Tannaka duality for stacks, after Lurie, Stefanich, Brandebourg, Schäpi, Gaitsgory, ... (Adrien)

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Slogan: many interesting schemes / stacks are 1-affine, i.e. affine in a categorified sense

$\to$ categorify the "functor of points" point of view

Let k be a comm ring, $\mathcal{O}$: k-algebra, $X = \mathrm{Spec}\,\mathcal{O}$

$\rightsquigarrow$ $X : \mathrm{CAlg}_k \longrightarrow \mathrm{Set}$

$$R \longmapsto X(R) := \mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}, R) = \mathrm{Hom}(\mathrm{Spec}\,R, X)$$

Schemes, and even alg spaces, can be defined to be functors of this form + condition / fppf topology

$X : \mathrm{Alg}_k \longrightarrow \mathrm{Set}$, covering $\{U_i \to U\}$ where U_i is affine, $U = \bigcup_i U_i$, each U_i is flat & locally of finite presentation

If $\mathcal{O}$ is an algebra $\Rightarrow$ $\mathcal{O}\text{-mod}_k$: sym mon cat for $\otimes_{\mathcal{O}}$, unit $\mathcal{O}$

$\Rightarrow$ $\mathrm{End}(\mathcal{O}) \underset{\mathrm{Alg}_k}{\cong} \mathcal{O}$: can recover $\mathcal{O}$ canonically from the category

This is a Grothendieck abelian category: all small colim, filtered colim are local

it has a generator G (such that $\mathrm{hom}(G, -)$ is faithful)

and $\forall M \in \mathcal{O}\text{-mod}$, $M \otimes -$ preserves colimits

If X is an alg space, $QC(X)$: cat of qcoh sheaves on $X = \bigcup_i U_i$

$\Rightarrow$ $QC(X) = \lim\limits_{i} (\mathcal{O}(U_i)\text{-mod})$

unit $\mathcal{O}_X$ = "the structure sheaf"

$\mathcal{O}(X) = \mathrm{End}(\mathcal{O}_X)$ = "algebra of global functions"

$$\begin{tikzcd} \Gamma_X : \mathrm{Hom}(\mathcal{O}_X, -) : QC(X) \arrow[rr] \arrow[dr] & & k\,\mathrm{Mod} \\ & \mathcal{O}(X)\text{-Mod} \arrow[ur] & \end{tikzcd}$$

Rq X is affine $\Longleftrightarrow$ $\Gamma_X : QC(X) \to \mathcal{O}(X)\text{-Mod}$ is an equivalence

Thm [Gabriel, Barr-Beck] X is affine $\Longleftrightarrow$ Γ_X preserves small colimits & is conservative: $\Gamma_X(M) \simeq 0 \Rightarrow M \simeq 0$

Algebraic stacks:

$X : \mathrm{CAlg}_k \longrightarrow \mathrm{Gpd}$ + condition

geometric stack: $\Delta : X \to X \times X$ is representable & affine, $\exists\, U \to X$ smooth & surjective, U affine geometric stack : $\Delta : X \to X \times X$ is representable & affine, ... is affine

ex 1) quasi-separated quasi-compact schemes (eg noetherian)

2) U : affine, G : affine gp scheme

$\Rightarrow U/G\,(R) :=$ groupoid ob = $U(R)$

mor : $x \xrightarrow[g \in G(R)]{} y$

In general, for a geom stack X

$$QC(X) := \lim_{\substack{U_i \to X \\ \text{aff open}}} \mathcal{O}(U_i)\text{-Mod}$$

If $X = \operatorname{Spec} k / G$, then $QC(X) = \operatorname{Rep}(G)$

In general, if $Y \to X$ is a morph of schemes / stacks / alg sps ...

$\Rightarrow \quad f^* : QC(X) \to QC(Y)$ is sym mon

ex $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, $f : A \to B \Rightarrow f^* = B \otimes_A (-)$

Thm [Lurie, Stefanich] $f \mapsto f^*$ is an equiv of gpds

$$\operatorname{Hom}(Y, X) \longrightarrow \operatorname{Hom}^{\otimes}(QC(X), QC(Y))$$

(sym mon colim functors)

Cor If $Y = \operatorname{Spec} R$, then $X(R) \simeq \operatorname{Hom}^{\otimes}(QC(X), R\text{mod})$

$=$ " $\operatorname{Spec} QC(X)$ "

There is a 2-category of sheaves of Groth cat on X, $2QC(X)$

the unit is $QC(X)$

$\Rightarrow \Gamma_X = \operatorname{Hom}_{\mathcal{G}}(QC(X), -)$

Thm [Stefanich] Γ_X is an equiv to $QC(X)$-mod