Leitmotiv: Understanding diffeomorphisms by how they act on curves

① Introduction: the standard curve complex

Let S be a surface. The simplicial complex $\mathcal{C}(S)$ is given by:

- vertices = isotopy classes of essential simple closed curves in S
  (essential = not homotopic to a point, a puncture or a boundary component)
- edges: there is an edge b/w a and b if $i(a,b) = 0$ (intersection $\#$)

$\mathcal{C}(S)$ is the flag complex of this graph ($\rightarrow$ add all the simplices generated by these edges)

Rk $\mathrm{Mod}(S)$ acts on $\mathcal{C}(S)$ by simplicial transfo

Thm 0 If $3g + n \geq 5$ then $\mathcal{C}(S_{g,n})$ is connected

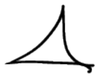
(Today: case $g \geq 1$)

ex $\mathcal{C}(S_{1,0})$: definitely not connected $\Rightarrow$ discrete!

It is more interesting to make an edge $a - b$ if $i(a,b) = 1$ instead

Thm [Masur - Minsky 85] $\mathcal{C}(S)$ is Gromov δ-hyperbolic

(It was recently shown that δ does not depend on S)

(δ = in a triangle  $\cup$ δ-nbhd of 2 edges contains the third)

$\mathcal{N}(S)$ = subgraph generated by non-separating simple closed curves

② Modified version $\hat{\mathcal{N}}(S)$

$\hat{\mathcal{N}}(S)$: vertices are the same as $\mathcal{N}(S)$

there is an edge $a - b$ if $i(a,b) = 1$

lemma (Lickorish) $\hat{\mathcal{N}}(S_{g,n})$ is connected for $g \geq 1$ and $n \geq 0$

proof Let a, b be vertices of $\hat{\mathcal{N}}(S)$. We need to find $c_0 = a, c_1, \dots, c_k = b$ such that $i(c_i, c_{i+1}) = 1$.

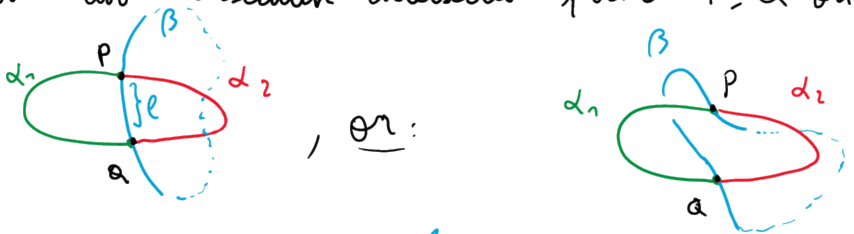
We construct this by induction on $i(a,b)$

- $i(a,b) = 1 \longrightarrow$ nothing to do
- suppose $i(a,b) = n \geq 2$, assume the result proved for lower values of i
  we look for γ st $c(a,\gamma), c(b,\gamma) < n$

* Suppose $i(a,b) = n \geq 2$, assume the result proven for lower values of i

we look for γ st $c(a,\gamma), c(b,\gamma) < n$

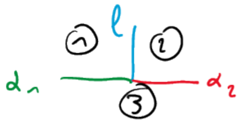
choose representatives α, β in minimal position, ie $\#(\alpha \cap \beta) = n$
& choose two consecutive intersection points P, Q on β



or more abstractly:

Let $\gamma_1 = \alpha_1 \cup \ell$, $\gamma_2 = \alpha_2 \cup \ell$.

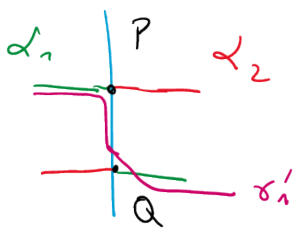
↳ one of these must be a non-separating closed curve:



three regions ①②③. If both γ_1 and γ_2 were separating,
then $① \cap ③ = \emptyset$, $② \cap ③ = \emptyset$
$\Rightarrow (① \cup ②) \cap ③ = \emptyset \Rightarrow \alpha$ is separating, contradiction

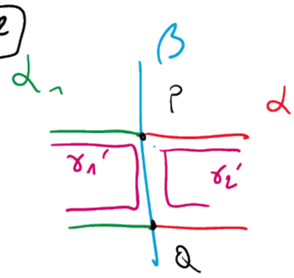
Pick γ_i the non-separating one & push it a bit inwards to get γ_i'

$\boxed{\text{case 1}}$



$\gamma_i' \pitchfork \alpha = 1$
$\gamma_i' \pitchfork \beta = n-1$

$\boxed{\text{case 2}}$

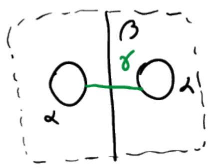


$\gamma_i' \pitchfork \alpha = 0$
$\gamma_i' \pitchfork \beta = n-2$

Moreover, since γ_i' is not separating, it is essential

* If $i(a,b) = 0$: two cases

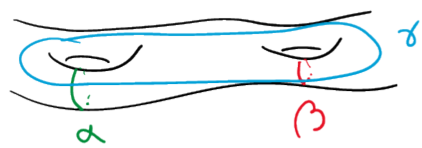
· If $a \cup b$ is separating:



$S \setminus \alpha$ locally

then $i(\alpha,\gamma) = i(\beta,\gamma) = 1$

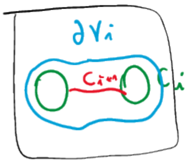
· If $a \cup b$ is not separating



Pf of thm 0: Let $a, b \in \mathcal{C}(S)$

* If a, b are not separating. From the lemma $\exists\, a = c, \dots c = b$

* If a, b are not separating. From the lemma, $\exists\, a = c_0, \dots, c_k = b$ st $i(c_i, c_{i+1}) = 1$. Let V_i be a regular nbh of $c_i \cup c_{i+1}$, then ∂V_i is a scc st $i(\partial V_i, c_i) = 0$, $i(\partial V_i, c_{i+1}) = 0$



Moreover, if ∂V_i was not essential, then $S = S_{1,1}$ $\rightarrow$ forbidden

$\rightarrow$ get a path $a = c_0, \partial V_0, c_1, \partial V_1, \dots, c_k = b$ in $\mathcal{C}(S)$

* If a is separating, $S \setminus a = S' \sqcup S'$. Both pieces have genus > 0 (b/c a is essential). Let a' be a non-separating curve in S' $\rightarrow$ then $i(a, a') = 0$. We can thus assume that a is not separating

Thm If $G \curvearrowright X$, X: connected graph, the action is transitive on vertices and on pairs of vertices related by an edge
Let $v, w \in Sk_0(X)$ st $v — w \in Sk_1(X)$. Let $h \in G$ st $h \cdot w = v$
Then G is generated by $h \cup Stab(v)$

Cor $Mod(S_g)$ is finitely generated