Balmer spectrum of discrete rings (Jordan)

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Classical algebraic geometry: $R \in CRing \Rightarrow Spec_Z(R) = \{ p \subseteq R \mid p \text{ prime} \}$ Zariski spectrum

$\hookrightarrow$ replace R by HR : spectrum

Given $R \in CRing$, $\exists$ E_∞-ring spectrum HR which represents $H^*(-;R)$

$$\pi_*(HR) = \begin{cases} R & * = 0 \\ 0 & o/w \end{cases}$$

Mod_{HR} ob = spectra M + action $HR \otimes_{S} M \longrightarrow M$ + ...

$\cup$

$Perf_{HR}$: compact objects

$\otimes_R : Mod_{HR} \times Mod_{HR} \longrightarrow Mod_{HR}$, $M \otimes_R N = |M \otimes_S HR^{\otimes \bullet} \otimes_S N|$

Spectral seq $E^2 = Tor_R(\pi_* M, \pi_* N) \Rightarrow \pi_*(M \otimes_{HR} N)$

Dictionary	Spectra	Derived
	Mod_{HR}	$D(R)$
	$\otimes_{HR}$	$\otimes^{\mathbb{L}}_R$
	$Perf(R)$	$D^{perf}(R) = \{ cx \simeq \text{bounded } cx \text{ of } \{\text{gen proj}\} \}$

It makes sense to ask about $Spc_B(Perf_R) =: Spec_B(R)$ Balmer spectrum

How does it compare to $Spec_Z(R)$?

Thm [Hopkins, Neeman] Suppose $R \in CRing$. Then there are mutually inverse bijections

$$\left\{ \begin{matrix} \text{thick subcat} \\ \text{of } Perf_R \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{matrix} \text{subsets of } Spec_Z(R) \\ \text{closed under specialization} \end{matrix} \right\}$$

- $S \subset Spec_Z(R)$ is closed under specialization if $\forall p \in S, \forall p' \supseteq p$, $p' \in S$
- Given $X \in Perf_R$, $supp(X) = \{ p \mid X_p \neq 0 \}$ where $X_p = X \otimes_{HR} R_p$ localization
- $f(\mathcal{T}) = \bigcup \{ supp(X) \mid X \in \mathcal{T} \}$

  $g(S)$ = smallest thick subcat containing R/p for $p \in S$

Neeman's proof: assume that R is noetherian

Thm (R nilpotence) $f : X \to Y \in Perf_R$. Suppose $\forall \alpha : R \to \mathbb{k}$, $\mathbb{k}$: field, we ... Then f is nilpotent, ie $\exists N > 0$ st $f^{\otimes N} : X^{\otimes N} \to Y^{\otimes N}$ is $\simeq 0$

Thm (R nilpotence) $f : X \to Y \in \mathrm{Perf}_R$. Suppose $\forall \alpha : R \to \mathbb{k}$, $\mathbb{k}$: field, we have $f \underset{HR}{\otimes} \mathbb{k} \simeq 0$. Then f is nilpotent, i.e. $\exists N > 0$ st $f^{\otimes N} : X^{\otimes N} \to Y^{\otimes N}$ is $\simeq 0$

proof sketch: • $X \xrightarrow{f} Y \rightsquigarrow R \to D(X) \otimes Y$

$\Rightarrow$ We can assume that $X = R$

• $R \xrightarrow{f} Y$, let $An(f^{\otimes n}) = \{x \in R \mid f^{\otimes n} \circ x \simeq 0\} \subset An(f^{\otimes 2}) \subset \cdots \subset R$

since R is noetherian, $\exists N > 0$ st $An(f^{\otimes N}) = An(f^{\otimes N+1})$

We can replace f by $f^{\otimes N}$ $\Rightarrow$ can assume $An(f) = An(f^{\otimes 2}) = An(f^{\otimes 3}) = \cdots$

• claim we have $An(f) = R$

if not, $\exists An(f) \subset \mathfrak{p} \subset R$ prime $\Rightarrow An(f)_\mathfrak{p} \subset R_\mathfrak{p}$ $\Rightarrow$ can assume R is local

can also have $\exists n \,/\, \mathfrak{p}^n \subset An(f) \subset \mathfrak{p}$.

↳ can choose $\mathfrak{p}$ minimal st $An\, f \subset \mathfrak{p}$

$R \xrightarrow{\pi} R/An(f)$, nilrad$(R/An(f)) = \pi(\mathfrak{p})$

$\mathfrak{p} = (x_1, \dots, x_n)$, $\Rightarrow \exists N_i$ st $\pi(x_i)^{N_i} = 0 \Rightarrow x_i^{N_i} \in \mathfrak{p}$, choose N very large

All this leads to f being nilpotent (difficult induction on Krull dim $\square$

Apply this to investigate thick subcat

lemma $X, Y \in \mathrm{Perf}_R$ st $\mathrm{supp}(Y) \subseteq \mathrm{supp}(X)$, then $Y \in \mathrm{thick}(X)$ (thick subcat gen by X)

Pf $X \xrightarrow{id} X \Leftrightarrow R \to X \otimes D(X)$, exact triangle $\Sigma^{-1}\mathrm{cofib}_X \to R \to X \otimes DX \to \mathrm{cofib}_X$

For $R \xrightarrow{\alpha} \mathbb{k}$, $\ker(\alpha) = \mathfrak{m}$, suppose $\mathfrak{m} \in \mathrm{supp}(X)$

$\Rightarrow (\mathfrak{m} \to X \otimes DX \otimes \mathfrak{m}) \neq 0$ b/c $\mathfrak{m} \in \mathrm{supp}(X)$

$(\mathbb{k} \to X \otimes DX \otimes \mathbb{k}) \neq 0$ (hard; derived Nakayama ?)

on π_*, this map is injective

$\Rightarrow (\Sigma^{-1}\mathrm{cofib}_X \otimes \mathbb{k} \to \mathbb{k}) \simeq 0$ for $\mathbb{k} = R/\mathfrak{m}$, $\mathfrak{m} \in \mathrm{supp}(X)$

? ——— ?

Nilpotence thm $\Rightarrow \Sigma^{-1}\mathrm{cofib}_X \to Y \otimes DY$ is nilpotent

General deduction $\Rightarrow$ can inductively show that Y is a retract of $\underbrace{\Sigma^{-1}\mathrm{cofib}_X \otimes X \otimes DX}_{\in \mathrm{thick}(X)}$

lemma f from the main thm is injective

lemma $\mathcal{T} \subset \mathrm{Perf}_R$ thick, $\mathrm{supp}(X) \subset f(\mathcal{T}) \Rightarrow X \in \mathcal{T}$

remark f from …

lemma $\mathcal{T} \subset \mathrm{Perf}_R$ thick, $\mathrm{supp}(X) \subset f(\mathcal{T}) \Rightarrow X \in \mathcal{T}$

let $\{p_i\}$ be the set of minimal primes of $\mathrm{supp}(X)$ (finite b/c R is noeth)

$Y_i \in \mathcal{T}$ st $p_i \in \mathrm{supp}(Y_i)$

$\Rightarrow \mathrm{supp}(X) = \mathrm{clos}(\{p_i\}) \subset \bigcup_i \mathrm{supp}(Y_i) \subset \mathrm{supp}(\bigoplus_i Y_i)$

$\Rightarrow X \in \mathrm{thick}(\bigoplus Y_i) \subset \mathcal{T}$

lemma $\forall p \in \mathrm{Spec}_Z(R)$, $\exists X \in \mathrm{Perf}_R$ st $\mathrm{supp}(X) = \overline{p}$