THE JOHNSON HOMOMORPHISMS

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1. N-series and associated graded of a f.g. group

Let G be a fintely generated group, and for subgroups $A, B \subset G$ let (A, B) be the subgroup generated by commutators $\{(a, b), a \in A, b \in B\}$.

Definition 1.1. *An N series for G is a sequence of subgroups*

$$G = \Phi^1 \supset \Phi^2 \supset \Phi^3 \dots$$

such that $(\Phi^m, \Phi^n) \subset \Phi^{m+n}$.

This implies at once that Φ^{m+1} is normal in G (hence in Φ^m) and that the quotient Φ^m/Φ^{m+1} is abelian. The main example of an N-series is the lower central series defined by $\Gamma^1 = G$ and

$$\Gamma^{m+1} = (G, \Gamma^m).$$

An N-series is in particular central, so that $\Gamma^m \subset \Phi^m$, hence the quotient G/Φ^m is nilpotent. In particular, the subset of torsion element is a (normal) subgroup. The rationalization of Φ is

$$\Phi^m_{\mathbb{Q}} = \{x \in G, x^n \in \Phi^m \text{ for some } n\}.$$

It has the property that $G/\Phi^m_{\mathbb{Q}}$ is the quotient of G/Φ^m by its torsion subgroup.

Definition 1.2. *The associated graded w.r.t the series Φ is*

$$\mathrm{gr}^{\Phi} G = \bigoplus_{m \geq 1} \mathrm{gr}^m G$$

where $\mathrm{gr}^m_{\Phi} G := \Phi^m/\Phi^{m+1}$. *We set* $\mathrm{gr}\, G := \mathrm{gr}_{\Gamma} G$.

Proposition 1.3. *The commutator induces on $\mathrm{gr}_{\Phi} G$ the structure of a graded $\mathbb{Z}$-Lie algebra. The inclusion $\Gamma^m \subset \Phi^m$ induces a graded Lie algebra map*

$$\mathrm{gr}\, G \longrightarrow \mathrm{gr}_{\Phi} G.$$

$\mathrm{gr}\, G$ is generated as a Lie algebra by $\mathrm{gr}_1 G$, i.e. the abelianization of G.

Sketch of proof. Let $g \in G$, $x \in \Phi^m, y, y' \in \Phi^n, z \in \Phi^p$ and for $a, b \in G$ set $a^b = aba^{-1}$. Then:

- by defintion, $(x, y) \in \Phi^{m+n}$ and $(x, y)^g = (x, y) \mod \phi^{m+n+1}$.
- $(x, yy') = (x, y)(x, y')^y$ so the commutator descends to a bilinear map
$$\mathrm{gr}^m G \times \mathrm{gr}^n G \longrightarrow \mathrm{gr}^{m+n}$$
- the Hall-Witt identity
$$((x, y), z^x)\,((z, x), y^z)\,((y, z), x^y) = 1$$
implies Jacobi.

□

Warning 1.4. *The map*

$$\mathrm{gr}\, G \longrightarrow \mathrm{gr}_{\Phi} G$$

is neither injective or surjective in general, although it's obviously surjective in degree 1.

Theorem 1.5 (Magnus). *The associated graded ot the free group on n generators is the free Lie algebra on n generators. In particular, if $\pi = \pi_1(S_{g,1})$ and if a_i, b_i is a symplectic basis of $H_1(S_{g,1})$, then $\operatorname{gr}\pi$ is the free Lie algebra on a_i, b_i.*

The associated graded of $\pi_1(S_g)$ is the quotient of the former by the relation

$$\sum [a_i, b_i] = 0.$$

2. Johnson homomorphisms

Let A be a subgroup of $\operatorname{Aut}(G)$. Since Γ^m is characteristic, there is a morphism

$$A \longrightarrow \operatorname{Aut}(G/\Gamma^m).$$

Definition 2.1. *The Johnson filtration is defined by:*

$$J^m := \ker(A \longrightarrow \operatorname{Aut}(G/\Gamma^{m+1})).$$

The Torelli group of A is $T_A := J^1$, a normal subgroup of A. The symmetry group of T_A is $A_0 = A/T_A$.

Proposition 2.2 (Kaloujnine). *J is an N series on T_A.*

For a graded Lie algebra $\mathfrak{g}$, let $\operatorname{Der}^+(\mathfrak{g})$ be the Lie algebra of positive derivations

$$\operatorname{Der}^+(\mathfrak{g}) := \bigoplus_{m \geq 1} \operatorname{Der}^m(\mathfrak{g})$$

where $\operatorname{Der}^m(\mathfrak{g})$ is the space of derivations of $\mathfrak{g}$ which maps $\mathfrak{g}^n$ to $\mathfrak{g}^{n+m}$. Note this Lie algebra is itself graded. The following is an infinitesimal analog of the action of T_A on G:

Theorem 2.3 (Johnson, Papadima). *There is a well-defined,* injective *map of graded Lie algebra*

$$\tau : \operatorname{gr}_J(T_A) \hookrightarrow \operatorname{Der}^+(\operatorname{gr} G)$$

called the Johson homomorphism, defined as follow: let $a \in J^m$, $x \in \Gamma^n$, then

$$\bar{a} \cdot \bar{x} := \overline{a(x)x^{-1}}.$$

Sketch of proof. Let $a \in J^m$, $x \in \Gamma^n$. First we claim that

$$a(x) \equiv x \mod \Gamma^{m+n}.$$

For $n = 1$ this is the definition, for $n \geq 1$ this is proved by induction. Therefore,

$$a(x)x^{-1} \in \Gamma^{m+n}.$$

Composing with the quotient map we get a map

$$\Gamma^n \longrightarrow \operatorname{gr}^{m+n} G.$$

For x, y in Γ^n, a direct computation shows that

$$a(xy)(xy)^{-1} \equiv (a(x)x^{-1})(a(y)y^{-1}) \mod \Gamma^{m+n+1}$$

hence this descends to an additive map

$$\operatorname{gr}^n G \longrightarrow \operatorname{gr}^{m+n} G.$$

By definition this map is the identity iff

$$\forall x \in \Gamma^n, a(x)x^{-1} \in \Gamma^{m+n+1}.$$

For $n = 1$ this says precisely that $a \in J^{m+1}$, and conversly every map in J^{m+1} satisfies this for all n. Hence this map is injective. The fact that a is a derivation, and that this map is a Lie algebra map follows from painful commutator computations. □

Remark 2.4. In a way the Johnson filtration is tailor made to make this map injective (it generally isn't for the lower central series).

The action of A on T_A by conjugation descends, essentially by construction, to an action of A_0 on $\mathrm{gr}_J(T_A)$ by graded Lie algebra automorphisms: for $a \in A$, $x \in T_A$,

$$\bar{a} \cdot \bar{x} := \overline{axa^{-1}}.$$

Likewise, it acts on $\mathrm{gr}\, G$ by

$$\bar{a} \cdot \bar{x} := \overline{a(x)},$$

hence on $\mathrm{Der}(\mathrm{gr}\, G)$ by the adjoint action:

$$\bar{a} \cdot d := \bar{a} \circ d \circ \bar{a}^{-1}$$

where $\circ$ is composition of endomorphisms.

Proposition 2.5. *The Johnson homomorphism is A_0-equivariant.*

3. Application to the actual Torelli groups

Let $S = S_{g,1}$ with a point $\star$ marked on the boundary and let $\pi = \pi_1(S, \star)$. Let a_i, b_i be a symplectic basis of $H = H_1(S)$ and let $\omega = \sum a_i \wedge b_i$ the bivector associated with the symplectic form. Let $\zeta = [\partial S] \in \pi$ and recall the following classical

Theorem 3.1 (Dehn)**.** *The natural map*

$$\mathrm{Mod}(S) \longrightarrow \mathrm{Aut}(\pi)$$

is injective, and its image is the subgroup of automorphisms which fix ζ.

Identifying $\mathrm{Mod}(S)$ with its image, the associated Torelli group in the sense of the previous section is the usual Torelli group T, and the symmetry group A_0 is $\mathrm{Sp}(H)$.

Remark 3.2. One striking illustration of how useful it is to “linearize” T in this way is that, as a consequence of a highly non-trivial theorem of Hain, we know that the Lie algebra $\mathbb{Q} \otimes \mathrm{gr}\, T$ is finitely presented (with an explicit presentation) for $g \geq 6$.

Remark 3.3. Since $\mathrm{Der}^+(\mathrm{gr}\, \pi)$ is torsion-free, and since $\mathrm{gr}_J T$ embeds into it, it is torsion free as well, which means that

$$J^m = J^m_{\mathbb{Q}}$$

and that

$$\Gamma^m_{\mathbb{Q}}(T) \subset J^m.$$

On the other hand, it is known that the abelianization of T has torsion so the map

$$\mathrm{gr}\, T \longrightarrow \mathrm{gr}_J T$$

is already not injective in degree 1, i.e.

$$\Gamma^2 \subsetneq J^2.$$

Johnson’s theorem below implies however that $\Gamma^2_{\mathbb{Q}}(T) = J^2$. One might hope this is true for $m \geq 3$, but it’s not: Hain has shown that the kernel of

$$\Gamma^2_{\mathbb{Q}}(T)/\Gamma^3_{\mathbb{Q}}(T) \longrightarrow J^2/J^3$$

is isomorphic to $\mathbb{Z}$.

Remark 3.4. It’s well known that for finitely generated free groups, one has

$$\bigcap_{m \geq 1} \Gamma^m_{\mathbb{Q}} = \{1\}$$

i.e. those are residually torsion-free-nilpotent. It implies at once that in the Torelli group

$$\bigcap_{m \geq 1} J^m = \{1\}$$

Since $\Gamma^m_{\mathbb{Q}}(T) \subset J^m$, T is itself residually-torsion-free nilpotent. This has cool consequences: it is in particular torsion free and residually nilpotent (but this is much stronger), residually p for all p, residually finite and bi-orderable.

Remark 3.5. One can check that the Johnson homomorphism in that case actually lands in the Lie algebra of symplectic derivations, i.e. those mapping

$$\omega = \sum a_i \wedge b_i \in H \subset \operatorname{gr} \pi$$

to 0. This is the infinitesimal counterpart of the fact that the mapping class group action on π preserves ζ.

Recall that T is generated by bounding pairs, i.e. elements of the form $T_\alpha T_\beta^{-1}$ where T_α is the Dehn twist along α and α, β are disjoint non-separating simple closed curves such that $[\alpha] = [\beta] \neq 0$. Recall also that if γ is a bounding simple closed loop then $T_\gamma \in T$.

Theorem 3.6 (Johnson)**.** *The images of the elements T_γ, γ a bounding simple closed curve, in $H_1(T)$ are 2-torsion, hence their image through τ_1 is 0. In fact they generate the kernel of the lift*

$$T \longrightarrow \operatorname{Der}^1(\operatorname{gr} \pi).$$

Theorem 3.7 (Johnson)**.** *Let S' be the component of $S \backslash (\alpha \cup \beta)$ which doesn't contain the base point. Let k be the genus of S' and let $\{a_i, b_i\}$ be a symplectic basis of H such that $[\alpha] = [a_{k+1}]$, and such that $\{a_1, b_1, \ldots, a_k, b_k, a_{k+1}\}$ is a basis of $H_1(S')$. Then*

$$\tau_1(f) = (\sum_{i=1}^{k} a_i \wedge b_i) \wedge a_{k+1}.$$

Since the Lie bracket is antisymmetric, and since there is no Jacobi relation in degree 2, the free $\mathbb{Z}$-module $\operatorname{gr}_2 \pi$ can be identified with $\wedge^2 H$. Therefore any linear map $H \to \wedge^2 H$ extends uniquely to a degree 1 derivation of $\operatorname{gr} \pi$, and using the symplectic form on H to identify $H \cong H^*$, we get an $\operatorname{Sp}(H)$-equivariant embedding

$$\wedge^3 H \hookrightarrow H \otimes \wedge^2 H \cong H^* \otimes \wedge^2 H \cong \operatorname{Der}^1(\operatorname{gr} \pi).$$

Theorem 3.8 (Johnson)**.** *The first Johnson homomorphism*

$$\tau_1 : H_1(T) \longrightarrow \operatorname{Der}^1(\operatorname{gr} \pi)$$

lands in $\wedge^3 H$, and induces an isomorphism

$$H_1(T, \mathbb{Q}) \cong \wedge^3 H \otimes \mathbb{Q}.$$

It descends to an isomorphism

$$H_1(T_g, \mathbb{Q}) \cong (\wedge^3 H \otimes \mathbb{Q}) / \langle h \wedge \omega, h \in H \rangle$$

where T_g is the Torelli group of the closed surface S_g and π_g its fundamental group.

4. Geometric Johnson homomorphisms

4.1. **Abel–Jacobi map.** Let $\mathbb{T}$ be the complex torus $\mathbb{C}^g / \mathbb{Z}^{2g}$. Since this is a $K(\mathbb{Z}^{2g}, 1)$, the abelinanization map $\pi \to \mathbb{Z}^{2g}$ determines a unique homotopy class of maps

$$S \longrightarrow \mathbb{T}.$$

The Abel–Jacobi map can be thought of as a way of picking representatives in that homotopy class in a way that interacts well with the action of T, by using complex structures on S. Fix once and for all a surface $\bar{S}$ obtained by gluing a disc to the boundary of S and fix a marked point inside that disc with a unit tangent vector at it. By a complex structure on S we'll mean a pair of a marked compact Riemann surface C and of a diffeomorphism $h : C \xrightarrow{\sim} \bar{S}$ which preserves the basepoint and its tangent vector. Two complex structures (C, h) and (C', h') on S are isotopic if $h^{-1} \circ h'$ is isotopic (rel. basepoint and tangent vector) to a holomorphic diffeomorphism.

Recall that the cotangent bundle of C has a canonical holomorphic structure, called the canonical line bundle K, so that global sections $H^0(C, K)$ are identified with holomorphic one forms on C. It is well-known that this space is isomorphic as a real vector space to $H^1(\bar{S}, \mathbb{R})$.

If α is such a form, and γ a path on C, define

$$\int_\gamma \alpha := \int_0^1 \gamma^*\alpha \in \mathbb{C}.$$

Integration of forms gives a non-degenerate pairing

$$H_1(C,\mathbb{C}) \times H^0(C,K) \longrightarrow \mathbb{C},$$

hence an embedding $H_1(C) \subset H^0(C,K)^*$.

Definition 4.1. *The Jacobian of C is $J(C) := H^0(C,K)^*/H_1(C)$. The choice of a symplectic basis of H_1 induces an identification $J(C) \cong \mathbb{T}$. The Abel–Jacobi map*

$$J : C \longrightarrow J(C)$$

is defined for $y \in C$ by picking a path γ from the marked point to y, and mapping y to

$$\alpha \mapsto \int_\gamma \alpha.$$

Note that if γ' is another path to y, then $\int_{\gamma^{-1}\gamma'} \alpha$ is 0 in $J(C)$, hence this is well-defined. It's also clear that this map induces the abelianization of π.

4.2. **Bundles over the Torelli space.** Let Teich be the Teichmüller space, whose points are sotopy classes of complex structures on S as above. An important fact about

Theorem 4.2. *The space* Teich *is homeomorphic to $\mathbb{R}^{6g-3}$ and carries a free action of the Torelli group T.*

Definition 4.3. *The Torelli space $\mathcal{T}$ is the quotient* Teich $/T$.

A point in $\mathcal{T}$ is thus a pair (a diffeomorphism class of complex structures on S, a symplectic basis of $H_1(S)$). Note it carries a residual action of Sp by changing the basis. It follows from the theorem that $\mathcal{T}$ is a $K(T,1)$, hence we have:

$$H_*(\mathcal{T}) \cong H_*(T).$$

Let $\mathcal{T}^* = (S \times \text{Teich})/T$ and let

$$p : \mathcal{T}^* \longrightarrow \mathcal{T}$$

be the projection. This is the homotopy quotient of S by T, i.e. it is the universal fiber bundle over $\mathcal{T}$ with fiber over any point identified with S. Let now $\mathcal{J}^*$ be the trivial bundle

$$(\mathbb{T} \times \text{Teich})/T \longrightarrow \mathcal{T}$$

where T acts on $\mathbb{T}$ trivially (hence this is indeed a trivial bundle). Fixing once and for all a symplectic basis of $H_1(S)$, for any $(C,h) \in$ Teich, using h this canonically fixes an identification $J(C) \simeq \mathbb{T}$. Therefore the Abel-Jacobi maps assemble into a map

$$S \times \text{Teich} \longrightarrow \mathbb{T} \times \text{Teich}$$

which is clearly T-equivariant, since by construction the action of T preserves any choice of a symplectic basis of $H_1(S)$. In other words, we get an Sp-equivariant bundle map

$$\mathcal{T}^* \longrightarrow \mathcal{J}^*.$$

Composing this map with the projection $\mathcal{J}^* \to \mathbb{T}$ we get an Sp-equivariant diagram

$$\mathcal{T} \longleftarrow \mathcal{T}^* \longrightarrow \mathbb{T}.$$

Definition 4.4. *Let $f : M \longrightarrow N$ be a smooth map between compact oriented manifolds of dimensions $m+k$ and m respectively. The pull-back along f in homology is the composition*

$$f^* : H_i(N) \xrightarrow{\text{Poincaré duality}} H^{m-i}(N) \xrightarrow{\text{pull-back along } f} H^{m-i}(M) \xrightarrow{\text{Poincaré duality}} H_{i+k}(M).$$

Definition 4.5. *Let $n = 6g - 3$ and $1 \leq i \leq n$. The ith geometric Johnson homomorphism is the composition*

$$\tau_i' : H_i(\mathcal{T}, \mathbb{Q}) \xrightarrow{p^*} H_{i+2}(\mathcal{T}^*, \mathbb{Q}) \longrightarrow H_{i+2}(\mathbb{T}, \mathbb{Q}) \cong \bigwedge^{i+2} H_{\mathbb{Q}}.$$

Theorem 4.6 (Johnson, Church–Farb)**.** $\tau_1' = \tau_1$.

4.3. **Mapping tori and τ_1'.** There is a fun way to compute $\tau_1 = \tau_1'$ this way which bypass the use of pull-back in homology. Let $\sigma \in H_i(\mathcal{T})$. Suppose we are given a map $B \longrightarrow \mathcal{T}$ and a class $x \in H_i(B)$ whose image in $H_i(T$ is σ.

Let f be a diffeomorphism of S lifting an element of T and fix a basepoint $x \in \mathcal{T}$. By definition f induces a loop in $\mathcal{T}$, i.e. a map $\gamma : S^1 \longrightarrow \mathcal{T}$. We can then form the pull-back bundle, whose total space is

$$M_f = \{(x,t) \in \mathcal{T}^* \times S^1 \mid p(x) = \gamma(t)\}.$$

Note that this is nothing but the mapping torus of f. Since f acts trivially on $H_1(S)$, there is a canonical decomposition

to be continued..

$$H_1(M_f) = H_1(S) \times H_1(S^1).$$

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