Last week:

Thm For every filtration of a chain complex (C,d), there is an associated spectral sequence (E^n_{pq}). If the filtration is bounded, then the spectral sequences converges to the homology:

$$E^n_{pq} \Rightarrow H_{p+q}(C) \qquad (\text{ie } E^\infty_{p,q} = F^p H_{p+q} / F^{p-1} H_{p+q})$$

Particular case: given a double complex $(M_{p,q})_{p,q\geq 0}$, we let
$\mathrm{Tot}(M)_n = \bigoplus_{p+q=n} M_{p,q} \Rightarrow$ two filtrations
- vertical filtration: ${}^{I}(F^p\,\mathrm{Tot}(M))_n = \bigoplus_{i\leq p} M_{i,n-i}$
- horizontal filtration: ${}^{II}(F^q\,\mathrm{Tot}(M))_n = \bigoplus_{j\leq q} M_{n-j,j}$

Both converge: ${}^{I}E^n_{pq} \Rightarrow H_{p+q}(\mathrm{Tot}(M))$, ${}^{II}E_{pq} \Rightarrow H_{p+q}(\mathrm{Tot}(M))$

③ Homology of a group with coefficients in a chain complex

Reminder For a group G, let F be the bar construction (or any proj resol° of $\mathbb{Z}$)
then $H_i(G) := H_i(F_G)$ where $F_G = F/(gx - x)_{g\in G,\, x\in F}$

A basis of F_n is given by the $[g_1|\cdots|g_n]$, and we have maps

$$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n], & i=0 \\ [g_1|\cdots|g_ig_{i+1}|\cdots|g_n], & 0<i<n \\ [g_1|\cdots|g_{n-1}], & i=n \end{cases} \qquad \partial = \sum_{i=0}^n (-1)^i d_i$$

ex in low dimensions:
$$\cdots (F_G)_2 \xrightarrow{\partial} (F_G)_1 \xrightarrow{0} \mathbb{Z}$$
$$[g|h] \longmapsto [g] - [gh] + g\cdot[h]$$

$\Rightarrow H_0(G) \cong \mathbb{Z}$, $H_1(G) \cong G/[G,G] = G^{ab}$, etc

Now, if (C,d) is a chain complex of G-modules, the homology of G w/ coefficients in C is:
$$H(G;C) := H(F\otimes_G C) \qquad \text{where } F \text{ is the bar resol°}$$

$$(F\otimes_G C)_n = \bigoplus_{p+q=n} F_p \otimes_G C_q = (\mathrm{Tot}(F_\bullet \otimes_G C_\bullet))_n$$

$\Rightarrow$ we have two spectral sequences ${}^{I,II}E_{p,q} \Rightarrow H_{p+q}(G;C)$

Prop (i) ${}^{I}E^1_{p,q} = H_q(F_p\otimes_G C_\bullet) = F_p \otimes_G H_q(C)$

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${}^{I}E^2_{p,q} = H_p(G; H_q(C))$

(ii) ${}^{II}E_{p,q} = H_q(F \otimes_G C_p) = H_q(G; C_p)$

${}^{II}E_{p,q} = H_p(H_q(G; C_\bullet))$

proof Construct the assoc graded of the filtered C_*

ex If $H_i(G; C_p)$ $\forall p$, $\forall i > 0$, then by (ii), ${}^{II}E^1$ is concentrated in the first row

$\Rightarrow$ the SS collapses at ${}^{II}E^2$ and $H_p(G; C) \cong {}^{II}E^2_{p,0} = H_p(C_G)$

Thm (Hochschild-Serre Spectral sequence) For any group extension $1 \to H \to G \to Q \to 1$ and any G-module M, there is a spectral sequence:

$$E^2_{p,q} = H_p(Q; H_q(H; M)) \Longrightarrow H_{p+q}(G; M)$$

proof For such an extension, we can check that $C_p = F_p \otimes_H M$ is Q-flat. We can thus apply the previous proposition, plus the fact that

$$H(G; M) = H(C_Q), \quad H(H; M) = H(C).$$

coro Under the same hypotheses, there is an exact sequence

$$H_2(G; M) \to H_2(Q; M_H) \to H_1(H; M)_Q \to H_1(G; M) \to H_1(Q; M_H) \to 0$$

proof We have an exact sequence:

$$0 \to E^\infty_{2,0} \xrightarrow{1} E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \xrightarrow{2} H_1(G; M) \xrightarrow{3} E^\infty_{1,0} \to 0$$

1 $E^\infty_{2,0} = E^3_{2,0} = \ker(d^2) \hookrightarrow E^2_{2,0}$

2 $E^2_{0,1} \twoheadrightarrow E^\infty_{0,1} = F^0 H_1(G; M) \hookrightarrow H_1(G; M)$

3 $H_1(G; M) \twoheadrightarrow E^\infty_{1,0} = E^2_{1,0} = H_1(G; M) / E^\infty_{0,1}$

But now we have: $E^2_{2,0} = H_2(Q; H_0(H; M)) = H_2(Q, M_H)$

$E^2_{0,1} = H_0(Q; H_1(H; M)) = H_1(H; M)_Q$

$E^\infty_{2,0} = E^3_{2,0} = H_2(G; M) / E^2_{1,0}$

ex $G = \langle F | R \rangle$ finitely presented group, where $R \subset W(F)$, $|F| = n$, $|R| = m$

$\Rightarrow$ group extension $1 \to \overline{\langle R \rangle} \to \langle F \rangle \to G \to 1$

We can apply the corollary with $M = \mathbb{Q}$ $\Rightarrow$ exact sequence:

We can apply the corollary with $M = \mathbb{Q}$ $\Rightarrow$ exact sequence:

$$H_2(\langle F\rangle;\mathbb{Q}) \longrightarrow H_2(G;\mathbb{Q}) \longrightarrow H_1(\overline{\langle R\rangle};\mathbb{Q})_G \longrightarrow H_1(F;\mathbb{Q}) \longrightarrow H_1(G;\mathbb{Q}) \longrightarrow 0$$

Note that
- $H_2(\langle F\rangle;\mathbb{Q}) = 0$ since $\langle F\rangle$ is free
- $H_1(\langle F\rangle;\mathbb{Q}) = \langle F\rangle^{ab}\otimes\mathbb{Q} = \mathbb{Z}^n\otimes\mathbb{Q} = \mathbb{Q}^n$
- $H_1(\overline{\langle R\rangle};\mathbb{Q})_G = H_0(G; H_1(\overline{\langle R\rangle};\mathbb{Q})) = H_0(G; \overline{\langle R\rangle}^{ab}\otimes\mathbb{Q})$ trivial action

  $= \overline{\langle R\rangle}\otimes\mathbb{Q} = \mathbb{Q}^k$ for some $k\leq m$

↳ in the exact sequence: $0 \to H_2(G;\mathbb{Q}) \longrightarrow \mathbb{Q}^k \longrightarrow \mathbb{Q}^n \longrightarrow H_1(G;\mathbb{Q}) \longrightarrow 0$

$\Rightarrow$ proves that $\dim H_2(G;\mathbb{Q}) \leq m$

Rk If $\dim H_2(G;\mathbb{Q}) = \infty$, then G cannot be finitely presented

4 Equivariant group homology

G-complex = CW-complex endowed with an action of G that permutes the cells

it is acyclic if $H_*(X) = H_*(pt)$

Given a G-complex X, its equivariant homology is: $H_i^G(X) = H_i(G; C(X))$

where $C(X)$ is the cellular chain cx

If M is a G-module, we let $H_i^G(X;M) = H_i(G; C(X)\otimes M)$

ex If $X = pt$, then $H^G(X;M) \cong H(G;M)$

If G is the trivial gp, then $H^G(X;M) = H(X;M)$

prop If X is acyclic then there is a canonical iso $H_i^G(X;M) \xrightarrow{\cong} H_i(G;M)$

Pf For any G-complex X, the map $X \longrightarrow pt$ induces a canonical map

$H^G(X;M) \longrightarrow H^G(pt;M) = H(G;M)$

There is a SS $\quad E^2_{p,q} = H_p(G; H_q(X;M)) \Rightarrow H_{p+q}(X;M)$

Since X is acyclic, $H_q(X;M) \cong H_q(pt;M)$ $\to$ induces an iso on the E^2 page:

$$\begin{array}{ccc} H_p(G; H_q(X;M)) & \Longrightarrow & H^G_{p+q}(X;M) \\ \downarrow\cong & & \downarrow \\ H_p(G; H_q(pt;M)) & \Longrightarrow & H_{p+q}(G;M) \end{array}$$

$\Rightarrow$ the induced map on E^∞ is an iso and therefore also before gr.

If X is a G-complex and σ is a p-cell of X, then we let:

If X is a G-complex and σ is a p-cell of X, then we let:

$$G_\sigma = \{g \in G \mid g\cdot\sigma = \sigma\}$$

Let $\mathbb{Z}_\sigma$ be the orientation module, i.e. ($G_\sigma \curvearrowright \mathbb{Z}$) with action given by $\chi_\sigma(g) = \begin{cases} 1, & \text{if } g \text{ preserves the orientation;} \\ -1, & \text{o/w.} \end{cases}$

Let $M_\sigma = \mathbb{Z}_\sigma \otimes M$: G_σ-module with action twisted by χ_σ

$$\Rightarrow C_p(X) \otimes M \cong \bigoplus_{\sigma \in X_p} \mathbb{Z}_\sigma \otimes M = \bigoplus_{[\sigma] \in X_p/G} \bigoplus_{g \in G} \mathbb{Z}_{g\cdot\sigma} \otimes M$$

$$\cong \bigoplus_{[\sigma]\in X_p/G} \mathrm{Ind}_{G_\sigma}^{G} M_\sigma$$

Lemma (Shapiro's lemma) If $H \subset G$ and M is an H-module, then

$$H(H; M) = H(G; \mathrm{Ind}_H^G M)$$

(cf Clovis's talk)

$$\Longrightarrow H_q(G; C_p(X; M)) \cong \bigoplus_{[\sigma]\in X_p/G} H_p(G_\sigma; M_\sigma)$$

$\Rightarrow$ the SS becomes: ${}^{\mathrm{II}}E^1_{p,q} = \bigoplus_{[\sigma]\in X_p/G} H_p(G_\sigma; M_\sigma) \Longrightarrow H^G_{p+q}(X; M)$

If X is acyclic, the limit is $H_{p+q}(G; M)$