Plan
1) Idea & definition
2) Exact couples and double complexes
3) Group cohomology with coefficient in a chain cx
4) Equivariant homology

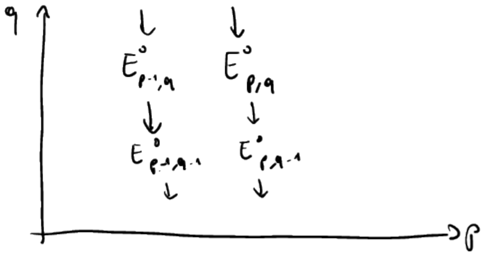
Reference: Rotman, "Homological algebra"
Brown, Group cohomology

1) Ideas & definitions

Spectral sequence = tool for computing homology in successive approximations

Begin with a family $(E^0_{p,q})_{p,q\geq 0}$ of R modules

+ differentials $d^0_{p,q} : E^0_{p,q} \longrightarrow E^0_{p,q-1}$



Columns are chain complexes
$\Rightarrow$ we can take homology to get a new family of R-modules:
$E^1_{p,q} = \left(H(E^0_{p,q}), d^0\right)_{p,q\geq 0}$

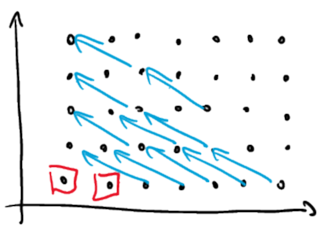
In a spectral sequence, there is a new diff $d^1_{p,q} : E^1_{p,q} \longrightarrow E^1_{p-1,q}$

Now the rows are chain complexes $\Rightarrow$ can take homology to get $E^2_{p,q}$

and in a SS we have $d^2_{p,q} : E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$

Iterating this process, we construct a family $(E^r_{p,q})$ + differentials
$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$

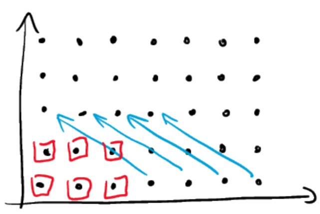
ex $d^2_{p,q} : E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$



E^2

Red part: not affected by d^2
$E^3_{0,0} = E^2_{0,0}$, $E^3_{0,1} = E^2_{0,1}$

For $r\geq 3$:



$d^3 : E^3_{p,q} \longrightarrow E^3_{p-3,q+2}$

Red part not affected by d^3

$\Rightarrow$ As r increases, more and more entries stabilize

$E^r_{p,q}$ stabilize at page $r = p+q+2$ (or before) def A (first quadrant, homological) spectral sequence is a family $(E^r_{p,q})_{p,q,r\geq 0}$ of R-modules endowed with differentials $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ such that: $\begin{cases} d^r \circ d^r = 0 \\ E^{r+1} = H(E^r, d^r) \end{cases}$

Every entry stabilizes at some page r, we denote $E^\infty_{p,q}$ the stable value

A spectral sequence $(E^r_{p,q})$ converges to a sequence of R-modules $(H_m)_{m\geq 0}$ written $E^R_{p,q} \Rightarrow H_{p+q}$ if there exists a filtration $0 = F^{-1}H_m \subset F^0 H_m \subset F^1 H_m \subset \cdots \subset F^m H_m = H_m$

such that $E^\infty_{p,q} \cong F^p H_{p+q} / F^{p-1} H_{p+q}$

def A spectral sequence collapses at $r \geq 2$ if there is a unique nonzero row or column in $\{E^r_{p,q}\}_{p,q\geq 0}$

In this case, we can read H_m off this page $\rightarrow$ the unique $E^r_{p,q}$ at $p+q=m$

② Exact couples and double complexes

def An exact couple is a triangle
$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$
such that

- D, E are bigraded modules
- the triangle is exact at each vertex

def A filtration of a chain cx (C, d) is a family of subcomplexes $(F^pC)_{p\in\mathbb{Z}}$ such that $F^pC_m \subset F^{p+1}C_m$ & $d(F^pC_m) \subset F^pC^m$

prop Every filtration $(F^pC)_p$ of a chain complex induces an exact couple with $\deg\alpha = (1,-1)$, $\deg\beta = (0,0)$, $\deg\gamma = (-1,0)$

pf For each $p \in \mathbb{Z}$, we have a SES: $0 \rightarrow F^{p-1}C \xrightarrow{\iota} F^pC \xrightarrow{\pi} F^pC/F^{p-1}C \rightarrow 0$

which induces a LES: $\cdots \rightarrow H_{p+q}(F^{p-1}C) \xrightarrow{\iota_*} H_{p+q}(F^pC) \xrightarrow{\pi_*} H_{p+q}(F^pC/F^{p-1}C)$
$\xrightarrow{\partial} H_{p+q-1}(F^{p-1}C) \xrightarrow{\iota_*} \cdots$

$\rightarrow$ define $D_{p,q} = H_{p+q}(F^pC)$, $E_{p,q} = H_{p+q}(F^pC/F^{p-1}C)$

$\alpha = \iota_*$, $\beta = \pi_*$, $\gamma = \partial$

(α,α')

$\llcorner \qquad \alpha = i_*, \quad \beta = \pi_*, \quad \gamma = \partial$

prop If
$$\begin{tikzcd} D \arrow[rr, "\alpha\ (a,a')"] & & D \arrow[dl, "\beta\ (b,b')"] \\ & E \arrow[ul, "\gamma\ (c,c')"] & \end{tikzcd}$$
is an exact couple, then $d = \beta \circ \gamma : E \to E$ is a differential and there exists a new exact couple
$$\begin{tikzcd} D' \arrow[rr, "\alpha'\ (a,a')"] & & D' \arrow[dl, "\beta'\ (b-a,\,b'-a')"] \\ & E' \arrow[ul, "\gamma'\ (c,c')"] & \end{tikzcd}$$
with $E' = H(E, d)$

in blue : degrees

Pf Let $D' = \operatorname{im} \alpha \subset D$, $\alpha' = \alpha|_{D'}$, $\beta' : D' \to E'$, $y \mapsto [\beta(\alpha^{-1}(y))]$

and $\gamma' : E' \to D'$, $[z] \mapsto \gamma(z)$

$\Rightarrow$ can check that this is well-defined and gives a new exact couple

We call this new exact couple the derived exact couple of the original one

Iterating this process gives a family $(E^0 = E,\ E^{r+1} = (E^r)',\ d^r = \beta \circ \gamma)$

Thm Let (C, d) be a chain cx

(i) Every filtration of (C,d) yields a spectral sequence (iterated derived couples)

(ii) If the filtration is bounded, then the associated spectral sequence converges to $H(C)$

ie $E^1_{p,q} \Rightarrow H_{p+q}(C)$

A double complex is a bigraded module $(M_{p,q})_{p,q \geq 0}$ equipped with differentials $d', d'' : M \to M$ of bidegrees $(-1,0)$ and $(0,-1)$ such that $d'd'' + d''d' = 0$

ie
$$\begin{tikzcd} M_{p-1,q} \arrow[d] & M_{p,q} \arrow[l] \arrow[d] \arrow[dl, phantom, "\ominus"] \\ M_{p-1,q-1} & M_{p,q-1} \arrow[l] \end{tikzcd}$$
anticommute

The total complex associated to it is $\operatorname{Tot}(M) = \left(\bigoplus_{p+q=n} M_{p,q},\ D = d' + d'' \right)_{n \geq 0}$

$\operatorname{Tot}(M)$ admits two filtrations :

- vertical filtration : ${}^{I}F^{p} \operatorname{Tot}(M)_{p,q} = \bigoplus_{i \leq p} M_{i,n-i}$
- horizontal filtration : ${}^{II}F^{q} \operatorname{Tot}(M)_{p,q} = \bigoplus_{j \leq q} M_{n-j,j}$

If (M,d) is a first quadrant double complex, then both filtrations are bounded $\Rightarrow$ both SS converge to $H(\operatorname{Tot} M)$