $n \in \mathbb{N} \cup \{\infty\}$