Johnson homomorphism : $\tau_1 : H_1(\mathcal{T}; \mathbb{Q}) \xrightarrow{\simeq} \Lambda^3 H \otimes \mathbb{Q}$

1) N-series on groups

G : fg group

An N-series on G is a sequence $\varphi^1 = G \supset \varphi^2 \supset \varphi^3 \supset \dots$ st
$[\varphi^m, \varphi^n] \subset \varphi^{m+n}$
$= \langle [x,y] \mid x \in \varphi^m, y \in \varphi^n \rangle$

ex Lower central series : $\Gamma_1 = G$, $\Gamma_{m+1} = [G, \Gamma_m]$

Rk If φ is an N-series, then $\Gamma^m \subset \varphi^m$ for all m
G/φ^m is nilpotent of class m

By construction, the quotients $\varphi^m / \varphi^{m+1}$ is abelian $\forall m$

Its "rationalization" is $\varphi^m_{\mathbb{Q}} = \{ x \in G \mid \exists n \text{ st } x^n \in \varphi^m \} \Rightarrow$ subgroup
$\varphi^m_{\mathbb{Q}} / \varphi^{m+1}_{\mathbb{Q}}$ is the torsion-free part of $\varphi^m / \varphi^{m+1}$

The associated graded is $gr^{\varphi} G = \bigoplus_{m \geq 1} \varphi^m / \varphi^{m+1}$

ex $gr\, G = gr^{\Gamma} G$

Prop The commutator induces a Lie algebra structure on $gr^{\varphi} G$ & the bracket preserves the degree

Pf Let $x \in \varphi^m$, $y, y' \in \varphi^n$, $z \in \varphi^p$

- By definition, $[x,y] \in \varphi^{m+n}$
- For any $g \in G$, $[x,y]^g \equiv [x,y] \pmod{\varphi^{m+n+1}}$
  Since $[x, yy'] = [x,y] \cdot [x,y']^y \Rightarrow$ linear in $gr\,\varphi$
- Hall-Witt identity :
  $[[x,y], z^x] \cdot [[z,x], y^z] \cdot [[y,z], x^y] = 1$ in G
  $\Rightarrow$ Jacobi in $gr\,\varphi$

⚠ There is a map of Lie algebras $gr\,\Gamma \longrightarrow gr\,\varphi$, but in general it is neither injective nor surjective

Thm [Magnus] Let F_n be a free group on n generators. Then
$gr\, F_n$ is the free Lie algebra on n generators

$\mathrm{gr}\, F_n$ is the free Lie algebra on n generators

(Rk: $F_g = \pi_1(S_{g,1})$)

For S_g, $\mathrm{gr}\,\pi_1(S_g) = \mathrm{Lie}(H_1(S_g)) / (\sum_i [a_i, b_i] = 0)$

where a_i, b_i is a symplectic basis of $H_1(S_g)$

2) Johnson homomorphism

Let $A \subset \mathrm{Aut}(G)$ be a subgroup. The Johnson filtration is given by:

$J^m = \ker(A \longrightarrow \mathrm{Aut}(G/\Gamma^{m+1}))$

Rk: Γ^{m+1} is a characteristic gp $\Rightarrow$ preserved by A

$T_A = J^1$ = Torelli group of A

prop [Kaloujnine] $(J^m)_{m\geq 1}$ is an N-series on T_A

Let $A_0 = A/T_A$ be the symmetry group of T_A $\Rightarrow$ not abelian in general

For a Lie algebra $\mathfrak{g}$, let:

$\mathrm{Der}^+(\mathfrak{g}) = \bigoplus_{m\geq 1} \mathrm{Der}^m(\mathfrak{g})$ where $\mathrm{Der}^m(\mathfrak{g}) = \{$derivations of $\mathfrak{g}$ of degree $m\}$

$\Rightarrow$ Lie algebra structure given by the commutator

Thm [Johnson-Papadima] There is a graded Lie algebra map $\mathrm{gr}^J T_A \xrightarrow{\tau} \mathrm{Der}^+(\mathrm{gr}\, G)$

which is injective $\Rightarrow$ Johnson homomorphism

It is defined by: given $a \in J^m$, $x \in \Gamma^n$,

$\bar{a} \cdot \bar{x} = \overline{a(x) \cdot x^{-1}}$

Rk $\bar{a} \in \mathrm{Der}^+(\mathfrak{g})$ is trivial $\iff \forall x \in \Gamma^n$, $a(x)x^{-1} \in \Gamma^{m+n+1}$

The action of A on T_A induces an action of A_0 on $\mathrm{gr}^J T_A$

given by $\bar{a} \cdot \bar{x} = \overline{a x a^{-1}}$

There is also an action of A_0 on $\mathrm{gr}\, G$ by $\bar{a} \cdot \bar{x} = \overline{a(x)}$

$\Rightarrow A_0$ acts on $\mathrm{Der}^+(\mathrm{gr}\, G)$ by $\bar{a} \cdot d = \bar{a} \circ d \circ \bar{a}^{-1}$

prop The Johnson homomorphism $\tau : \mathrm{gr}^J T_A \longrightarrow \mathrm{Der}^+(\mathrm{gr}\, G)$

is A_0-equivariant

3) Actual Torelli group

Let $S = S_{g,1}$, $\pi = \pi_1(S_{g,1}, *)$ where $* \in \partial S_{g,1}$

Let (a_i, b_i) be a symplectic basis of $H = H_1(S_{g,1})$, $\omega = \sum_i a_i \wedge b_i \in \Lambda^2 H$

and $\zeta = [\partial S]$

Thm [Dehn] The action of $\mathrm{Mod}(S)$ on π is faithful

$\mathrm{Mod}(S) = \{ f \in \mathrm{Aut}(\pi) \mid f(\zeta) = \zeta \}$

General setting : Torelli gp $\rightsquigarrow$ Torelli gp

$A_0 \rightsquigarrow Sp(H)$

Johnson homomorphism $\rightsquigarrow$ $gr^J T \xrightarrow{\tau} \mathrm{Der}^+(\underbrace{gr\,\pi}_{\mathrm{Lie}(H)})$

$\Rightarrow$ $gr^J T_A$ is torsion - free $\Rightarrow$ $J^m_{\mathbb{Q}} = J^m \supset \Gamma^m_{\mathbb{Q}}(T)$

Rk It's known that $H_1(T)$ has torsion

$\Rightarrow \Gamma^2 \subsetneq J^2$ even though $\Gamma^2_{\mathbb{Q}} = T^2$

[Hain] $\Gamma^2_{\mathbb{Q}} / \Gamma^3_{\mathbb{Q}} \longrightarrow J^2/J^3$ has kernel $\cong \mathbb{Z}$

[Hain] If $g \geq 6$, $gr\, T \otimes \mathbb{Q}$ is finitely presented as a Lie algebra

In the free group F_n, $\bigcap_{m \geq 1} \Gamma^m_{\mathbb{Q}} = \{1\}$ $\Rightarrow$ F_n is residually torsion free nilpotent

$\Rightarrow \bigcap_{m \geq 1} J^m_{\mathbb{Q}} = \{1\}$ in T

$\Rightarrow$ T is residually torsion - free nilpotent

The Johnson homomorphism $\tau : gr^J \longrightarrow \mathrm{Der}^+(gr\,\pi)$ lands in symplectic derivations that preserve $\omega \in \Lambda^2 H = gr^2 \pi$

Recall that T is generated by bounding pairs $T_\alpha T_\beta^{-1}$, α, β : simple closed curves $[\alpha] = [\beta]$

If γ is a s.c.c. st. $[\gamma] = 0$ then $T_\gamma \in T$

Theorem [Johnson]

1) (T_γ) generate the kernel of $T \longrightarrow H_1(T) \longrightarrow \mathrm{Der}^+(gr\,\pi)$ and their image in $H_1(T)$ is 2 - torsion

2) Explicit formulas for the images of $T_\alpha T_\beta^{-1}$

$\mathrm{Der}^1(gr\,\pi) \cong H^* \otimes \Lambda^2 H \cong H \otimes \Lambda^2 H$

$$\mathrm{Der}^1(\mathrm{gr}\,\pi) \cong H^* \otimes \Lambda^2 H \underset{\text{using } \omega}{\cong} H \otimes \Lambda^2 H$$

Thm [Johnson] The image of τ_1 is $\Lambda^3 H \subset H \otimes \Lambda^2 H$ and induces an iso $H_1(\mathcal{T};\mathbb{Q}) \cong \Lambda^3 H \otimes \mathbb{Q}$