Mapping class groups and the symplectic representation (Laura Marino)

mercredi 29 novembre 2023 09:15

$S_{g,n}^{b}$: compact connected oriented surface of genus g with b boundary components and n punctures

$\mathrm{Mod}(S_{g,n}^{b})$ = group of isotopy classes of orientation preserving homeomorphisms of S that fix ∂S
$= \pi_0 \mathrm{Homeo}_\partial^+(S)$

<u>Rmk</u> : • we can replace homeos by diffeos

• we can replace isotopy classes with homotopy classes

We will define generators for $\mathrm{Mod}(S)$:

Dehn twist : for an annulus $A = S^1 \times I$

$\Rightarrow$ Twist map $T : A \longrightarrow A$
$(e^{2i\theta}, t) \longmapsto (e^{2i(\theta + t)}, t)$

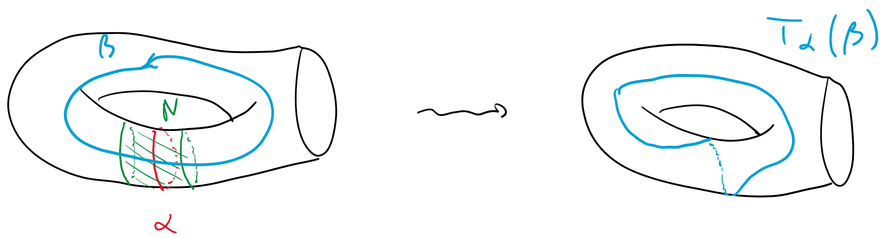
visually :  "turn left"

for a general surface : fix a simple closed curve (scc) $\alpha \subset S$ and a regular neighborhood N of α and an orientation-preserving homeo $\varphi : A \longrightarrow N$

$\Rightarrow$ Dehn twist along α : $T_\alpha : S \longrightarrow S$

$$x \longmapsto \begin{cases} \varphi \circ T \circ \varphi^{-1}(x) & \text{if } x \in N \\ x & \text{o/w} \end{cases}$$

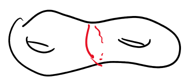
For a scc β, T_α acts like this :



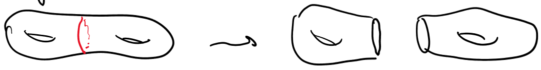
$T_\alpha \in \mathrm{Mod}(S)$ does not depend on the choice of N or φ

It only depends on the isotopy class of α

<u>Change of coordinates principle</u>

Recall : an scc is separating if $S \setminus \alpha$ is disconnected 

Recall: an scc is separating if $S \setminus \alpha$ is disconnected 

$S \searrow \alpha = S \setminus \mathrm{nbhd}(\alpha)$ 

Given α, β scc, define the geometric intersection number of α and β to be $i(\alpha, \beta) = \min\{ |\alpha' \cap \beta'| \mid \alpha' \text{ isotopic to } \alpha,\ \beta' \text{ isotopic to } \beta \} \in \mathbb{N}$

Thm
- If α and β are non-separating, then there is an orientation-preserving homeo $\varphi: S \to S$ st $\varphi(\alpha) = \beta$
- If α and β are separating and if the cut surfaces $S \searrow \alpha$, $S \searrow \beta$ are homeo, then there is an or-preserving homeo $\varphi: S \to S$ st $\varphi(\alpha) = \beta$
- If (α, β) and (α', β') are pairs of scc in S st $i(\alpha, \beta) = i(\alpha', \beta')$ and $S \searrow (\alpha, \beta)$ is homeo to $S \searrow (\alpha', \beta')$ then there is an or-preserving homeo of S taking (α, β) to (α', β')

Some facts about Dehn twists:

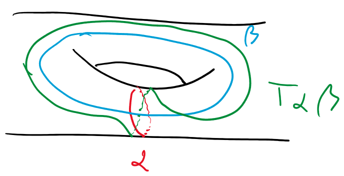
prop If α is a scc homotopic to a point or puncture of S then T_α is trivial (in $\mathrm{Mod}(S)$)

pf 

If we twist along α, we can "untwist" in the disk D bounded by α

prop If α is not homotopic to a point or puncture, then T_α is not trivial

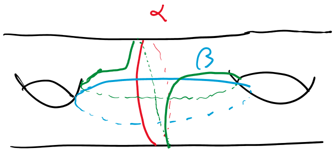
pf * If α is not separating, then we can find a scc β st $i(\alpha, \beta) = 1$ by the change of coordinates principle

Then $i(T_\alpha(\beta), \beta) = 1$ 

$\Rightarrow T_\alpha(\beta) \neq \beta$

* If α is separating and essential (not homotopic to boundary comp or puncture) then we can find β st $i(\alpha, \beta) = 2$

then we can find β st $i(\alpha,\beta) = 2$



then $i(T_\alpha(\beta), \beta) = 4$
$\Rightarrow T_\alpha(\beta) \neq \beta$

* If α is homotopic to a boundary component
  let $\bar{S}$ be the double of $S = S \cup_{\partial S} S$
  then in $\bar{S}$, α is essential $\Rightarrow$ we can conclude by the first two cases
  (b/c if T_α were trivial in S, it would be trivial in $\bar{S}$ too)

prop If α, β are essential, then $\forall k \in \mathbb{Z}$, $i(T_\alpha^k(\beta), \beta) = |k| \cdot i(\alpha,\beta)^2$
$\Rightarrow$ Dehn twists are of infinite order in $\mathrm{Mod}(S)$

prop $\forall \alpha, \beta$, $\forall i, j \in \mathbb{Z}$, $T_\alpha^i = T_\beta^j \Leftrightarrow \alpha$ is isotopic to β and $i = j$

prop If $f \in \mathrm{Mod}(S)$, then $T_{f(\alpha)}^j = f T_\alpha^j f^{-1}$ $\forall j \in \mathbb{Z}$

prop $f T_\alpha^j = T_\alpha^j f \Leftrightarrow f(\alpha)$ is isotopic to α (for any $j \neq 0$)

prop If α, β are non-separating scc, then T_α and T_β are conjugate in $\mathrm{Mod}(S)$

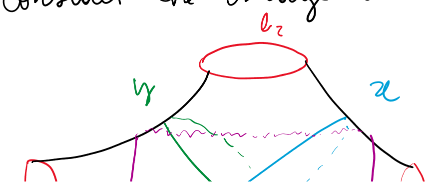
pf By the change of coordinates principle, $\exists f \in \mathrm{Mod}(S)$ st $f(\alpha) = \beta$
$\Rightarrow$ follows from previous results

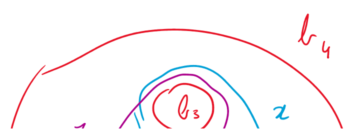
prop $i(\alpha,\beta) = 0 \Leftrightarrow T_\alpha(\beta) = \beta \Leftrightarrow T_\alpha T_\beta = T_\beta T_\alpha$
$i(\alpha,\beta) = 1 \Leftrightarrow T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$ ("braid relation")

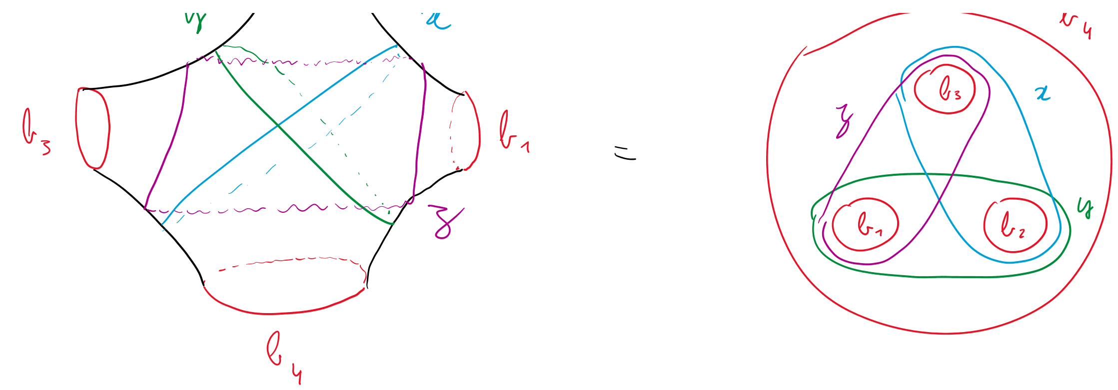
In general, if $S' \subset S$ is a subsurface, then there is a homomorphism $\mathrm{Mod}(S') \to \mathrm{Mod}(S)$

If $g \geq 2$, then S contains a subsurface homeo to S_0^4 (sphere w/ 4 boundary components)

prop (Lantern relation) Suppose that we have an embedding $S_0^4 \hookrightarrow S$
and consider the image in S of the seven curves below:



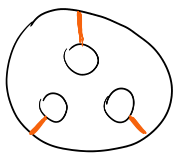




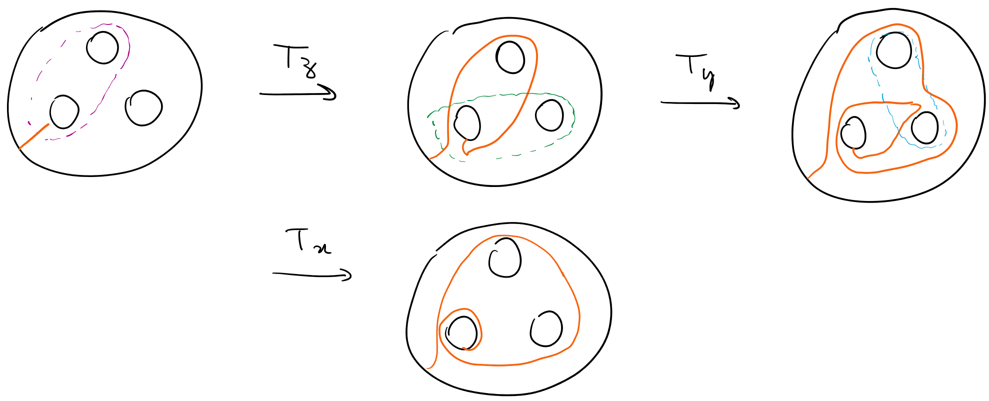
Then $T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}$

<u>proof</u> ("Alexander method") The action of an element of Mod(S) is often determined by its action of a well-chosen collection of curves and arcs in S

For S_0^4, take these three arcs:



It is enough to check that the lantern relation holds for these three arcs



Do the same for $T_{b_1} T_{b_2} T_{b_3} T_{b_4}$ $\Rightarrow$ same arc in the end

To be continued next week!