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Structure des espaces de configuration

Mémoire d'Habilitation à Diriger des Recherches

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Jury

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1 Introduction

This manuscript is part of the author's *Habilitation à Diriger des Recherches* (HDR). Like in most HDR memoirs, our goal is to give a synthetic, contextualized presentation of the author's research work over the last few years. We hope that the reader will not be disappointed by the lack of new results in this manuscript.

General context Before delving into the concrete mathematical results, let us first explain the general context. Roughly speaking, the research presented in this memoir is within the field of algebraic topology. The objects we will study are topological in nature, such as manifolds (locally Euclidean spaces) and spaces constructed out of manifolds. We are most interested in properties of spaces that are invariant under a notion of "deformation" known as homotopy. The tools we use, on the other hand, are algebraic; they include the well-known notions of groups, rings, or their representations, as well as notions that would deserve to be better known such as operads and their representations. Very often, homotopy invariant features of topological spaces are reflected in the algebraic objects; or sometimes, equivalent to properties of the algebraic objects – the dream of any algebraic topologist. Algebraic topology lies at the meeting point of many fields of mathematics, some of which will also appear (perhaps briefly) in this manuscript: homological algebra, category theory, mathematical physics.

Most of the research presented here stems from a desire to understand configuration spaces, related topological objects, and especially the algebraic structure of these objects. Configuration spaces are ubiquitous in algebraic topology. They have a very simple definition: collections (ordered or not) of pairwise distinct points in a given ambient space. This simple definition hides a complexity that becomes evident once one starts to try and compute algebraic invariants of these spaces. For example, the fundamental group of the unordered configuration spaces of n points in the plane is isomorphic to the nth braid group, the study of which being a deep topic by itself.

As often in algebraic topology, it becomes easier to study configuration spaces if they are viewed through the lens of a big machinery (a term used positively), instead of studying them one by one. In this case, the machinery is that of the theory of operads. Indeed, configuration spaces of manifolds are intimately related to a family of operads called the little disks operads. These operads were initially introduced in the sixties to study iterated loop spaces; they quickly grew beyond this use case and have found many uses since then. These operads, and their cousins, act on configuration spaces up to homotopy. This action is at the heart of many deep results, and we also use it heavily in many of our proofs.

Studying this machinery quickly leads down several deep rabbit holes. One such hole of particular interest is Koszul duality. Initially introduced for rings and algebras, Koszul duality has been generalized with great success to operads about thirty years ago. Nowadays, Koszul duality of operads serves many purposes; one of them is to produce resolutions (in the sense of homological algebra). Resolutions are useful to compute homotopy invariant constructions, which circles back to our initial goal of studying homotopy invariants of spaces reflected by algebras.

Outline The results presented in this memoir can be, roughly speaking, divided into three categories, which correspond to the three subsequent chapters. Note that many of the results mentioned in the remainder of this section are part of joint research works with other authors (see Section 1.1 or the precise references given for each theorem in the next chapters).

The first chapter is about configuration spaces themselves. We discuss the homotopy invariance conjecture, and the results we have obtained in this direction in characteristic zero. We moreover explain how the real homotopy type of configuration spaces of simply connected manifolds is completely encoded by a finite amount of data, the Lambrechts–Stanley CDGA. We show how this CDGA can be used to perform concrete, algorithmic computations, such as the search for nontrivial Massey products.

The second chapter is about operads. After explaining what operads are and why one might take an interest in their study, we discuss how the results of the previous chapter interact with operadic structures. We take this opportunity to delve deeper into the proof of one of the main results of the first chapter. This proof, which uses graph complexes, could be written without mentioning operads at all; but operads are very much present in the intuition behind them. Finally, we discuss some results of purely operadic nature regarding the Swiss-Cheese operads and their variants.

The third and last chapter is about resolutions. Computing algebraic invariants of spaces or manifolds often requires one to find a resolution (e.g., projective or injective) of some object. This is for example the case of factorization homology, which can be described as a derived tensor product. We discuss some of the results we obtained in that direct, namely, in the study of curved Koszul duality for algebras over operads, and in the study of bar, cobar, and W constructions. We finish with some concrete computations using some of the resolutions and algebraic models of configuration spaces and other applications.

What is not here In addition to a review of past work, each chapter also includes a few conjectures or questions for future research. Something missing from this memoir, however, is the author's current long-term project. Indeed, it has not yet led to publishable results, and expanding on future research is not necessarily the goal of an HDR memoir. Let us say a few brief words about it (that should probably be read after all three chapters to make sense).

Many homotopy invariants of unordered configuration spaces exhibit a remarkable property: they are eventually constant as the number of points grows compared to the invariant's dimension [125, 150]. Ordered configuration spaces are more intricate: one must take into account the action of symmetric groups, and the eventual constancy condition needs an adaptation in terms of group representations [39]. This property was reinterpreted algebraically [40] and stems from properties of (strong) polynomial functors [50, 51, 139]. These insights allowed the generalization of "representation stability" to other algebraic settings involving e.g., braids [155].

Recent works [128, 90] on secondary stability, which involve homotopy invariants of different dimensions, hints at the fact that operads must play a role in this story. (Operads are explicitly used in the initial discovery of secondary stability [76].) In particular, operadic right modules (which are starkly different from operadic left modules) are intimately connected to polynomial functors (see Def. 3.1.39) and should play a central role in the theory.

The main goals of the research project would be to reinterpret primary and secondary stability using operadic right modules, obtain "higher" kinds of stability, and establish relations between the various stabilization maps. This does raise a number of questions, though, such as whether the Noetherian property [148] of the category of operadic right modules holds. Our hope is that since we have access to more structure using right modules, we have smaller presentation, and thus, smaller amount of data to deal with. Thanks to the explicit, combinatorial models of configuration spaces and their operadic structure obtained, applying concrete techniques from e.g., Koszul duality theory (or perhaps some generalizations from the world of rewriting methods) appears doable. As an objective, an efficient computation of factorization homology of non-free algebras could hopefully give rise to stability results for homology with twisted coefficients (a difficult question, see e.g. recent results of [126, 127]).

1.1 Works presented in this document

This list contains the articles presented in this memoir. Some older articles are not included; for a complete list, see my website, arXiv:idrissi_n_1 or HAL:najib-idrissi.

[1] R. Campos, J. Ducoulombier, and N. Idrissi. "Boardman–Vogt resolutions and bar/cobar constructions of (co)operadic (co)bimodules". In: *High. Struct.* 5.1 (2021), pp. 293–366. DOI: 10.21136/HS.2021.09. arXiv: 1911.09474.

[2] R. Campos, J. Ducoulombier, N. Idrissi, and T. Willwacher. *A model for framed configuration spaces of points*. Version 2. 2018. arXiv: 1807.08319. Pre-published.

[3] R. Campos, N. Idrissi, P. Lambrechts, and T. Willwacher. *Configuration Spaces of Manifolds with Boundary*. Astérisque 449. Soc. Math. Fr., 2024. ISBN: 978-2-85629-990-6. DOI: 10.24033/ast.1222. arXiv: 1802.00716.

[4] R. Campos, N. Idrissi, and T. Willwacher. *Configuration Spaces of Surfaces*. Version 2. 2019. arXiv: 1911.12281. Pre-published.

[5] N. Idrissi. "The Lambrechts–Stanley Model of Configuration Spaces". In: *Invent. Math.* 216.1 (2019), pp. 1–68. DOI: 10.1007/s00222-018-0842-9. arXiv: 1608.08054.

[6] N. Idrissi. "Formality of a higher-codimensional Swiss-Cheese operad". In: *Algebr. Geom. Topol.* 22.1 (2022), pp. 55–111. DOI: 10.2140/agt.2022.22.55. arXiv: 1809.07667.

[7] N. Idrissi. "Curved Koszul duality of algebras over unital versions of binary operads". In: *J. Pure Appl. Algebra* 227.3 (2023). DOI: 10.1016/j.jpaa.2022.107208. arXiv: 1805.01853.

[8] N. Idrissi and E. Rabinovich. "Homotopy Prefactorization Algebras". In: *Res. Math. Sci.* 11.45 (2024). DOI: 10.1007/s40687-024-00456-9. arXiv: 2304.13011.

[9] N. Idrissi and R. V. Vieira. "Non-formality of Voronov's Swiss-Cheese operads". In: *Q. J. Math.* 75.1 (2024), pp. 63–95. DOI: 10.1093/qmath/haad041. arXiv: 2303.16979.

1.2 Conventions

Unless otherwise specified, we work over the base field $\mathbb{Q}$ and (co)homology of spaces is taken with rational coefficients.

Differential-graded vector spaces (or "dg-modules") are graded cohomologically, i.e., we write them as families of the form $V = (V^i)_{i\in\mathbb{Z}}$; differentials have degree $+1$. The homology of a space X is concentrated in non-positive degrees while the cohomology is in nonnegative degrees. The degree shift of a graded module is written as $(V[k])^i := V^{k+i}$.

A commutative differential graded algebra (CDGA) is a dg-module equipped with a graded-commutative associative product. For a dg-module V, the free CDGA on V is denoted $S(V)$. It is isomorphic to the tensor product of the polynomial algebra on the even part of V with the exterior algebra on the odd part of V.

Given objects X, Y (topological spaces, dg-modules, CDGAs...), we write $X \cong Y$ if they are isomorphic, and $X \simeq Y$ if they are weakly equivalent, i.e., connected by a zigzag of weak equivalences.

2 Configuration Spaces

We now introduce the main objects considered in this memoir: configuration spaces of manifolds. We define them, we explain some of their uses, and we describe the problem of homotopy invariance. We then describe the various results obtained towards the conjecture of rational homotopy invariance as well as concrete computations of homotopy types, including a computer computation of Massey products.

2.1 Definition

Configuration spaces are classical objects of algebraic topology whose study dates back to the 1960s [57, 63]. The simplicity of their definition, which we give next, hides their rich structure and their many applications in topology and geometry.

Definition 2.1.1. Given a topological space M and an integer $r \geq 0$, the rth (ordered) configuration space of M is defined by:

$$\mathrm{Conf}_M(r) \coloneqq \{(x_1, \dots, x_r) \in M^r \mid \forall i \neq j,\, x_i \neq x_j\}. \tag{2.1.2}$$

In other words, an element of $\mathrm{Conf}_M(r)$ is an ordered collection of r pairwise distinct points in M. As the notation indicates, we are most often be interested in the case where M is a manifold, although the definition makes sense for any topological space.

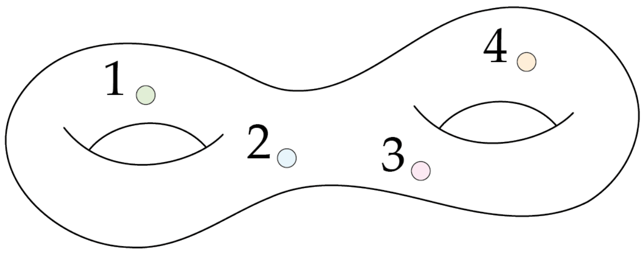


Figure 2.1: An element of the 4th configuration space of an oriented surface of genus 2.

There exists numerous variations on this definition. A common one is to consider unordered configuration spaces $\mathrm{UConf}_M(r)$, which are defined as the quotient of $\mathrm{Conf}_M(r)$ by the action of the symmetric group Σ_r on M^r by permutation of indices. Another is to consider the space of "non-k-equal" configuration space

$$\mathrm{Conf}_M^{<k}(r) \coloneqq M^r \setminus \{(x_1, \dots, x_r) \in M^r \mid \exists 1 \leq i_1 < \dots < i_k \leq r,\, x_{i_1} = \dots = x_{i_k}\}. \tag{2.1.3}$$

The study of configuration spaces finds its roots in the work of Hurwitz [95] on the topology of Riemann surfaces (see Magnus [121]) and rose to prominence in the study of braid groups in the work of Artin [14]. The connection between configuration spaces and braid groups is made explicit by the following fact:

Theorem 2.1.4 (Artin [14])**.** *The fundamental group of* $\mathrm{UConf}_{\mathbb{R}^2}(r)$ *is isomorphic to the* r*-strand braid group* B_r*.*

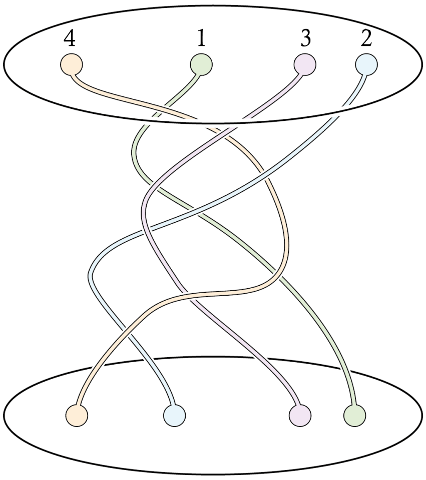


Figure 2.2: An element of the 4th braid group B_4.

This hints at the notion that the computation of homotopy invariants of configuration spaces is a difficult but worthwhile endeavour.

Remark 2.1.5. There is a morphism of groups $\pi \colon \mathrm{B}_r \to \Sigma_r$ which records the permutation of the strands by a braid. For example, the braid of Fig. 2.2 goes to the cycle (132). The fundamental group of $\mathrm{Conf}_{\mathbb{R}^2}(r)$ is isomorphic to the pure braid group $\mathrm{PB}_r \coloneqq \ker(\pi)$.

2.2 Homotopy invariance

The question of homotopy invariance of configuration spaces is a natural one that can be stated as follows: given two manifolds M and N, if M and N are homotopy equivalent, is it then true that $\mathrm{Conf}_M(r)$ and $\mathrm{Conf}_N(r)$ are homotopy equivalent? A positive answer to that question would allow one to use standard techniques from algebraic topology to compute homotopy invariants of configuration spaces, e.g., reduce the manifold to a simpler one. The question is not an obvious one: a homotopy equivalence $f \colon M \leftrightarrows N \colon g$ is rarely injective in both directions, and even if it is, the homotopy between $f \circ g$ and $g \circ f$ and the relevant identity maps are rarely isotopies.

Stated naively, the answer to this question is evidently no. For example, the real line $\mathbb{R}$ is homotopy equivalent to the singleton $\{0\}$, but while $\mathrm{Conf}_{\{0\}}(r)$ is empty for $r \geq 2$, the space $\mathrm{Conf}_{\mathbb{R}}(r)$ is never empty. Up to homotopy, $\mathrm{Conf}_{\mathbb{R}}(r)$ is discrete with $r!$ connected components; but $\mathbb{R}$ is homotopy equivalent to $\mathbb{R}^2$, and $\mathrm{Conf}_{\mathbb{R}^2}(r)$ is the classifying space of the (pure) braid group on r strands.

A first refinement of the question is to restrict our attention to closed manifolds. Since two homotopy equivalent closed manifolds have the same dimension, the failure observed for $\{0\} \simeq \mathbb{R} \simeq \mathbb{R}^2 \simeq \dots$ is avoided. Evidence of a positive answer for this refined question abounds: if two closed manifolds M and N are homotopy equivalent, then for all $r \geq 0$, the spaces $\mathrm{Conf}_M(r)$ have the same homology groups, the same homotopy groups, the same loop spaces, and the same stable homotopy types [32, 22, 113, 10]. Variations of the question were also shown to have a positive answer for some classes of manifolds, such as smooth projective complex varieties [108, 160] (in characteristic zero, i.e., for rational homotopy types).

Longoni–Salvatore [119], however, proved that even this refined question also has a negative answer. More precisely, they proved that the configuration spaces of the lens spaces $L_{7,1}$ and $L_{7,2}$ (which are both quotients of S^3 under two different actions of $\mathbb{Z}/7\mathbb{Z}$ and which are homotopy equivalent) have different homotopy types using the theory of Massey products. This discovery led to further refinement of the question:

Conjecture 2.2.1. *If two simply connected closed manifolds M and N have the same homotopy type, then so do $\mathrm{Conf}_M(r)$ and $\mathrm{Conf}_N(r)$ for all $r \geq 0$.*

This restrictions eliminates the counterexample of Longoni–Salvatore. There is good reason to believe that it is true: when manifolds are simply connected, homotopy equivalence is the same as simple-homotopy equivalence (in the sense of [44]). This could lead to the use of more geometrical methods. Yet, to this day, this conjecture remains open.

As is often the case in algebraic topology, this conjecture has been also restricted to slices of the homotopy types involved. In this case, a great deal of research has been performed about the *rational* homotopy types of configuration spaces. Rational homotopy, pioneered by Quillen [138] and Sullivan [157] (and based on earlier insights of Serre [151]), is concerned with the homotopy invariants of a space that are defined in characteristic zero. More precisely, rational homotopy equivalence (between simply connected spaces) $f\colon M \to N$ is a continuous map which induces isomorphisms $f_*\colon \pi_*(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(N) \otimes_{\mathbb{Z}} \mathbb{Q}$. Two spaces are rationally equivalent if they can be connected by a zigzag of rational equivalences. The following conjecture is also still open:

Conjecture 2.2.2 (Félix–Halperin–Thomas [59, §39 Problem 8])**.** *If two simply connected closed manifolds M and N have the same rational homotopy type, then so do $\mathrm{Conf}_M(r)$ and $\mathrm{Conf}_N(r)$ for all $r \geq 0$.*

Note that while a positive answer to this conjecture would give a weaker understanding of configuration spaces than their full homotopy types, the starting data is also weaker (as M and N are merely assumed to be rationally equivalent). This conjecture is thus, overall, neither stronger nor weaker than the previous one.

A slight refinement of this conjecture consists in considering *real* homotopy types, rather than rational homotopy types. While real homotopy types contain slightly less information than their rational counterparts, they are sufficient for almost every computation (including Betti numbers, algebra structure on cohomology, ranks of homotopy groups, Whitehead brackets). We settled this slightly refined conjecture, simultaneously as Campos–Willwacher [34].

Theorem 2.2.3 ([5], Campos–Willwacher [34])**.** *Let M and N be two simply connected, smooth, closed manifolds. If M and N have the same real homotopy types, then so do $\mathrm{Conf}_M(r)$ and $\mathrm{Conf}_N(r)$ for all $r \geq 0$.*

Remark 2.2.4. The proof of the above theorem is completely different depending on whether $\dim(M) \geq 4$ or $\dim(M) \leq 3$. If $\dim(M) \leq 3$, then there are only a few cases to consider up to diffeomorphism (thanks to the classification of surfaces and the Poincaré conjecture): the singleton $\{0\}$, the sphere S^2, and the sphere S^3. Homotopy invariance then obviously holds.

The question of homotopy invariance can also be asked about compact manifolds with boundary. To avoid aberrations such as $[0,1] \simeq \{0\}$, the question must be phrased as follows: given compact manifolds with boundary M and N, if the pair $(M, \partial M)$ is weakly homotopy equivalent to the pair $(N, \partial N)$, is it then true that $\mathrm{Conf}_M(r) \simeq \mathrm{Conf}_N(r)$ for all $r \geq 0$? We gave a partial answer with Campos, Lambrechts, and Willwacher :

Theorem 2.2.5 ([3])**.** *If M is a simply connected compact manifold of dimension ≥ 4, then the real homotopy type of $\mathrm{Conf}_M(r)$ only depends on the real homotopy type of the map $\partial M \to M$.*

Remark 2.2.6. These results have since then been improved by Willwacher [172] to deal with rational homotopy types rather than real homotopy types, with the restriction that manifolds need to be framed.

2.3 Real homotopy theory

Homotopy invariance, as presented above, is good enough to tell us that if we want to know about the configuration spaces of a given manifold M, we can apply algebro-topological techniques to simplify the manifold M up to homotopy, and then compute the configuration spaces of the simplified manifold. However, by itself, it does not give us any way to answer specific questions about $\mathrm{Conf}_M(r)$, such as “what is the rank of the fourth cohomology group,” or “what is the cup length of the ring $H^*(\mathrm{Conf}_M(r))$.” To answer this kind of question, we need much more concrete information.

As we mentioned above, we are mainly concerned with rational homotopy types of manifolds. According to the theory of Sullivan [157], the rational homotopy type of a given (simply connected) space X is fully encoded by a certain quasi-isomorphism class of commutative differential-graded algebras (CDGAs), called “(rational) models.” The notion of model uses the CDGA of piecewise polynomial forms (PL forms) $\Omega^*_{\mathrm{PL}}(\_)$, which is defined (over $\mathbb{Q}$) using a mix of singular cohomology and differential forms on standard simplices. We refer to [59, 61, 60] for references on the matter.

Definition 2.3.1. A model of X is a CDGA $A = (A, d)$ which is quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(X)$, i.e., there exists a zigzag of quasi-isomorphisms of CDGAs over $\mathbb{Q}$:

$$A \xleftarrow{\sim} \dots \xrightarrow{\sim} \Omega^*_{\mathrm{PL}}(X). \tag{2.3.2}$$

Two spaces have the same rational homotopy type if and only if they have quasi-isomorphic models. Indeed, a continuous map $f \colon X \to Y$ between topological spaces induces a morphism of CDGAs $f^* \colon \Omega^*_{\mathrm{PL}}(Y) \to \Omega^*_{\mathrm{PL}}(X)$. If f is a rational homotopy equivalence, then f^* is a quasi-isomorphism. The power of rational homotopy theory comes from the converse: any quasi-isomorphism between $\Omega^*_{\mathrm{PL}}(X)$ and $\Omega^*_{\mathrm{PL}}(Y)$ comes from a homotopy class of rational homotopy equivalences between X and Y. This leads to the following theorem:

Theorem 2.3.3 (Sullivan [157])**.** *Let X, Y be two simply connected spaces with finite dimensional cohomology in each degree. The CDGAs $\Omega^*_{\mathrm{PL}}(X)$ and $\Omega^*_{\mathrm{PL}}(Y)$ are quasi-isomorphic if and only if X and Y have the same rational homotopy type.*

Remark 2.3.4. Using models, we can also state more clearly the difference between real and rational homotopy types. Two spaces X, Y are said to have the same real homotopy type if $\Omega^*_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$ and $\Omega^*_{\mathrm{PL}}(Y) \otimes_{\mathbb{Q}} \mathbb{R}$ can be connected through a zigzag of CDGAs with real coefficients. Note that unlike in the rational case, such a zigzag does not necessarily come from a homotopy class of continuous maps.

Remark 2.3.5. For a smooth manifold M, we have that $\Omega^*_{\mathrm{PL}}(M) \otimes_{\mathbb{Q}} \mathbb{R}$ is quasi-isomorphic to $\Omega^*_{\mathrm{dR}}(M)$, the CDGA of de Rham forms on X.

Remark 2.3.6. In the proofs of results in the sequel, we actually need to use piecewise *semi-algebraic* (PA) forms Ω^*_{PA}. These forms, initially introduced by Kontsevich–Soibelman [107] and developed by Hardt–Lambrechts–Turchin–Volić [89], are defined for semi-algebraic sets, and give real models for compact such sets. While the definition of Ω^*_{PA} is immensely more involved than that of Ω^*_{PL}, the two functors share many properties. Moreover, if X is a compact semi-algebraic set, then $\Omega^*_{\mathrm{PA}}(X)$ is quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$. PA forms have the advantage that they enable the computation of integrals along fibers of semi-algebraic bundles of some classes of forms, an operation that satisfies a version of Stokes' formula.

If A is a model of X, then the cohomology of A is isomorphic to the rational cohomology of X as a graded ring, i.e., $H^*(A) \cong H^*(X)$. However, A contains much more information. For example, the rational duals of the homotopy groups of X can be recovered as the Harrison homology:

$$\mathrm{Hom}(\pi_k(X), \mathbb{Q}) \cong H^{\mathrm{Har}}_{k-1}(A). \tag{2.3.7}$$

One can also compute other pieces of information about X using A, such as the Massey products or the Whitehead brackets.

2.4 Models of configuration spaces

Let us now turn back to configuration spaces. Since all manifolds based on the building block $\mathbb{R}^n$, it is natural to first focus on $M = \mathbb{R}^n$. The cohomology of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ is well-known, and is given by the following theorem:

Theorem 2.4.1 (Arnold [11])**.** *Let $n \geq 2$ and $r \geq 1$ be integers. The cohomology of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ is given by the following graded commutative ring, where the generators ω_{ij} are of degree $n-1$:*

$$H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) = \frac{S(\omega_{ij};\ 1 \leq i, j \leq r)}{(\omega_{ji} - (-1)^n \omega_{ij},\ \omega_{ii},\ \omega_{ij}^2,\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij})}. \tag{2.4.2}$$

Remark 2.4.3. This computation actually holds over $\mathbb{Z}$, and the result is a free abelian group, so by the universal coefficients theorem, it holds over any ring.

The relations in the above theorem are called the *Arnold relations*. The proof of this theorem hinges on the existence of the Fadell–Neuwirth [57] fibration, which allows one to work by induction on r (and which happens to admit a section for $M = \mathbb{R}^n$):

$$M \setminus \{*\} \to \mathrm{Conf}_M(r) \to \mathrm{Conf}_M(r-1). \tag{2.4.4}$$

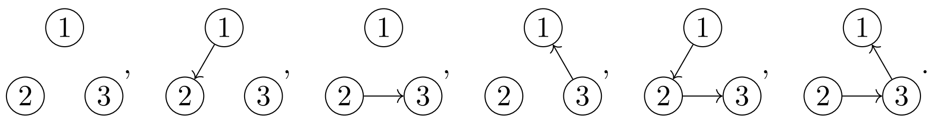
The cohomology of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ is a free CDGA on the generators ω_{ij}, modulo some relations. It is instructive to understand the elements of the cohomology ring in terms of *graphs*. Let us consider directed graphs with vertices $\{1, \dots, r\}$ endowed with a total order on the set of its edge, as well as formal linear combinations of such graphs. The degree of a graph is the number of edges multiplied by $(n-1)$. To such a graph, we can naturally associate a monomial in the generators ω_{ij}, by taking the product of the ω_{ij} corresponding to the edges of the graph, in the order given by the total order on the edges. For example, $\omega_{13}\omega_{32}$ is the image of the graph with two edges, one from 1 to 3 and one from 3 to 2. The relations in the cohomology ring of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ can be interpreted as follows:

- The relation $\omega_{ji} = (-1)^n \omega_{ij}$ corresponds to the fact that reversing the direction of an edge changes the sign of the corresponding monomial.
- The relation $\omega_{ii} = 0$ corresponds to the fact that a graph cannot have a loop.
- The relation $\omega_{ij}^2 = 0$ corresponds to the fact that a graph cannot have two edges between the same two vertices.
- The relation $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$, which is the most interesting one, corresponds to a certain local relation between graphs. If three graphs Γ_1, Γ_2, Γ_3 are identical except for the edges between the vertices i, j, k, such that Γ_1 contains the edge $\{i \to j, j \to k\}$, Γ_2 contains the edge $\{j \to k, k \to i\}$, and Γ_3 contains the edge $\{k \to i, i \to j\}$, then the monomials corresponding to these graphs sum to zero.

The vector space spanned by graphs modded out by this relation is isomorphic to $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$. The product of the algebra can also be described in terms of graphs: the product of two graphs is obtained by taking the disjoint union of edges (gluing vertices together).

Example 2.4.5. Let us consider n odd and $r = 3$. The classes of the following graphs form a basis of $H^*(\mathrm{Conf}_{\mathbb{R}^n}(3))$:

(2.4.6)



These graphs correspond respectively to the elements 1, ω_{12}, ω_{23}, ω_{31}, $\omega_{12}\omega_{23}$, and $\omega_{23}\omega_{31}$. Note that since n is assumed odd, the degree of ω_{ij} is even, so the order of the edges do not matter. However, the symmetry relation reads $\omega_{ji} = -\omega_{ij}$, so that we need to orient edges. If we had assumed n even instead, then the degree of ω_{ij} would be odd, and the order of the edges would matter; but the symmetry relation would read $\omega_{ji} = \omega_{ij}$, so that we would not need to orient edges.

Under the Arnold relations, every graph can be reduced to a linear combination of the above graphs. For example, the graph corresponding to $\omega_{31}\omega_{12}$ is equal to $-\omega_{12}\omega_{23} - \omega_{23}\omega_{31}$. Any graph with more than two edges vanishes because of the Arnold relation.

The above theorem allows us to fully understand the cohomology of $\mathrm{Conf}_{\mathbb{R}^n}(r)$. However, in principle, it is not enough to understand the full rational homotopy type of $\mathrm{Conf}_{\mathbb{R}^n}(r)$. It turns out that the configuration spaces of $\mathbb{R}^n$ satisfy a strong property, called *formality*, which precisely means that the rational homotopy type of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ is determined by its cohomology ring.

Definition 2.4.7. A space X is called *formal* if $(H^*(x), d = 0)$ is quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(X)$.

Remark 2.4.8. Formality over $\mathbb{Q}$ is equivalent to formality over any field of characteristic zero.

Theorem 2.4.9 (Arnold [11] for $n = 2$, Kontsevich [105] for all n)**.** *The configuration spaces* $\mathrm{Conf}_{\mathbb{R}^n}(r)$ *are* $\mathbb{Q}$*-formal for all* $n \geq 1$ *and* $r \geq 0$*.*

Arnold's proof for $n = 2$ is relatively direct. We can view $\mathbb{R}^2$ as the space of complex numbers $\mathbb{C}$. This allows us to find explicit representatives of the classes $\omega_{ij} \in H^1(\mathrm{Conf}_{\mathbb{C}}(r); \mathbb{C})$ (for $i \neq j$) which satisfy "on the nose" the relations of the cohomology ring:

$$H^*(\mathrm{Conf}_{\mathbb{C}}(r); \mathbb{C}) \to \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_{\mathbb{C}}(r); \mathbb{C}), \quad \omega_{ij} \mapsto d\log(z_j - z_i) = \frac{dz_j - dz_i}{z_j - z_i}. \tag{2.4.10}$$

However, for $n \geq 3$, the situation is much more complicated. It is unknown (and unlikely) that there exist forms in $\Omega^1_{\mathrm{dR}}(\mathrm{Conf}_{\mathbb{R}^n}(r))$ (or $\Omega^1_{\mathrm{PL}}(\mathrm{Conf}_{\mathbb{R}^n}(r))$) which represent the classes ω_{ij} and satisfy the Arnold relations. It is thus necessary to find an intermediate CDGA between $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ and $\Omega^*_{\mathrm{dR}}(\mathrm{Conf}_{\mathbb{R}^n}(r))$ in which the Arnold relations are relaxed up to homotopy; that is, a resolution (in the sense of homological algebra) of $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$. One such type of resolutions, based on graph complexes, was introduced by Kontsevich [105]. Another one for $n = 2$, based on infinitesimal braids, was introduced by Tamarkin [158] based on earlier work on Kohno [103] and Drinfeld [55].

In both cases, the structure of configuration spaces plays an essential role in the construction of the resolution. It is not sufficient to merely consider each configuration space of $\mathbb{R}^n$ separately. Indeed, one needs to consider the whole family of configuration spaces $\mathrm{Conf}_{\mathbb{R}^n}(r)$ at once, as well as the maps between them. The structure maps, and the relations between them, are neatly encoded by the notion of operad, which we present in Sec. 3.

* * *

Models According to Th. 2.2.3, for closed manifolds M satisfying appropriate hypotheses, the real homotopy type of $\mathrm{Conf}_M(r)$ only depends on the real homotopy type of M. However, by itself, this invariance result does not tell us much about $\mathrm{Conf}_M(r)$. To truly understand $\mathrm{Conf}_M(r)$, we need more.

The first step is to find a model of M, that is, a CDGA A which is quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(M)$. All our models and methods of computation for $\mathrm{Conf}_M(r)$ rely heavily on the fact that M is a manifold. Indeed, while the study of configuration spaces of non-manifolds (e.g., graphs, see Fig. 2.3) is fascinating, it goes beyond the scope of this memoir.

One of the most fundamental features of manifolds is Poincaré duality. Under Poincaré duality, the cohomology of a closed oriented manifold M is isomorphic to the dual of the homology of M. This feature is reflected on certain models of M.

Definition 2.4.11. A Poincaré duality model of an oriented closed n-manifold M is a CDGA A which is quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(M)$ and which is equipped with a linear map $\varepsilon \colon A^n \to \mathbb{R}$ (called the *orientation*), satisfying the following properties:

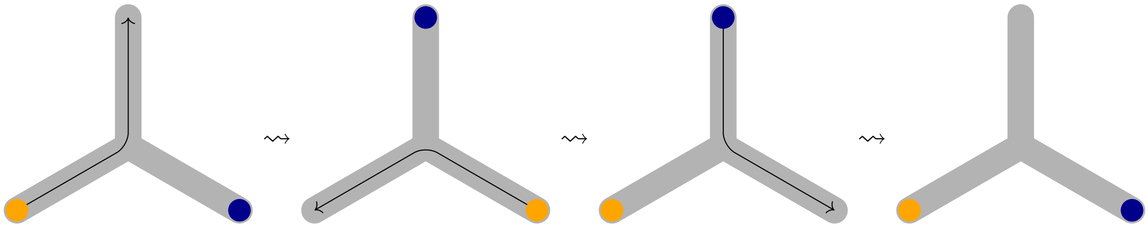


Figure 2.3: The configuration space of two points in a tripod is connected.

- The orientation vanishes on cocycles, that is, $\varepsilon(dx) = 0$ for all $x \in A^n$.
- The orientation induces a non-degenerate pairing $A^k \otimes A^{n-k} \to \mathbb{R}$, $a \otimes b \mapsto \varepsilon(ab)$ for all $k \in \mathbb{Z}$.

Lambrechts–Stanley [111] proved that such models always exist for all simply connected closed manifolds. Hájek [88] proved that such models also exist for connected closed manifolds, with an additional technical condition if $\dim(M)$ is even.

Let A be a Poincaré duality model of M. For $1 \leq i \leq r$, we let $\iota_i \colon A \to A^{\otimes r}$ be the inclusion of the ith factor, that is, $a \mapsto 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(r-i)}$. Moreover, we let Δ_A be the diagonal class, that is, the dual of the fundamental class of the diagonal $\Delta \subseteq M^{\times 2}$. More precisely, if $(x_\alpha)_\alpha$ is a graded basis of A and $(x^*_\alpha)_\alpha$ is its dual basis under the bilinear form induced by the orientation, then $\Delta_A = \sum_\alpha (-1)^{|x_\alpha|} x_\alpha \otimes x^*_\alpha$. Note that $(a \otimes 1)\Delta_A = (1 \otimes a)\Delta_A$ for all $a \in A$. Finally, we let $\Delta_{ij} = (\iota_i \cdot \iota_j)(\Delta_A)$ be the dual of the fundamental class of the diagonal $\{x_i = x_j\} \subseteq M^r$.

Example 2.4.12. Let $M = S^n$ be a sphere. Any sphere is formal, so we can take $A = H^*(M) = \Lambda(x) = \mathbb{Q}\langle 1, x\rangle$ as our Poincaré duality model of M, where $\deg x = n$. The diagonal class of A is $\Delta_A = 1 \otimes x + (-1)^n x \otimes 1$. Note that due to the Koszul rule of signs, we indeed have:

$$(x \otimes 1)\Delta_A = (x \otimes 1)(1 \otimes x + (-1)^n x \otimes 1) = x \otimes x, \tag{2.4.13}$$

$$(1 \otimes x)\Delta_A = (1 \otimes x)(1 \otimes x + (-1)^n (x \otimes 1)) = x \otimes x. \tag{2.4.14}$$

Taking a Poincaré duality model as initial data, Lambrechts–Stanley [110] constructed a CDGA $\mathsf{G}_A(r)$ whose cohomology is isomorphic to $H^*(\mathrm{Conf}_M(r))$ as a Σ_r-module, and they conjectured that $\mathsf{G}_A(r)$ is a model of $\mathrm{Conf}_M(r)$. This CDGA has a long history; variations of it appeared in the works of Cohen–Taylor [43], Bendersky–Gitler [22], Kriz [108], Félix–Thomas [62], and Berceanu–Markl–Papadima [23].

Definition 2.4.15. The CDGA $\mathsf{G}_A(r)$ is defined by:

$$\mathsf{G}_A(r) \coloneqq \Big(A^{\otimes r} \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) / \big(\iota_i(a)\omega_{ij} = \iota_j(a)\omega_{ij}\big)_{a \in A, 1 \leq i \neq j \leq r}, d\omega_{ij} = \Delta_{ij} \Big). \tag{2.4.16}$$

Roughly speaking, the idea behind the CDGA $\mathsf{G}_A(r)$ is that since we have:

$$\mathrm{Conf}_M(r) = M^r \setminus \bigcup_{1 \leq i \neq j \leq r} \Delta_{ij}, \tag{2.4.17}$$

a model for $\mathrm{Conf}_M(r)$ "should" be obtained by starting from a model of M^r and adding generators to kill the fundamental classes of the diagonals. The CDGA $\mathsf{G}_A(r)$ is a candidate for such a model. It was known to be an actual CDGA model in a few cases: for smooth projective complex varieties [108]; for 2-connected manifolds and $r = 2$ [109]; for even-dimensional simply connected manifolds and $r = 2$ [45]. Moreover, Lambrechts–Stanley [110] proved that $\mathsf{G}_A(r)$ computes the cohomology of $\mathrm{Conf}_M(r)$ as a Σ_r-module for simply connected closed manifolds.

We have obtained the following result, proved independently by Campos–Willwacher [34]:

Theorem 2.4.18 ([5, 34])**.** *Let M be a simply connected smooth closed manifold. Then for any Poincaré duality model A of M and any $r \geq 0$, the CDGA $\mathsf{G}_A(r)$ is a model of $\mathrm{Conf}_M(r)$ over $\mathbb{R}$.*

Since then, this statement has been shown to hold over $\mathbb{Q}$ by Willwacher [172] for framed manifolds of dimension at least 4. The assumption that M is framed comes from the way that Willwacher's result, like Th. 2.4.18, is obtained. Indeed, just like Kontsevich's proof [105] of formality of configuration spaces of $\mathbb{R}^n$ relies on the structure of configuration spaces, the proof of Th. 2.4.18 relies on the operadic structure of configuration spaces of M, which requires M to be framed. For Th. 2.4.18, the framed assumption does not appear because the key constructions, that of the propagator and the graph complexes, can still be defined even when M is not framed.

Example 2.4.19. Let $r = 2$ and A be a Poincaré duality CDGA. Then as a graded vector space, $\mathsf{G}_A(2)$ is isomorphic to:

$$\mathsf{G}_A(2) \cong (A \otimes A \otimes 1) \oplus (A \otimes \omega_{12}). \tag{2.4.20}$$

The multiplication is given by:

$$(a \otimes a' \otimes 1 + a'' \otimes \omega_{12})(b \otimes b' \otimes 1 + b'' \otimes \omega_{12}) = \pm ab \otimes a'b' \otimes 1 + (aa'b'' \pm bb'a'')\omega_{12}. \tag{2.4.21}$$

The differential is given by:

$$d(a \otimes a' \otimes 1) = d(a \otimes a') \otimes 1, \quad d(a \otimes \omega_{12}) = d(a) \otimes \omega_{12} \pm (a \otimes 1)\Delta_A \otimes 1. \tag{2.4.22}$$

This CDGA is thus isomorphic to the cone of the map $A \to A \otimes A$, $a \mapsto (a \otimes 1)\Delta_A$. Since that map is injective, the cone is quasi-isomorphic to the quotient, that is,

$$\mathsf{G}_A(2) \simeq A^{\otimes 2}/(\Delta_A). \tag{2.4.23}$$

This echoes the classical result that $H^*(\mathrm{Conf}_M(2)) = H^*(M^2 \setminus \Delta_M) \cong H^*(M^2)/(\Delta_M)$.

The computation of $\mathsf{G}_A(r)$ becomes significantly harder when $r \geq 3$. Let us give an example for $r = 3$, which is the smallest difficult case. Let us denote by $\mathsf{G}_A(r)^{(w)}$ the subspace of $\mathsf{G}_A(r)$ spanned by the elements of the form $[\alpha \otimes \omega]$ where $\alpha \in A^{\otimes r}$ and $\omega \in H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ is a product of w generators.

Example 2.4.24. Suppose that $M = S^n$ is a sphere. Any sphere is formal, so we can take $A = H^*(S^n) = \Lambda(x)$ as above. Let us denote by $x_i = \iota_i(x)$ the image of the generator x under the inclusion of A as the ith factor of $A^{\otimes 3}$. Then $\mathsf{G}_A(3) = \mathsf{G}_A(3)^{(0)} \oplus \mathsf{G}_A(3)^{(1)} \oplus \mathsf{G}_A(3)^{(2)}$ is a direct sum of subspaces consisting of terms of three types:

- Terms in $\mathsf{G}_A(3)^{(0)} = A^{\otimes 3}$. These form a subspace of dimension $2^3 = 8$. They are all cycles, and correspond to the image of the dual of the inclusion $\mathrm{Conf}_M(3) \hookrightarrow M^3$.
- Terms of the form $\iota_i(a)\iota_k(b)\omega_{ij} \in \mathsf{G}_A(3)^{(1)}$, where $\{i, j, k\} = \{1, 2, 3\}$. Note that $\omega_{ji} = -\omega_{ij}$, so there are $\binom{3}{1} \times 2^2 = 12$ terms of this form. The differential is given on elements of the form $\iota_1(a)\iota_3(b)\omega_{12}$ by:

$$\begin{aligned} d(\omega_{12}) &= x_2 + (-1)^n x_1, & d(x_1\omega_{12}) &= (-1)^n x_1 x_2, \\ d(x_3\omega_{12}) &= x_2 x_3 + (-1)^n x_1 x_3, & d(x_1 x_3 \omega_{12}) &= (-1)^n x_1 x_2 x_3. \end{aligned} \tag{2.4.25}$$

  The differential is defined similarly on the other terms.
- Terms of the form $\iota_i(a)\omega_{ij}\omega_{jk} \in \mathsf{G}_A(3)^{(2)}$, where $\{i, j, k\} = \{1, 2, 3\}$. Because of the (anti)-symmetry of ω_{ij}, the two-term relation, and the three-term relation, there are $2 \times 2 = 4$ such terms. The differential is given by:

$$d(\omega_{ij}\omega_{jk}) = x_j \omega_{jk} + (-1)^n x_i \omega_{jk} - (-1)^n x_k \omega_{ij} - x_i \omega_{ij}, \tag{2.4.26}$$

$$d(x_i \omega_{ij}\omega_{jk}) = x_i x_j \omega_{jk} + (-1)^n x_i x_k \omega_{ij}. \tag{2.4.27}$$

We thus see interesting phenomena occur in $H^*(\mathsf{G}_A(3))$ depending on the parity of n.

- If n is odd, then the image of the differential $d\colon \mathsf{G}_A(3)^{(1)} \to \mathsf{G}_A(3)^{(0)}$ is generated by relations of the form $\iota_i(x) = \iota_j(x)$ for all i, j, so that in cohomology, the class of $[\iota_1(x)]$ remains nonzero. In fact, if A is the cohomology of an odd sphere, then $\mathsf{G}_A(r)$ is quasi-isomorphic to $A \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n}(r-1))$ for odd n, using the map defined on generators by:

$$A \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n}(r-1)) \to \mathsf{G}_A(r),\ a \otimes 1 \mapsto \iota_r(a),\ 1 \otimes \omega_{ij} \mapsto \omega_{ij}. \tag{2.4.28}$$

  This is consistent with the fact that for odd n, the Fadell–Neuwirth fibration $\mathrm{Conf}_{S^n}(r) \to S^n, x \mapsto x_1$, whose fiber is $\mathrm{Conf}_{\mathbb{R}^n}(r-1)$, splits rationally for all r. For $n \in \{0, 1, 3, 7\}$ this follows from the fact that S^n is an H-space. For other odd values of n, one observes that the splitting map remains well-defined rationally.
- If n is even, then the image of the above differential is generated by relations of the form $\iota_i(x) = -\iota_j(x)$ for all i, j. But then, we have that $[\iota_1(x)] = -[\iota_2(x)] = [\iota_3(x)] = -[\iota_1(x)]$, so that $[\iota_1(x)] = 0$. It follows that only the unit $1 \in \mathsf{G}_A(3)^{(0)}$ survives in cohomology. The only other surviving class is $[x_3\omega_{12} - x_2\omega_{23} - x_1\omega_{13}]$ in degree $2n-1$. It follows that $\mathrm{Conf}_{S^n}(3)$ is a rational $(2n-1)$-sphere. The Fadell–Neuwirth fibration $\mathrm{Conf}_{S^n}(3) \to S^n$, whose fiber is $\mathrm{Conf}_{\mathbb{R}^n}(2) \simeq S^{n-1}$ is a rational version of the Hopf fibration.

* * *

Framed case The homotopy type of the configuration spaces of a manifold provide important information about the structure of that manifold. However, this information is sometimes insufficient for some applications.

One instance of this phenomenon appears in the computation of embedding spaces through the theory of embedding calculus (without the claim of being exhaustive, let us mention [85, 84, 30, 161, 83, 31, 13, 12]). The goal of embedding calculus is to compute the homotopy type of the space $\mathrm{Emb}(M, N)$ of embeddings $f\colon M \hookrightarrow N$ for some manifolds M, N (possibly with some restrictions, e.g., compact support). To perform this computation, the rough idea of embedding calculus is to replace the functor $\mathrm{Emb}(\_, N)$ by polynomial approximations, in the sense that a functor is polynomial of degree d if (roughly speaking) its value on a space X can be recovered from its value on a covering of X by d open balls and their intersections. Under good conditions (e.g., when $\dim N - \dim M \geq 3$), the limit of this procedure as $d \to \infty$ recovers the homotopy type of $\mathrm{Emb}(M, N)$.

Modern definitions of polynomial approximations use maps between configuration spaces [31, 161] and their operadic structures (Sec. 3). However, if one only has access to plain configuration spaces, then the machinery of embedding calculus can only be applied to framed manifolds and framed embeddings. If one wants to deal with general manifolds and arbitrary embeddings, it is necessary to use framed configuration spaces instead.

Definition 2.4.29. Let M be a smooth manifold and $\mathrm{Fr}_M \to M$ be the oriented frame bundle of M. The rth framed configuration space of M is the pullback:

$$\begin{tikzcd} \mathrm{Conf}^{\mathrm{fr}}_M(r) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fr}_M^{\times r} \arrow[d, ->>] \\ \mathrm{Conf}_M(r) \arrow[r, hook] & M^{\times r}. \end{tikzcd} \tag{2.4.30}$$

That is, a point of $\mathrm{Conf}^{\mathrm{fr}}_M(r)$ is a configuration of r points in M, each endowed with a basis of its tangent space.

Unfortunately, the local structure of framed configuration spaces is much harder to understand than for plain configuration spaces. As a simple example of the added complexity compared to the unframed case, note that the action of $\mathrm{SO}(n)$ on S^{n-1} is not formal for odd $n \geq 3$ [98, Rem. 9.5], which means that the study of just two points in $\mathbb{R}^n$ colliding at a single position requires careful consideration. More generally, the local structure of framed configuration spaces is encoded by the framed little disks operads (Def. 3.2.25). The homotopy type of these operads are more complex than that of plain little disks operads (Th. 3.3.9). Nevertheless, with Campos, Ducoulombier, and Willwacher, we were able to prove the following in the preprint [2]:

Theorem 2.4.31 ([2])**.** *There is a model over* $\mathbb{R}$ *of* $\mathrm{Conf}^{\mathrm{fr}}_M(r)$ *based on graph complexes, which depends on a Maurer–Cartan element defined through integrals in the framed graph complex.*

The Maurer–Cartan elements mentioned in the previous theorem are difficult to compute. Their definition involve integrals given by "Feynman rules," and no general method exists to find their values. We will see that in the case of a surface $M = S_g$, using some completely different methods, the Maurer–Cartan element can be shown to vanish up to homotopy to yield a much simpler model of $\mathrm{Conf}^{\mathrm{fr}}_{S_g}(r)$.

* * *

Surfaces While closed surfaces are closed manifolds, Th. 2.4.31 does not apply to most of them, as only the 2-sphere is simply connected. Th. 2.4.52 does apply to surfaces, but the result is not explicit: it depends on the calculation of certain integrals that are hard to perform in general. These integrals have been computed by Campos–Willwacher [34], but only for the 2-sphere. Since these integrals remain a mystery, it is a priori not obvious that the graph complex obtained in Th. 2.4.52 is related to some kind of "small" model (of the form of G_A from Th. 2.4.18).

However, surfaces are notoriously well-behaved manifolds. In particular, oriented closed surfaces are smooth projective complex manifolds (complex curves). The results of Kriz [108] and Totaro [160] thus apply: for such a surface S, the CDGA $\mathsf{G}_{H^*(S)}(r)$ is a rational model of $\mathrm{Conf}_M(r)$ for any $r \geq 0$.

With Campos and Willwacher, in [4], we studied the framed configuration spaces of closed oriented surfaces. Given the genus g surface S_g, let us denote the generators of its cohomology as follows, where $\deg a_i = \deg b_i = 1$ for all i:

$$A_g \coloneqq H^*(S_g) = S(a^1, \dots, a^g, b^1, \dots, b^g)/(a^i b^i - a^j b^j, a^i a^j, a^i b^j, b^i b^j)_{1 \leq i \neq j \leq g}. \tag{2.4.32}$$

We also let $\nu \in H^2(S_g)$ be the volume form, that is, $\nu = [a^1 b^1]$. The diagonal class Δ_{A_g} is equal to $1 \otimes \nu + \nu \otimes 1 - \sum_i (a^i \otimes b^i - b^i \otimes a^i)$. Note that if we apply the product of the algebra $A_g \otimes A_g \to A_g$ to Δ_{A_g}, we recover $(2-2g)\nu = \chi(S_g)\nu$, as expected. Then the frame bundle Fr_{S_g} has a rational model given as follows, where $\deg \theta = 1$:

$$A_g^{\mathrm{fr}} \coloneqq (A_g \otimes S(\theta), \, d\theta = (2-2g)\nu). \tag{2.4.33}$$

Theorem 2.4.34 (Bezrukavnikov [26])**.** *A rational model for the framed configuration space* $\mathrm{Conf}^{\mathrm{fr}}_{S_g}(r)$ *is given by the following CDGA:*

$$\mathsf{G}^{\mathrm{fr}}_{A_g^{\mathrm{fr}}}(r) \coloneqq (\mathsf{G}_{A_g}(r) \otimes S(\theta_1, \dots, \theta_r)/(\theta_i \omega_{ij} = \theta_j \omega_{ij})_{1 \leq i \neq j \leq r}, \, d\theta_i = (2-2g)\nu_i). \tag{2.4.35}$$

In [4], we prove that this model is compatible with the action of the framed little disks operad. We go back to operads in Sec. 3, so we postpone our discussion of this result there. Simply note for now that our methods are completely different than the ones we used for Th. 2.4.18 or Th. 2.4.52. We use that any surface can be decomposed into two kinds of surfaces: a sphere with holes, and cylinders (attached to the holes). Gluing back simple models for configuration spaces of these two kinds of manifold, we recover the model for the configuration spaces of the whole surface, while preserving the operadic structure.

The above models for $\mathrm{Conf}_{S_g}(r)$ and $\mathrm{Conf}^{\mathrm{fr}}_{S_g}(r)$ are quite nice: they are quadratic Koszul [26] (with inhomogeneous differential for framed configuration spaces). We refer to Sec. 4 for a discussion of Koszul duality. These models thus lend themselves to concrete computations.

Example 2.4.36. By computing the C_∞-structure on cohomology using the homotopy transfer theorem, we can explicitly prove that $\mathrm{Conf}_{S_g}(r)$ admits nontrivial Massey products when $g \geq 1$ and $r \geq 3$. In particular, this implies that $\mathrm{Conf}_{S_g}(r)$ is not formal, recovering [26, Corollary 1].

To exhibit the fact that the models are well-suited to computations, we implemented in Mathematica the computation of the transferred C_∞-structure – see Appendix 5. We find that for a suitable choice of deformation retract of $B \coloneqq \mathsf{G}_{A_1}(3)$ onto its cohomology, if we let $a_i^j = \iota_i(a^j)$ and $b_i^j = \iota_i(b^j)$, then we get a nonzero ternary product:

$$\begin{aligned}
u &= 2\alpha_1^1 - 8\alpha_2^1 + 6\alpha_3^1 + \frac{8}{3}\beta_1^1 + 8\beta_3^1 - \frac{32}{3}\beta_2^1 \in H^1(B); \\
v &= \frac{8}{3}\alpha_1^1 + 8\alpha_3^1 - \frac{32}{3}\alpha_2^1 + \frac{4}{3}\beta_2^1 - \beta_3^1 - \frac{1}{3}\beta_1^1 \in H^1(B); \\
w &= \frac{3}{2}\alpha_1^1 - 6\alpha_2^1 + \frac{9}{2}\alpha_3^1 - \beta_1^1 + 4\beta_2^1 - 3\beta_3^1 \\
m_2(u,v) &= 0 \in H^2(B);\ m_2(v,w) = 0 \in H^2(B); \\
m_3(u,v,w) &= 192\alpha_1^1\omega_{1,2} + 192\alpha_1^1\omega_{1,3} - 192\alpha_1^1\omega_{2,3} - 192\alpha_2^1\omega_{1,3} \\
&\quad + 192\alpha_2^1\omega_{2,3} - 192\alpha_3^1\omega_{1,2} - 24\beta_1^1\omega_{1,2} - 24\beta_1^1\omega_{1,3} \\
&\quad + 24\beta_1^1\omega_{2,3} + 24\beta_2^1\omega_{1,3} - 24\beta_2^1\omega_{2,3} + 24\beta_3^1\omega_{1,2} \neq 0 \in H^2(B)/(u,w).
\end{aligned}$$

Remark 2.4.37. Strictly speaking, in Ex. 2.4.36, we are computing the transferred A_∞-structure, not the C_∞-structure. Thanks to a theorem of Saleh [142], formality as an associative dg-algebra is equivalent to formality as a CDGA in characteristic zero. In fact, two CDGAs are quasi-isomorphic if and only if they are so as DGAs [33].

Question 2.4.38. *Can the cohomology of $\mathsf{G}_A(r)$ be described more generally (e.g., can a basis be found)? Can the transferred C_∞-algebra structure be computed in general?*

* * *

Manifolds with boundary Computing the homotopy type of configuration spaces of manifolds with boundary is, for some aspects, more arduous than those of closed manifolds. In general, the homotopy type of $\mathrm{Conf}_M(r)$ (for a manifold M with boundary) cannot possibly depend on just the homotopy type of M, even in simple cases. For example (see [101, Sec. 1.1]), the torus with a point removed $S_{1,1}$ and the sphere with three points removed $S_{0,3}$ are both homotopy equivalent to the wedge sum $S^1 \vee S^1$, but their configuration spaces $\mathrm{Conf}_{S_{1,1}}(r)$ and $\mathrm{Conf}_{S_{0,3}}(r)$ are not homotopy equivalent for $r \geq 2$. It is, instead more reasonable to hope that $\mathrm{Conf}_M(r)$ can be determined from the homotopy type of the pair $(M, \partial M)$, or from the proper homotopy type of M (in the sense of [134]).

However, configuration spaces of manifolds with boundary admit much more structure than those of closed manifolds. In addition to applying operadic operations (Sec. 3), which are common to both kinds of manifolds, it is also possible to *glue* configuration spaces of manifolds with boundary. More precisely, if a manifold X is equal to $M \cup_N M'$, where $\partial M = \partial \bar{M}' = N$, then one can define:

- on the symmetric sequence $\mathrm{Conf}_{N\times\mathbb{R}} = \{\mathrm{Conf}_{N\times\mathbb{R}}(r)\}_{r\geq 0}$, the structure of a monoid up to homotopy, by gluing cylinders end-to-end;
- on the symmetric sequence Conf_M (resp., $\mathrm{Conf}_{M'}$), the structure of a right (resp., left) module up to homotopy over $\mathrm{Conf}_{N\times\mathbb{R}}$;
- a natural equivalence from the "homotopy tensor product" $\mathrm{Conf}_M \times_{\mathrm{Conf}_{N\times\mathbb{R}}} \mathrm{Conf}_{M'}$ to Conf_X.

This structure is pervasive in the literature. One common avatar is the operation "adding a point at infinity" $\mathrm{Conf}_M(r) \to \mathrm{Conf}_M(r+1)$, which appears e.g., in the study of homological stability [125, 150]. This operation is simply given by choosing a base point $\infty \in \mathrm{Conf}_{N\times\mathbb{R}}(1)$ and plugging it into the operation $\mathrm{Conf}_M(r) \times \mathrm{Conf}_{N\times\mathbb{R}}(1) \to \mathrm{Conf}_M(r+1)$. The monoid/module structure also plays an important role in the axioms characterizing factorization homology [18], which are themselves inspired by the definition of topological quantum field theories [15].

There are few computations regarding the homotopy types of configuration spaces of manifolds with boundary. A notable approach is the one of Petersen [132], who provided a method to compute the cohomology with compact support of $\mathrm{Conf}_M(r)$ using a twisted CDGA model for M.

In [3], with Campos, Lambrechts, and Willwacher, we extended the results of [34, 5] to provide models for configuration spaces of manifolds with boundary. The general idea is similar to the one for closed manifolds. To even define the analogue of the small model G_A for manifolds with boundary, we needed new tools.

In particular, we generalized the notion of Poincaré duality models to manifolds with boundary, and defined the notion of "Poincaré–Lefschetz duality (PLD) models."

Definition 2.4.39. A PLD model of a pair $(M, \partial M)$, where M is a manifold with boundary of dimension n, is a morphism of CDGAs $\lambda \colon B \to B_\partial$ which is quasi-isomorphic to the restriction map $\Omega^*(M) \to \Omega^*(\partial M)$ such that:

- The CDGA B_∂ is equipped with an augmentation $\varepsilon_\partial \colon B_\partial^{n-1} \to \mathbb{R}$ that makes it into a Poincaré duality model of ∂M;
- The morphism λ is surjective;
- There is an augmentation $\varepsilon \colon B^n \to \mathbb{R}$ satisfying Stokes' formula ($\varepsilon(dx) = \varepsilon_\partial(\lambda(x))$);
- The augmentation ε induces a perfect pairing of degree n between the kernel of λ (which models relative forms $\Omega^*(M, \partial M)$) and the quotient P of B by the dg-ideal $\{x \mid \forall y \in \ker(\lambda), \varepsilon(xy) = 0\}$.

Example 2.4.40. A PLD model for $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is given by:

$$B_\partial = (\langle 1, v_{n-1} \rangle, dv_{n-1} = 0), \qquad \varepsilon_\partial(v_{n-1}) = 1; \tag{2.4.41}$$
$$B = (\langle 1, v_{n-1}, w_n \rangle, dv_{n-1} = w_n), \qquad \varepsilon(w_n) = 1. \tag{2.4.42}$$

and λ is the obvious quotient map. The kernel of λ is spanned by w while the quotient P mentioned above is spanned by 1.

Example 2.4.43. More generally, let M' be a simply connected closed n-manifold and let A be a Poincaré duality model of M'. Let M be the manifold with boundary obtained from M' by removing an open disk. Then a PLD model of $(M, \mathbb{S}^{n-1})$ is given by:

$$B_\partial = (\langle 1, v_{n-1} \rangle, dv = 0), \qquad \varepsilon_\partial(v) = 1; \tag{2.4.44}$$
$$B = (A \oplus \langle v_{n-1} \rangle, dv = \mathrm{vol}_A), \qquad \varepsilon(\mathrm{vol}_A) = 1. \tag{2.4.45}$$

By adapting the arguments of Lambrechts–Stanley [111] to the case of manifolds with boundary and combining them with results of Cordova Bulens–Lambrechts–Stanley [46], the following result was obtained:

Theorem 2.4.46 ([3])**.** *Any simply connected manifolds with simply connected boundary of dimension ≥ 7 admits a PLD model.*

Remark 2.4.47. In [3], it is also proved that any manifold admitting a surjective pretty model (see [47] for the definition) also admits a PLD model. According to the results of Cordova Bulens–Lambrechts–Stanley [47], examples of manifolds admitting surjective pretty models include even-dimensional disk bundles over simply connected manifolds, manifolds $(M, \partial M)$ such that ∂M retracts rationally onto its half-skeleton, and complements of high-codimensional 2-connected subpolyhedra in simply connected closed manifolds.

Question 2.4.48. *Can the assumptions that $\dim M \geq 7$ and/or the that M is simply connected be dropped from Th. 2.4.46?* (It would be interesting to investigate if methods from Hodge theory, as in [88], could allow one to answer this question.)

We then used PLD models to define a "small" Lambrechts–Stanley-type CDGA. This CDGA is defined almost exactly as G_A above. Let $(\lambda : B \to B_\partial, \varepsilon, \varepsilon_\partial)$ be a PLD model of some manifold with boundary M, and let K (resp., P) be the kernel (resp., quotient) from Def. 2.4.39.

$$\mathsf{G}_P(r) \coloneqq (P^{\otimes r} \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))/(\iota_i(a)\omega_{ij} = \iota_j(a)\omega_{ij})_{a\in A, 1\le i\neq j\le r}, d\omega_{ij} = (\Delta_P)_{ij}), \tag{2.4.49}$$

where Δ_P is the image of the element $\Delta_{KP} \in K \otimes P$ that witnesses the perfect pairing between K and P, under the composite map $K \otimes P \hookrightarrow B \otimes P \twoheadrightarrow P \otimes P$. Using techniques similar to the ones of Lambrechts–Stanley [110], we prove that the homology $\mathsf{G}_P(r)$ is isomorphic to the homology of $\mathrm{Conf}_M(r)$ as a Σ_r-module.

However, it can be seen heuristically that the homotopy type of G_P is incorrect [3, Ex. 8.11]. Instead, we introduce a deformation $\tilde{\mathsf{G}}_P(r)$ which is isomorphic to $\mathsf{G}_P(r)$ as a dg-Σ_r-space, but has a different algebra structure. Let $\sigma = \sum_\alpha \sigma'_\alpha \otimes \sigma''_\alpha \in P \otimes B_\partial$ be an element dual to a linear section composed with the projection, $B_\partial \to B \twoheadrightarrow P$. The CDGA $\tilde{\mathsf{G}}_P(r)$ is defined as the quotient of $P^{\otimes r} \otimes S(\tilde{\omega}_{ij})_{1\le i\neq j\le r}/(\tilde{\omega}_{ij}^2)$ by two kinds of relations:

- For all $b \in B$ and $1 \le i < j \le r$,
$$\iota_i(b)\tilde{\omega}_{ij} - (-1)^n \iota_j(b)\tilde{\omega}_{ij} + \sum_{i,j,\alpha,\beta} \pm\varepsilon_\partial(\lambda(b)\sigma''_\alpha\sigma''_\beta)\iota_i(\sigma'_\alpha)\iota_j(\sigma'_\beta) = 0. \tag{2.4.50}$$
- And for all $b \in B$ and $1 \le i < j < k \le r$,
$$\iota_i(b)\tilde{\omega}_{ij}\tilde{\omega}_{ik} + \iota_j(b)\tilde{\omega}_{ji}\tilde{\omega}_{jk} + \iota_k(b)\tilde{\omega}_{ki}\tilde{\omega}_{kj} + \sum_{i,j,k,\alpha,\beta,\gamma} \pm\varepsilon_\partial(\lambda(b)\sigma''_\alpha\sigma''_\beta\sigma''_\gamma)\iota_i(\sigma'_\alpha)\iota_j(\sigma'_\beta)\iota_k(\sigma'_\gamma) = 0. \tag{2.4.51}$$

While these relation might appear complicated, they have simple interpretations. For $b = 1$, Eq. (2.4.50) says that $\tilde{\omega}_{ij}$ is almost equal to $\pm\tilde{\omega}_{ji}$, with a lower-weight corrective term (if weight is the number of $\tilde{\omega}$ in an expression). For arbitrary b, Eq. (2.4.50) says that ι_* and $\tilde{\omega}$ almost satisfy the symmetry relation that allows one to move a decoration around on a connected component of a graph. Eq. (2.4.51) has a similar interpretation with respect to the Arnold three-terms relation. Note that these relations are tantalizingly similar to a "decorated" version of the generalized Jacobi relation in the cohomology of non-k-equal configuration spaces of $\mathbb{R}^n$ [52, Eq. (3.3)] – investigating this similarity would be worthwhile.

By extending and generalizing the techniques employed in the case of closed manifolds, we have obtained (in joint work with Campos, Lambrechts, Willwacher):

Theorem 2.4.52 ([3])**.** *Let M be a simply connected smooth manifold with simply connected boundary. For any PLD model of M, the CDGA $\tilde{\mathsf{G}}_P(r)$ is a model of $\mathrm{Conf}_M(r)$ over $\mathbb{R}$.*

The hypothesis that ∂M is simply connected appears in the previous theorem for technical reasons. However, it is absent from Th. 2.2.5. The following question seems reasonable:

Question 2.4.53. *Can the condition that ∂M is simply connected be dropped from Th. 2.4.52?*

3 Operads

Most of the research that is presented in this memoir is closely related to the theory of operads. In this part, after introducing the main notions of that theory, we explain how the results presented in the previous part involve operadic methods, as well as present formality results that are purely about operads. The theory of operads has a long and rich history, and we refer to [117] or [67, Part I(a)] for more details.

3.1 Introduction to operads

In this section, we give a brief introduction to the theory of operads, and especially of the parts relevant to the statements we want in connection to configuration spaces.

Let us fix $\mathcal{C}$ a symmetric monoidal category. An operad in $\mathcal{C}$ is an object that encodes a certain category of algebras in $\mathcal{C}$. Classical examples when $\mathcal{C}$ is the category of vector spaces over some field include the operad that encodes associative algebras in $\mathcal{C}$, the one that encodes commutative algebras, or the one that encodes Lie algebras.

Notation 3.1.1. We denote FB the category of finite sets and bijections. For a pair $W \subseteq U$ of sets, we let U/W be the quotient set, and $U \to U/W$, $u \mapsto [u]$ be the quotient map.

Definition 3.1.2. A symmetric collection (or FB-module) is a functor $\mathsf{P}\colon \mathsf{FB}^{\mathrm{op}} \to \mathcal{C}$.

Remark 3.1.3. The category of symmetric collections is equivalent to the (perhaps more familiar) category of symmetric sequences, which consist of sequences $\mathsf{P} = \{\mathsf{P}(n)\}_{n\geq 0}$ of objects of $\mathcal{C}$ equipped with a right action of the symmetric group Σ_n on $\mathsf{P}(n)$. Given $\underline{n} = \{1, \dots, n\}$, then any symmetric collection P defines a symmetric sequence by $\mathsf{P}(n) \coloneqq \mathsf{P}(\underline{n})$.

Example 3.1.4. Let M be a manifold. The configuration spaces of M form a symmetric collection: for a finite set U,

$$\mathrm{Conf}_M(U) \coloneqq \{(x_u) \in M^{\times U} \mid \forall u \neq v,\, x_u \neq x_v\}, \tag{3.1.5}$$

where $M^{\times U}$ is the set of maps $U \to M$.

Definition 3.1.6. An operad is a symmetric collection P endowed with the following structure:

- for any singleton $\{*\}$, a morphism from the monoidal unit I of $\mathcal{C}$ (invariant under bijections of singletons):
$$\eta\colon I \to \mathsf{P}(\{*\}); \tag{3.1.7}$$
- for any pair of sets $W \subseteq U$, a morphism:
$$\circ_W\colon \mathsf{P}(U/W) \otimes \mathsf{P}(W) \to \mathsf{P}(U); \tag{3.1.8}$$

satisfying the following axioms:

- (unit) for any $U \in \mathsf{FB}$, and any $u \in U$, if we identify $U/\{u\} = U$ and $U/U = \{*\}$, then the following diagrams commute:

$$
\begin{tikzcd}
\mathsf{P}(U) \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[dr, equal] & \mathsf{P}(U/\{u\}) \otimes \mathsf{P}(\{u\}) \arrow[d, "\circ_{\{u\}}"] \\
& \mathsf{P}(U),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P}(U) \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, equal] & \mathsf{P}(U/U) \otimes \mathsf{P}(U) \arrow[d, "\circ_{U}"] \\
& \mathsf{P}(U);
\end{tikzcd}
\tag{3.1.9}
$$

- (sequential composition) for any triple of sets $W \subseteq V \subseteq U$, identifying $U/V = (U/W)/(V/W)$, the following diagram commute:

$$
\begin{tikzcd}
\mathsf{P}(U/V) \otimes \mathsf{P}(V/W) \otimes \mathsf{P}(W) \arrow[r, "\circ_W"] \arrow[d, "\circ_{V/W}"] & \mathsf{P}(U/V) \otimes \mathsf{P}(V) \arrow[d, "\circ_V"] \\
\mathsf{P}(U/W) \otimes \mathsf{P}(W) \arrow[r, "\circ_W"] & \mathsf{P}(U);
\end{tikzcd}
\tag{3.1.10}
$$

- (parallel composition) for any disjoint subsets $W, W' \subseteq U$, identifying $U/W/W' = U/W'/W$, the following diagram commutes:

$$
\begin{tikzcd}
\mathsf{P}(U/W/W') \otimes \mathsf{P}(W) \otimes \mathsf{P}(W') \arrow[r, "\circ_W"] \arrow[d, "\circ_{W'}"] & \mathsf{P}(U/W') \otimes \mathsf{P}(W') \arrow[d, "\circ_{W'}"] \\
\mathsf{P}(U/W) \otimes \mathsf{P}(W) \arrow[r, "\circ_W"] & \mathsf{P}(U).
\end{tikzcd}
\tag{3.1.11}
$$

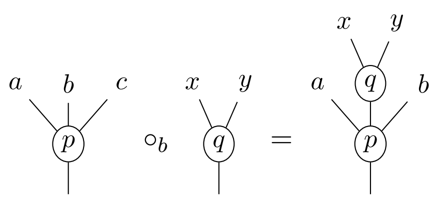
Example 3.1.12. The most fundamental and most illuminating example of operad is the endomorphism operad of an object $X \in \mathcal{C}$. This is in general an operad in the category of sets (though if $\mathcal{C}$ is closed or enriched, one can also produce an endomorphism operad in other categories). The endomorphism operad End_X is given by:

$$\mathrm{End}_X(r) \coloneqq \mathcal{C}(X^{\otimes U}, X), \tag{3.1.13}$$

where $X^{\otimes U}$ is the colimit of the diagram indexed by total orders on U (with a single arrows between any two orders), with constant value $X \otimes \cdots \otimes X$ ($\#U$ times) and which uses the symmetrical monoidal structure to permute factors on arrows of the diagram. Symmetric groups act by permuting input variables. The unit is simply $\mathrm{id}_X \in \mathrm{End}_X(1)$. The operadic structure maps are composition of multi-variable maps.

Based on this example, given an operad P, it is customary to view an element $p \in \mathsf{P}(U)$ as an operation with multiple inputs labeled by U, and a single input. The symmetric group action relabels inputs, and partial composition is viewed as partial composition of operations. For example, if $p \in \mathsf{P}(\{a, b, c\})$ and $q \in \mathsf{P}(\{x, y\})$:

(3.1.14)



Example 3.1.15. The trivial operad is the set operad I given by $\mathsf{I}(U) = \{\mathrm{id}\}$ if $\#U = 1$ and $\mathsf{I}(U) = \emptyset$ otherwise.

Example 3.1.16. The (unital) associative operad is the set operad uAss given by $\mathsf{uAss}(U) = \mathsf{FB}(U, U)$ for all U. The action of FB is by precomposition, and the unit is $\mathrm{id}_{\{*\}} \in \mathsf{uAss}(\{*\})$. Operadic composition is given by block composition of bijections.

Example 3.1.17. The (unital) commutative operad is the set operad uCom given by $\mathsf{uCom}(U) = \{*\}$ for all U. The structure maps are the only possible ones.

Remark 3.1.18. The previous operads admit, of course, analogues in the category of vector spaces over a field, in the category of topological spaces, etc. These analogue operads are usually be given the same name and notation.

Remark 3.1.19. If P is an operad, then $\mathsf{P}(\{*\})$ is naturally a monoid in $\mathcal{C}$. Conversely, if M is a monoid and $\mathcal{C}$ admits an absorbing element $\emptyset$ for the monoidal product, then we can define an operad P by by setting $\mathsf{P}(U) = M$ for $\#U = 1$ and $\mathsf{P}(U) = \emptyset$ for other U.

Remark 3.1.20. The above definition of operads is written in terms of partial composition [123]. The original definition of operads is written in terms of total composition [124] and is equivalent to the previous definition in the presence of units. Given a symmetric collection P, an operad structure on P is the data of a unit and, for any map of finite sets $f\colon U \to V$, of operations:

$$\gamma_f\colon \mathsf{P}(V) \otimes \bigotimes_{v\in V} \mathsf{P}(f^{-1}(v)) \to \mathsf{P}(U), \tag{3.1.21}$$

satisfying equivariance, unit, and associativity axioms. Yet another point of view is to see an operad as a monoid in the category of symmetric collections for the monoidal product given by the plethysm. For symmetric collections P, Q, their plethysm is defined, for a finite set U, by the coend:

$$(\mathsf{P} \circ \mathsf{Q})(U) \coloneqq \int^{V\in\mathsf{FB}} \Bigl(\mathsf{P}(V) \otimes \Bigl(\bigoplus_{f\colon U\to V} \bigotimes_{v\in V} \mathsf{Q}(f^{-1}(v))\Bigr)\Bigr). \tag{3.1.22}$$

The primary purpose of an operad is usually to define a representation category, whose objects are called the algebras over the operad.

Definition 3.1.23. Let P be an operad and A be an object of $\mathcal{C}$. A P-algebra structure on A is the data of structure maps, for all finite sets U:

$$\gamma\colon \mathsf{P}(U) \otimes A^{\otimes U} \to U. \tag{3.1.24}$$

These structure maps must satisfy equivariance, unit, and associativity axioms.

Notation 3.1.25. Given a P-algebra A, an element $p \in \mathsf{P}(U)$, and a tensor $\bigotimes_u a_u \in A^{\otimes U}$, we denote:

$$p(\otimes_u a_u) \coloneqq \gamma\Bigl(p \otimes \bigotimes_u a_u\Bigr). \tag{3.1.26}$$

Remark 3.1.27. If $\mathcal{C}$ is closed, then a P-algebra structure is the same thing as a morphism of operads $\mathsf{P} \to \underline{\mathrm{End}}_A$, where $\underline{\mathrm{End}}_A(U)$ is the internal hom object $\underline{\mathcal{C}}(A^{\otimes U}, A)$.

Example 3.1.28. Let $\mathcal{C}$ be the category of sets. An algebra over I is just a set. An algebra over uAss is a monoid. An algebra over uCom is a commutative monoid.

Notation 3.1.29. For a symmetric sequence E and an object $X \in \mathcal{C}$ viewed as a symmetric sequence concentrated in arity zero, we will write

$$\mathsf{E}(X) \coloneqq (\mathsf{E} \circ X)(0) = \int^{V\in\mathsf{FB}} (\mathsf{P}(V) \otimes X^{\otimes V}). \tag{3.1.30}$$

Remark 3.1.31. The map γ from Def. 3.1.23 can be reinterpreted as a map $\gamma\colon \mathsf{P}(A) \to A$.

Example 3.1.32. Let P be an operad and V be an object of $\mathcal{C}$. There is a natural P-algebra structure on $\mathsf{P}(V)$. This algebra satisfies a universal property that makes it the *free* P-algebra on V.

In the definition of an algebra over an operad, the operad acts "on the left." For example, a set X is obviously an algebra over End_X via $(f, (x_i)) \mapsto f(x_i)$. The following notion is in that sense mirrored.

Definition 3.1.33. Let P be an operad. A right P-module is a symmetric collection M endowed, for every pair of finite sets $W \subseteq U$, with structure maps:

$$\circ_W : \mathsf{M}(U/W) \otimes \mathsf{P}(W) \to \mathsf{M}(U), \tag{3.1.34}$$

satisfying equivariance, unit, and associativity axioms.

Remark 3.1.35. Thanks to the presence of units in P, this definition is equivalent to the data of a right module over the monoid P in the category of symmetric collections and plethysm (see Rem. 3.1.20).

Example 3.1.36. For any $X, Y \in \mathcal{C}$, there is a natural right End_X-module defined by:

$$\mathrm{End}_{X,Y}(U) \coloneqq \mathcal{C}(X^{\otimes U}, Y). \tag{3.1.37}$$

Example 3.1.38. A right I-module is just a symmetric collection. A right uCom-module is a functor on the category of finite sets and all maps (rather than just bijections).

A key use of operadic right modules is the definition of functors.

Definition 3.1.39. Suppose that $\mathcal{C}$ has small colimits. Let P be an operad and M a right P-module. There is a functor:

$$S_{\mathsf{M}} : \mathsf{P}\text{-Alg} \to \mathcal{C}, \qquad A \mapsto \mathrm{coeq}((\mathsf{M} \circ \mathsf{P})(A) \rightrightarrows \mathsf{M}(A)). \tag{3.1.40}$$

This property of operadic right modules, and the fact that plethysm of symmetric collections is linear on the left but nonlinear on the right, makes right modules markedly different from their cousins, left modules.

Definition 3.1.41. Let P be an operad. A left P-module is a symmetric collection N endowed, for every map of finite sets $f : U \to V$, with structure maps:

$$\gamma_f : \mathsf{P}(V) \otimes \bigotimes_{v \in V} \mathsf{N}(f^{-1}(v)) \to \mathsf{N}(U), \tag{3.1.42}$$

satisfying equivariance, unit, and associativity axioms.

Definition 3.1.43. Given operads P, Q, a (P, Q)-bimodule is a symmetric collection equipped with a left P-action and a right Q-action that commute.

Remark 3.1.44. Since a left module has (in general) no unit, the notion of left module is *not* equivalent to the one given in terms of partial compositions, called *infinitesimal* left modules. Infinitesimal right modules are however the same thing as right modules.

Example 3.1.45. Let $X, Y \in \mathcal{C}$ be objects. Then $\mathrm{End}_{X,Y}$ (Ex. 3.1.36) is and $(\mathrm{End}_Y, \mathrm{End}_X)$-bimodule. It is not, however, an infinitesimal left End_Y-module in general.

3.2 Little disks operads and configuration spaces

The proofs of the results of Sec. 2 on the homotopy types of configuration spaces involve operads in some way. The relationship between operads and configuration spaces already appeared at the very beginning of the development of operad theory under the form of the little cubes operads [27, 124]. In this section, we introduce these operads and their relationship to configuration spaces.

As a matter of personal preference (and to simplify some definitions in Sec. 3.5), we deal with their equivalent cousins, the little disks operads (introduced in [78]).

Notation 3.2.1. We write $\mathbb{D}^n$ for the closed unit n-disk, whose interior is denoted $\mathring{\mathbb{D}}^n$.

Definition 3.2.2. A standard embedding $\mathbb{D}^n \to \mathbb{D}^n$ is one that is obtained as a composite of a positive rescaling and a translation.

Definition 3.2.3. Let $n \geq 1$ be an integer. The little n-disks operad D_n is defined as a symmetric collection, for a finite set U, by:

$$\mathsf{D}_n(U) \coloneqq \left\{ c\colon \mathbb{D}^n \times U \to \mathbb{D}^n \;\middle|\; \begin{array}{l} \text{each } c(\_, u) \text{ is a standard embedding,} \\ \forall u \neq v,\ c(\mathring{\mathbb{D}}^n, u) \cap c(\mathring{\mathbb{D}}^n, v) = \emptyset \end{array} \right\}. \tag{3.2.4}$$

Morphisms of FB act by precomposition on the second factor. The unit is $\mathrm{id}_{\mathbb{D}^n} \in \mathsf{D}_n(1)$. Partial composition is given by composition of embeddings. See Fig. 3.1 for an illustration

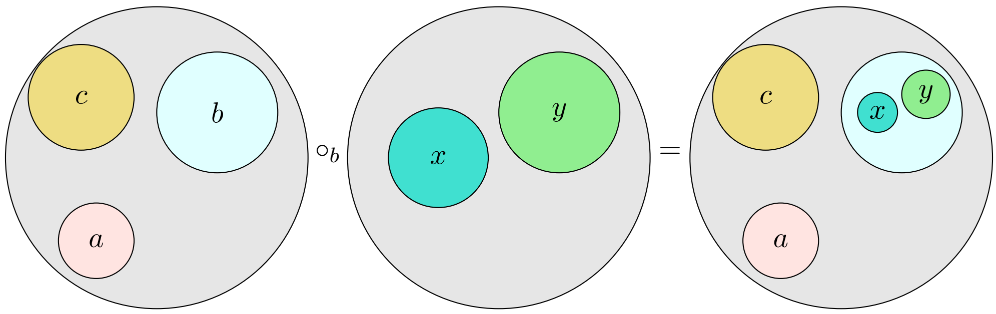


Figure 3.1: Example of the structure map $\circ_b\colon \mathsf{D}_2(\{a,b,c\}) \times \mathsf{D}_2(\{x,y\}) \to \mathsf{D}_2(\{a,x,y,c\})$.

Remark 3.2.5. The operad D_n is almost the endomorphism operad of $\mathbb{D}^n$ in the subcategory of Top given by disks and standard embeddings, with monoidal product given by disjoint union. However, since the boundaries of two distinct disks are allowed to meet, this statement is not quite true.

Remark 3.2.6. This family of operads admit many variants. Any topological operad which is weakly equivalent to D_n is called an E_n-operad. Examples include the little cubes operads C_n [124] or the (Axelrod–Singer–)Fulton–MacPherson operads [79, 75, 17] (see Def. 3.4.18), or the Kontsevich operads [105].

Remark 3.2.7. Note that $\mathsf{D}_n(\varnothing)$ is a singleton, not the empty set. It is sometimes necessary to consider the sub-operad obtained by removing the point of $\mathsf{D}_n(\varnothing)$.

* * *

While we will not really need it in the study of configuration spaces, let us now explain what the little disks operads were initially used for: the study of iterated loop spaces. We will briefly come back to this in Sec. 3.5 when dealing with the Swiss-Cheese operad.

Definition 3.2.8. Let (X, x_0) be a based space. The loop space ΩX is the space of continuous loops $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$, endowed with the compact open topology. The n-fold iterated loop space $\Omega^n X$ (for $n \geq 0$) is defined recursively by $\Omega^0 X = X$ and $\Omega^{n+1} X = \Omega(\Omega^n X)$.

Remark 3.2.9. An element of $\Omega^n X$ can be viewed as a map $\gamma\colon [0,1]^n \to X$ such that $\gamma(\partial[0,1]^n) = \{x_0\}$.

Proposition 3.2.10. *For any based space (X, x_0), the n-fold iterated loop space is an algebra over the little cubes operad C_n (see Rem. 3.2.6), with structure maps defined by:*

$$\begin{aligned} \mathsf{C}_n(U) \times (\Omega^n X)^{\times U} &\to \Omega^n X, & c(\gamma)\colon [0,1]^n &\to X, \\ (c, \gamma) &\mapsto c(\gamma); & x &\mapsto \begin{cases} \gamma_u(y), & \text{if } \exists (y,u) \text{ s.t. } x = c(y,u), \\ x_0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2.11}$$

This structure defines a rich structure on the homology of an iterated loop space. Note that if P is a topological operad, then $H_*(\mathsf{P})$ is a linear operad, and if A is a P-algebra, then $H_*(A)$ is an $H_*(\mathsf{P})$-algebra. Since C_n and D_n are (weakly) homotopy equivalent as topological operads, their homologies are isomorphic.

Proposition 3.2.12. *The homology of* D_1 *is the (linear version of the) operad* uAss *of Ex. 3.1.16.*

Theorem 3.2.13 (Cohen [42])**.** *Let $n \geq 2$. The homology of* D_n *is the operad* uPois_n *of unital n-Poisson algebras, which encodes objects A equipped with a commutative product of degree 0, a Lie bracket of (cohomological) degree $1-n$, such that the bracket is a biderivation with respect to the product, and a unit which is central for the Lie bracket.*

Remark 3.2.14. In positive characteristic, the homology of $\Omega^n X$ inherits even more structure than that of an n-Poisson algebra, because of the symmetric group actions. One recovers Dyer–Lashof operations.

Heuristically speaking, a D_n-algebra is an algebra equipped with a product that is associative up to strong homotopy, i.e.: there is a homotopy between $(a,b,c) \mapsto (ab)c$ and $(a,b,c) \mapsto a(bc)$; this induces two distinct homotopies between $(a,b,c,d) \mapsto a(b(cd))$ and $(a,b,c,d) \mapsto ((ab)c)d$, and there is a homotopy between; and so on, in every homotopical degree. Moreover, an algebra over D_n is homotopy commutative up to degree $n-1$, that is, in D_2 there is a homotopy between $(a,b) \mapsto ab$ and $(a,b) \mapsto ba$, but the composition may be a nontrivial loop (Fig. 3.2); in D_3, this loop is filled by a homotopy, but this may define a nontrivial sphere; and so on. At the level of homology, the mere existence of a homotopy makes the product into a strictly commutative one, but the nontrivial $(n-1)$-sphere is witnessed by the Lie bracket.

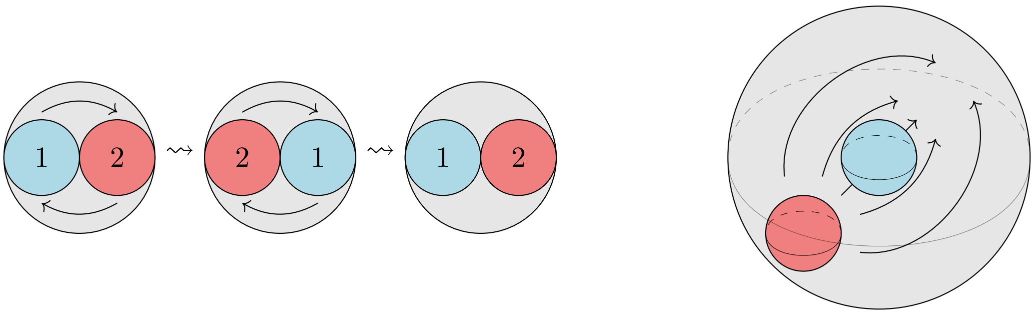


Figure 3.2: A homotopy between the product and the opposite product in D_2 yields a nontrivial loop. It becomes trivial in D_3, but the homotopy making it trivial (second picture) then yields a nontrivial sphere when composed with its mirror image.

The following recognition principle is a kind of converse to Prop. 3.2.10:

Theorem 3.2.15 (Stasheff [156] for $n=1$, Boardman–Vogt [28] and May [124] for all n)**.** *Let Y be a* D_n*-algebra (in a suitable category of topological spaces). Assume that the induced monoid structure on $\pi_0 Y$ defines a group. Then there exists a based space X and a zigzag of weak equivalences of* D_n*-algebras $Y \simeq \Omega^n X$.*

* * *

Let us now get back to the main point of this section. The relationship between D_n and configuration spaces comes from the following result:

Proposition 3.2.16. *For each $n \geq 0$ and finite set U, the "center" map defined next is a homotopy equivalence:*

$$\mathrm{ctr}\colon \mathsf{D}_n(U) \to \mathrm{Conf}_{\mathbb{R}^n}(U), \quad c \mapsto (c(0,u))_{u \in U}. \tag{3.2.17}$$

Moreover, if a manifold M is framed (i.e., it is equipped with a trivialization of its tangent bundle), one can also define an operadic structure on the configuration spaces of M up to homotopy.

Definition 3.2.18. Let M be a closed smooth manifold of dimension n equipped with a framing, i.e., a bundle isomorphism $\tau \colon TM \to M \times \mathbb{R}^n$. The symmetric collection D_M is defined on a finite set U as the set of pairs of smooth maps and numbers $(c \colon \mathbb{D}^n \times U \to M,\ \lambda > 0)$ such that:

- for all $u \in U$, $c(\_, u)$ is an embedding;
- for all $u \neq v \in U$, the sets $c(\mathring{\mathbb{D}}^n, u)$ and $c(\mathring{\mathbb{D}}^n, v)$ are disjoint;
- for all $u \in U$, the following diagram commutes:

$$\begin{tikzcd} T\mathbb{D}^n \times U \arrow[rr, "dc"] \arrow[d, "\text{canon.}"', "\cong"] & & TM \arrow[d, "\cong"', "\tau"] \\ \mathbb{D}^n \times \mathbb{R}^n \times U \arrow[rr, "(x,v,u) \mapsto (c(x,u), \lambda v)"'] & & M \times \mathbb{R}^n. \end{tikzcd} \tag{3.2.19}$$

Proposition 3.2.20. *Let M be as in the previous definition. For all U, the "center" map $\mathrm{ctr} \colon \mathsf{D}_M(U) \to \mathrm{Conf}_M(U)$ is a homotopy equivalence.*

Proposition 3.2.21. *Let M be as above. Composition of embeddings makes D_M into a right D_n-module.*

If the manifold M is not framed but merely oriented, the above definition does not make sense. However, it is possible to build an operadic structure out of the framed configuration spaces of M (Eq. (2.4.30)) and the framed little disks operads.

Definition 3.2.22 (Salvatore–Wahl [147])**.** Let G be a (topological) group and P be an operad in the category of G-spaces. The semidirect product $\mathsf{P} \rtimes G$ is the topological operad whose underlying symmetric collection is given by:

$$(\mathsf{P} \rtimes G)(U) \coloneqq \mathsf{P}(U) \times G^{\times U}. \tag{3.2.23}$$

The unit is $(\mathrm{id}_{\mathsf{P}}, 1)$ and the morphisms of FB act diagonally on the product (by permutation of $G^{\times(\_)}$). The partial composite of $(p, (g_{[u]})) \in \mathsf{P}(U/W) \times G^{\times(U/W)}$ and $(q, (h_w)) \in \mathsf{P}(W) \times G^{\times W}$ is given by:

$$(p, (g_{[u]})) \circ_W (q, (h_w)) \coloneqq (p \circ_W g_{[W]} q, (k_u)), \tag{3.2.24}$$

where $(k_u) \in G^{\times U}$ is defined by $k_u = g_u$ for $u \notin W$ and $k_w = g_{[W]} h_w$ for $u \in W$.

Definition 3.2.25. The framed little n-disks operad $\mathsf{D}_n^{\mathrm{fr}}$ is the semidirect product $\mathsf{D}_n \rtimes \mathrm{SO}(n)$, where $\mathrm{SO}(n)$ acts on $\mathsf{D}_n(U)$ by postcomposition.

Concretely, an element of $\mathsf{D}_n^{\mathrm{fr}}(U)$ is a configuration of interior-disjoint little n-disks obtained by positive rescaling, translation, and viewed as being postcomposed by an oriented isometry (which does not change the image of the embedding). Composition is the same as in D_n, except that when a configuration of disks is inserted into a slot, the whole configuration is moved by the isometry at that slot. See Fig. 3.3

Definition 3.2.26. Let M be a closed oriented smooth manifold. The symmetric collection $\mathsf{D}_M^{\mathrm{fr}}$ is defined just like D_M (Def. 3.2.18), but the requirement that the differential of each embedding is a positive rescaling is dropped and replaced with the condition that each embedding preserves orientation.

Let $(e_i)_{1 \leq i \leq n}$ be the canonical basis of $T_0\mathbb{D}^n = \mathbb{R}^n$.

Proposition 3.2.27. *The "center + frame" map defined next is a homotopy equivalence:*

$$\mathsf{D}_M^{\mathrm{fr}}(U) \to \mathrm{Conf}_M^{\mathrm{fr}}(U), \qquad c \mapsto \bigl(c(0,u), (dc((0,u), e_i))_{1 \leq i \leq n}\bigr). \tag{3.2.28}$$

Proposition 3.2.29. *Composition of embeddings define a right $\mathsf{D}_n^{\mathrm{fr}}$-module structure on the symmetric collection $\mathsf{D}_M^{\mathrm{fr}}$.*

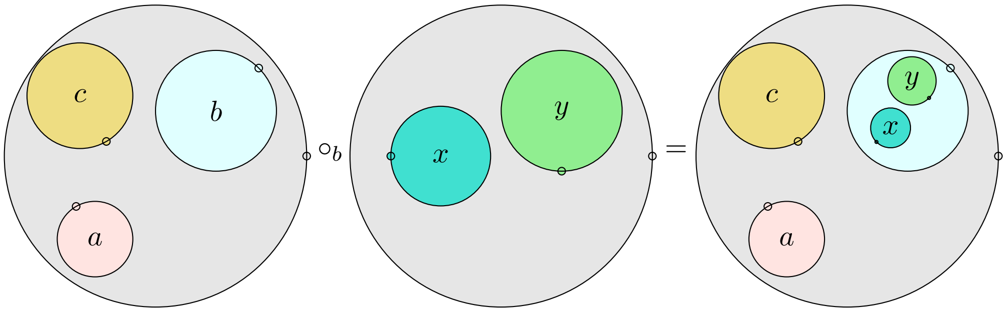


Figure 3.3: Example of the structure map $\circ_b\colon \mathsf{D}_2^{\mathrm{fr}}(\{a,b,c\}) \times \mathsf{D}_2^{\mathrm{fr}}(\{x,y\}) \to \mathsf{D}_2^{\mathrm{fr}}(\{a,x,y,c\})$. The small notches represent rotations compared to the horizontal axis.

3.3 Homotopy types of operads and modules

The results presented in Sec. 2, such as the formality of $\mathrm{Conf}_{\mathbb{R}^n}(U)$ or the computation of the real homotopy type of $\mathrm{Conf}_M(U)$, were all presented purely in terms of homotopy types of spaces. All these results can be upgraded in operadic terms: the models for Conf_M given are compatible with the operadic structures involved.

Properly defining rational or real homotopy theory for operads and modules is, however, somewhat tricky. The main issue is that the functor of forms Ω^* (either Ω^*_{PL} or Ω^*_{PA}, see Sec. 2.3) is contravariant. One could think that if P is a topological operad, then $\Omega^*(\mathsf{P})$ would form a cooperad:

Definition 3.3.1. Let $\mathcal{C}$ be a symmetric monoidal category. A cooperad in $\mathcal{C}$ is a symmetric collection C endowed with a counit $\varepsilon\colon \mathsf{C}(\{*\}) \to I$ and, for all pairs of finite sets $W \subseteq U$, with cocomposition maps:

$$\circ_W^\vee\colon \mathsf{C}(U) \to \mathsf{C}(U/W) \otimes \mathsf{C}(W), \tag{3.3.2}$$

satisfying equivariance, counit, and coassociativity conditions. Operadic left (resp., right, bi-) modules are defined analogously.

Definition 3.3.3. If $\mathcal{C}$ is the category of CDGAs, then cooperads and comodules in $\mathcal{C}$ are called "Hopf cooperads" and "Hopf comodules."

Remark 3.3.4. Working with cooperads is the main motivation for using symmetric collections instead of symmetric sequences. Tracking the numerical indices in the cooperad structure can become quite involved and error-prone.

In general, though, $\Omega^*(\mathsf{P})$ does *not* define a cooperad for an operad P. The issue is that the Künneth quasi-isomorphism goes in the wrong direction and is not strictly invertible, so that we only get a map in the homotopy category, i.e., a zigzag:

$$\Omega^*(\mathsf{P}(U)) \xrightarrow{\circ_W^*} \Omega^*(\mathsf{P}(U/W) \times \mathsf{P}(W)) \xleftarrow{\sim} \Omega^*(\mathsf{P}(U/W)) \otimes \Omega^*(\mathsf{P}(W)). \tag{3.3.5}$$

There exist several ways around this issue. One direction, taken in [112], is to define an ad-hoc notion of a "morphism into" $\Omega^*(\mathsf{P})$ and a notion of zigzag of quasi-isomorphisms (i.e., weak equivalence) between a genuine Hopf cooperad and $\Omega^*(\mathsf{P})$. Another (see e.g., [98]) is to view $\Omega^*(\mathsf{P})$ as a Hopf cooperad up to homotopy. While these approaches are fruitful, it remains to prove that the resulting homotopy category is equivalent to that of topological operads (involving simply connected spaces of finite) up to rational/real homotopy equivalence.

The most complete approach is the one from [68], who upgraded the functor Ω^*_{PL} into a functor $\Omega^*_\sharp$ from topological (or rather simplicial) operads P such that $\mathsf{P}(0) = \mathsf{P}(1) = \{*\}$ to the category of Hopf cooperads. A morphism into $\Omega^*_\sharp(\mathsf{P})$ is equivalent to a "morphism into" $\Omega^*_{\mathrm{PL}}(\mathsf{P})$ as defined in [112], and if P is cofibrant, then $\Omega^*_\sharp(\mathsf{P}(U)) \simeq \Omega^*_{\mathrm{PL}}(\mathsf{P}(U))$. Moreover, the functor $\Omega^*_\sharp$ is indeed an equivalence on homotopy category when restricted to operads with simply connected finite-type components. This construction was refined and generalized to apply to operads that do not necessarily satisfy $\mathsf{P}(1) = \{*\}$ [69]. The theory also applies *mutatis mutandis* if Ω^*_{PL} is replaced by Ω^*_{PA} [5, Rem. 7] and adaptations can be made for operadic right modules [73].

As mentioned in Sec. 2, the configuration spaces $\mathrm{Conf}_{\mathbb{R}^n}(U)$ are formal as topological spaces. But as we saw in Sec. 3.2, each $\mathrm{Conf}_{\mathbb{R}^n}(U)$ is homotopy equivalent to $\mathsf{D}_n(U)$, so that the cohomology of these configuration spaces are equipped with a Hopf cooperad structure. This structure is given, for a pair of finite sets $W \subseteq U$ and using the presentation of Th. 2.4.1, by the following maps [42]:

$$\begin{aligned} \circ^\vee_W \colon H^*(\mathsf{D}_n(U)) &\to H^*(\mathsf{D}_n(U/W)) \otimes H^*(\mathsf{D}_n(W)), \\ \omega_{uv} &\mapsto \begin{cases} 1 \otimes \omega_{uv}, & \text{if } u, v \in W, \\ \omega_{[u][v]} \otimes 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3.6}$$

The description of $H^*(\mathsf{D}_n)$ in terms of graphs (see the discussion following Th. 2.4.1) makes the cooperad structure map easier to handle. Given a graph $\Gamma \in H^*(\mathsf{D}_n(U))$, one has $\circ^\vee_W(\Gamma) = \Gamma_{U/W} \otimes \Gamma_W$, where Γ_W is the full subgraph of Γ on the vertex set W, and $\Gamma_{U/W}$ is the graph obtained from Γ by collapsing Γ_W to a single vertex.

Th. 2.4.9 can be upgraded to be a statement about the homotopy type of the operad D_n:

Theorem 3.3.7 (Kontsevich [105], Tamarkin [159], Lambrechts–Volić [112], Petersen [131], Fresse–Willwacher [72], and Boavida de Brito–Horel [29])**.** *For all $n \geq 1$, the operad D_n is formal over $\mathbb{Q}$, i.e., there exists a zigzag of weak equivalences of Hopf cooperads $\Omega^*_\sharp(\mathsf{D}_n) \simeq H^*(\mathsf{D}_n)$.*

Remark 3.3.8. An even stronger statement is true [72]: D_n is intrinsically formal, i.e., any $\mathbb{Q}$-good connected operad with the same homology as D_n as a Hopf operad, with an extra condition when $n \equiv 0 \pmod 4$, is formal and thus rationally equivalent to D_n.

For the framed little disks operads, the situation is subtler.

Theorem 3.3.9 (Giansiracusa–Salvatore [80] and Ševera [152] for $n = 2$, Moriya [130] for odd n, Khoroshkin–Willwacher [98] for all n)**.** *Let $n \geq 2$. The framed little n-disks operad $\mathsf{D}^{\mathrm{fr}}_n$ is formal if and only if n is even.*

Remark 3.3.10. Note that the space $\mathsf{D}^{\mathrm{fr}}_n(U) \simeq \mathrm{Conf}_{\mathbb{R}^n}(U) \times \mathrm{SO}(n)^{\times U}$ is formal for all $n \geq 1$ and U. The non-formality result for odd n is truly about the operadic structure.

Remark 3.3.11. In fact, a stronger statement is true: the (genuine) dg-operad given by the chains $C_*(\mathsf{D}^{\mathrm{fr}}_n)$ is not formal for odd n.

Remark 3.3.12. The proofs of the formality of $\mathsf{D}^{\mathrm{fr}}_2$ [80, 152] are fairly explicit. For even $n \geq 4$, though, the proof of [98] relies on obstruction-theoretical arguments.

The previous theorems and their proofs have important consequences, e.g., in deformation quantization [106] or embedding calculus [71].

The proof of Kontsevich [105] (as completed by Lambrechts–Volić [112]) is of particular interest to us. Recall that if M is a closed framed manifold, then the symmetric collection Conf_M is a right module over D_n up to homotopy. This structure should be reflected on models of Conf_M, i.e., it is possible to find models of Conf_M that assemble to right Hopf $H^*(\mathsf{D}_n)$-comodules. Similarly, if M is just oriented, then $\mathrm{Conf}^{\mathrm{fr}}_M$ is a right module over $\mathsf{D}^{\mathrm{fr}}_n$ up to homotopy, and one should be able to find models of these configuration spaces that assemble to right Hopf comodules over a model of $\mathsf{D}^{\mathrm{fr}}_n$. By adapting and generalizing the methods of [105, 112], we get:

Theorem 3.3.13 ([5, 34, 2, 3, 4])**.** *The models given in Th. 2.4.18, 2.4.31, 2.4.34, and 2.4.52 are compatible with the actions of D_n or $\mathsf{D}_n^{\mathrm{fr}}$ (depending of the theorem).*

In the work of Khoroshkin–Willwacher [98], the authors study the non-oriented framed little disks operad, $\mathsf{D}_n \rtimes \mathrm{O}(n)$ (rather than $\mathsf{D}_n \rtimes \mathrm{SO}(n)$). The following question is natural, but some technical difficulties arise from the disconnected nature of $\mathrm{O}(n)$:

Question 3.3.14. *Can the result of Th. 2.4.31 be upgraded to the non-oriented case?*

3.4 Graph complexes

The proofs of the results summarized by Th. 3.3.13 all involve graph complexes. Graph complexes are combinatorial objects introduced by Kontsevich [104] based on the perturbative expansion of Chern–Simons theory (see Axelrod–Singer [16, 17]). While they find their roots in mathematical physics, their uses have expanded far beyond, see Willwacher [171] for a survey.

In this section, we give a brief introduction to graph complexes, we explain how they relate to configuration spaces, and we give the results of some key computations necessary to prove the results of Sec. 2.

Remark 3.4.1. In what follows, we consider the cohomological versions of the graph complexes, i.e., differentials contract edges, as the primary objects. Most of the literature considers the homological versions as the primary objects. The homological version is linearly dual to the cohomological one. Since graph complexes are often infinite-dimensional, linear duality is a one-way road, so as a matter of personal preference, we define the predual rather than the dual. Notation regarding the restrictions on valences of vertices is also variable in the literature.

Let $n \in \mathbb{Z}$ be any integer (typically, the dimension of a manifold, but not necessarily). Let us define the cohomological version GC_n of the simplest version kind of graph complexes, due to Kontsevich [104]. To define it, we consider the set of *connected* graphs $\Gamma = (V_\Gamma, E_\Gamma)$ with some finite set of vertices V_Γ and some finite set of edges E_Γ. A half-edge of Γ is a pair (e, ε) where $e \in E_\Gamma$ and $\varepsilon \in \{+1, -1\}$. The degree of such a graph Γ is:

$$\deg(\Gamma) \coloneqq (n-1)\#E_\Gamma - n\#V_\Gamma - n. \tag{3.4.2}$$

The notion of "orientation set" of such a graph differs depending on the parity of n.

- If n is even, then the orientation set of Γ is the set of edges of Γ.
- If n is odd, then the orientation set of Γ is the pair of sets given by the vertices of Γ and the set of half-edges of Γ.

We consider the vector space $\mathrm{GC}_n^{\circlearrowleft}$ consisting of formal linear combinations of pairs (Γ, o), where Γ is a graph and o is an ordering of the orientation set of Γ, modulo the relation $(\Gamma, o) \sim \varepsilon(\sigma)(\Gamma, o \cdot \sigma)$ for all permutations σ of the orientation set (with $\varepsilon(\sigma)$ being the sign of σ).

Remark 3.4.3. We often allow ourselves to drop o from the notation and the pictures. However, we have to keep in mind it is necessary to consider it to get well-defined formulas and signs.

Remark 3.4.4. The behavior of the symmetries greatly differs depending on the parity of n. If n is even, for example, any graph containing a multiple edge (i.e., two or more edges with the same extremities) vanishes in the quotient, as exchanging the order of the edges yields an odd symmetry. Similarly, if n is odd, then any graph containing a tadpole (i.e., an edge whose two extremities are equal) vanishes in the quotient.

We now define a differential d on $\mathrm{GC}_n^{\circlearrowleft}$. Given some graph Γ as above, its differential is a sum of edge contractions:

$$d(\Gamma) \coloneqq \sum_{e \in E_\Gamma} \pm \Gamma/e, \tag{3.4.5}$$

where Γ/e is the graph obtained from Γ by contracting the edge e, i.e., by merging the two extremities of e and removing it from the new graph. See Fig. 3.4 for an example. One has to be mindful of the orientation set to define the $\pm$ sign properly.

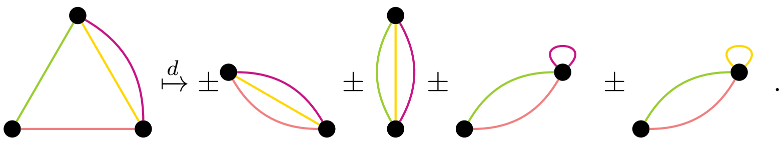


Figure 3.4: An example of differential on $\mathrm{GC}_n^{\circlearrowleft}$. We color the edges differently so that one can see better the differential, but depending on the parity of n, edges may or may not be ordered. Note that the last two graphs in $d(\Gamma)$ must vanish no matter the parity of n; the first two are equal up to sign.

There are two special cases for the differential d:

- If $\Gamma = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ consists of exactly two vertices and a single edge between then, then $d(\Gamma) = -\bullet$ is the opposite of the graph with a single vertex and no edge;
- If Γ is not the graph of the previous case, and if an edge e is adjacent to a vertex which is adjacent only to e, then Γ/e is dropped from the sum defining $d(\Gamma)$.

Remark 3.4.6. These oddities can be explained by the fact that the corresponding summands actually appear either three times (with alternating signs) or twice (with opposite signs) in the sums. See Remark 3.4.13.

Definition 3.4.7. The graph complex (with loops) in dimension n is the vector space $\mathrm{GC}_n^{\circlearrowleft}$ defined above equipped with the differential d.

Remark 3.4.8. The above definition is self-contained but rather "hands-on." It is possible to define graph complexes using the general theory of operadic twisting [53, 54].

We now define interesting quotients of $\mathrm{GC}_n^{\circlearrowleft}$ based on valence, i.e., the number of edges incident to a given vertex.

Lemma 3.4.9. *The quotient $\mathrm{GC}_n^{\geq 1}$ of $\mathrm{GC}_n^{\circlearrowleft}$ by the subspace spanned by graphs containing tadpoles and/or multiple edges is a quotient complex. The quotients $\mathrm{GC}_n^{\geq 2}$ (resp., $\mathrm{GC}_n^{\geq 3}$) of $\mathrm{GC}_n^{\geq 1}$ by the subspace spanned by graphs containing at least a vertex of valence < 2 (resp., < 3) is a quotient complex.*

Proposition 3.4.10. *The cochain complex $\mathrm{GC}_n^{\circlearrowleft}$ is equipped with a Lie coalgebra structure given by the sum of all possible ways of contracting subgraphs (not necessarily full):*

$$\delta \colon \mathrm{GC}_n^{\circlearrowleft} \to (\mathrm{GC}_n^{\circlearrowleft})^{\wedge 2}, \quad \Gamma \mapsto \sum_{\Gamma' \subseteq \Gamma} \pm \Gamma/\Gamma' \wedge \Gamma'. \tag{3.4.11}$$

This structure is compatible with the quotients defining $\mathrm{GC}_n^{\geq 1}$, $\mathrm{GC}_n^{\geq 2}$, and $\mathrm{GC}_n^{\geq 3}$.

Remark 3.4.12. As mentioned above, we are using slightly nonstandard notation. The linear dual of $\mathrm{GC}_n^{\geq 3}$ is what is usually called the (Kontsevich) graph complex. This dual is equipped with a more familiar Lie algebra structure, rather than a coalgebra structure.

Remark 3.4.13. With the Lie coalgebra structure, we can give the "true" definition of the differential of $\mathrm{GC}_n^{\circlearrowleft}$. Let $z : \mathrm{GC}_n^{\circlearrowleft} \to \mathbb{Q}$ be the map which evaluates to 1 on the graph with a single edge and a single vertex, and 0 on all other graphs. Then z is a Maurer–Cartan element for the Lie coalgebra

$(\mathrm{GC}_n^{\circlearrowleft}, d = 0)$, in the sense that $(z \otimes z)\delta = 0$. The differential on $\mathrm{GC}_n^{\circlearrowleft}$ is then the twist with respect to this Maurer–Cartan element, i.e.,

$$d(\Gamma) = (z \otimes \mathrm{id} + \mathrm{id} \otimes z)\delta(\Gamma). \tag{3.4.14}$$

This explains the "oddities" of Rem. 3.4.6. Indeed, for a subgraph $\Gamma' \subseteq \Gamma$, we have $(z \otimes \mathrm{id})\delta(\Gamma) = 0$, unless $\Gamma \setminus \Gamma'$ consists of a single edge. This happens twice if Γ is the graph consisting of two vertices and a single edge (once for each of the two vertices), and otherwise once for every edge attached to a univalent vertex.

Despite the apparent simplicity of the definition, the (co)homologies of the graph complexes are tremendously difficult to compute. Willwacher [168] proved that GC_n is quasi-isomorphic to $\mathrm{GC}_n^{\geq 2}$, and that the homology of $\mathrm{GC}_n^{\geq 2}$ is equal to that of $\mathrm{GC}_n^{\geq 3}$ plus an extra summand given by loop graphs. The heart of the difficult thus lies in $\mathrm{GC}_n^{\geq 3}$. As a striking example, let us mention:

Theorem 3.4.15 (Willwacher [168, Th. 1.1])**.** *The homology of the dual of* $\mathrm{GC}_2^{\geq 3}$ *in degree zero is isomorphic to the Grothendieck–Teichmüller Lie algebra* $\mathfrak{grt}_1$*.*

The Lie algebra $\mathfrak{grt}_1$ is the Lie algebra of a prounipotent group GRT_1 defined by Drinfeld [55]. This group and its variants appear in a dizzying array of contexts: initially in the study of the absolute Galois group of $\mathbb{Q}$ [86], in the study of quantum groups, deformation quantization of Lie bialgebras and Poisson manifolds, polyzeta values, and homotopy automorphisms of the little disks operad. See [149] or [67, Chapter 12] for more information.

* * *

Graph complexes feature prominently in the proof of the formality of D_n by Kontsevich [105]. To explain how they appear, we first introduce operads equivalent to the little n-disks operads, the Fulton–MacPherson operads FM_n. These operads are built out of compactifications of the configuration spaces of $\mathbb{R}^n$ that were initially introduced by Axelrod–Singer [17] (and that are analogous to constructions developed by Fulton–MacPherson [75] in the complex setting). We refer to Sinha [154] and Lambrechts–Volić [112] for an extensive treatment.

To explain these compactifications, let us consider the space $\mathrm{Conf}_{\mathbb{R}^n}(U)$. Heuristically, this space is noncompact for two reasons: a configuration can grow infinitely big or escape to infinity, and two or more point can converge to the same location.

The first reason comes from noncompactness of $\mathbb{R}^n$ and it is easy to deal with it: we can simply mod out by the group of translations and positive rescalings to obtain the $(n\#U - n - 1)$-dimensional manifold

$$\mathrm{Conf}_{\mathbb{R}^n}[U] \coloneqq \mathrm{Conf}_{\mathbb{R}^n}(U)/(\mathbb{R}^n \rtimes \mathbb{R}_{>0}). \tag{3.4.16}$$

Geometrically, one can imagine that we consider configurations of points with isobarycenter at the origin and diameter 1.

To deal with the second reason for noncompactness, though, another approach is needed. A class in the quotient is completely determined by the pairwise angles between two points, and by the relative distances between three points, i.e., by the images of the following maps (defined for $u \neq v \neq w \neq u \in U$):

$$\begin{aligned} \theta_{uv} \colon \mathrm{Conf}_{\mathbb{R}^n}[U] &\to S^{n-1} & \qquad \delta_{uvw} \colon \mathrm{Conf}_{\mathbb{R}^n}[U] &\to [0, +\infty] \\ [x] &\mapsto \frac{x_v - x_u}{\|x_v - x_u\|}; & [x] &\mapsto \frac{\|x_w - x_u\|}{\|x_v - x_u\|}. \end{aligned} \tag{3.4.17}$$

Definition 3.4.18. The space $\mathsf{FM}_n(U)$ is the closure of the image of $\mathrm{Conf}_{\mathbb{R}^n}[U]$ in $\left(S^{n-1}\right)^{\mathrm{Conf}_U(2)} \times [0, +\infty]^{\mathrm{Conf}_U(3)}$.

Intuitively, an element of $\mathsf{FM}_n(U)$ can be seen as a configuration of points, where two or more points are allowed to become infinitesimally close relative to the other points. However, in that situation, we keep all the information (angles, relative distances) necessary to reconstruct a configuration out of all the collided points, see Fig. 3.5. It is then possible that inside that infinitesimal configuration, some points become even closer together relative to the other points, and so on. In that way, $\mathsf{FM}_n(U)$ consists of nested configurations.

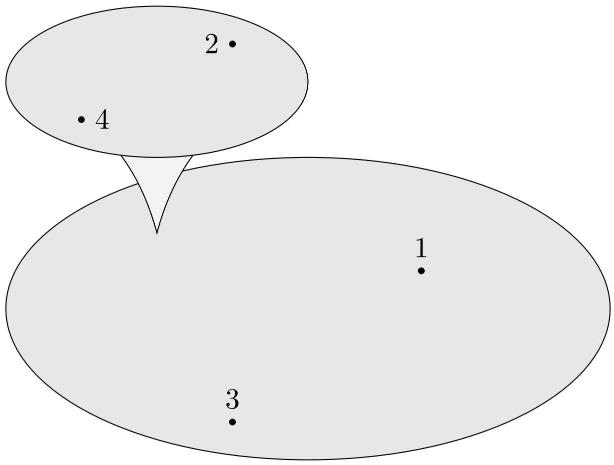


Figure 3.5: An element of $\mathsf{FM}_2(4)$. The points 2 and 4 are infinitesimally close to each other, with a macroscopic position indicated by the "looking glass."

Proposition 3.4.19. *The space* $\mathsf{FM}_n(U)$ *is a compact manifold with boundary of dimension* $n\#U-n-1$ *(or* 0 *if* $\#U \leq 1$*) whose interior is* $\mathrm{Conf}_{\mathbb{R}^n}[U]$*. The collection* FM_n *assembles to form a topological operad.*

Given $x \in \mathsf{FM}_n(U/W)$ and $y \in \mathsf{FM}_n(W)$, the element $x \circ_W y \in \mathsf{FM}_n(U)$ is a configuration composed of the points of x, with $x_{[W]}$ replaced with an infinitesimal configuration consisting of the points of y. For example, the element of Fig. 3.5 is in the image of $\circ_{\{2,4\}} \colon \mathsf{FM}_2(\{1,3,*\}) \times \mathsf{FM}_2(\{2,4\}) \to \mathsf{FM}_2(4)$. The operad structure maps are defined explicitly in terms of the "coordinates" θ_{uv} and δ_{uvw}, and it is instructive to figure out the formulas (see [112]).

Theorem 3.4.20 (Salvatore [144])**.** *The operad* D_n *is weakly equivalent to* FM_n*.*

Example 3.4.21. The spaces $\mathsf{FM}_n(0)$ and $\mathsf{FM}_n(1)$ are singletons. The map $\theta_{12} \colon \mathsf{FM}_n(2) \to S^{n-1}$ is a homeomorphism.

Example 3.4.22. The space $\mathsf{FM}_1(U)$ is a union of $(\#U)!$ copies of associahedra, which were introduced by Stasheff [156].

The boundary of $\mathsf{FM}_n(U)$ is particularly interesting. It decomposes into a union of "faces" (of codimension 1 in $\mathsf{FM}_n(U)$) whose pairwise intersections are of codimension ≥ 2, i.e., we may view $\mathsf{FM}_n(U)$ as a manifold with corners. These faces are precisely the images of the structure maps $\circ_W \colon \mathsf{FM}_n(U/W) \times \mathsf{FM}_n(W) \to \mathsf{FM}_n(U)$ for $\#W \geq 2$.

Moreover, let us consider the projection $\pi \colon \mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)$ which, for finite sets U, I, forgets the points indexed by I in a configuration. This is a fiber bundle, but not a submersion in general [112, Ex. 5.9.1]. The fiberwise boundary of π is the subspace:

$$\mathsf{FM}_n^{\partial}(U; I) \coloneqq \bigcup_{x \in \mathsf{FM}_n(U)} \partial\pi^{-1}(\{x\}). \tag{3.4.23}$$

This is a subspace of $\partial\mathsf{FM}_n(U \sqcup I)$ which consists of the faces $\mathrm{im}(\circ_W)$ for $W \subseteq U \sqcup I$ satisfying $\#W \leq 2$, and $U \subseteq W$ or $\#(W \cap U) \leq 1$.

The heart of the proof of Kontsevich [105] (reformulated by Lambrechts–Volić [112]) is then to build a CDGA $\mathsf{Graphs}_n(U)$ out of graphs which sits in a zigzag of quasi-isomorphisms:

$$H^*(\mathsf{D}_n(U)) \xleftarrow{\sim} \mathsf{Graphs}_n(U) \xrightarrow{\sim} \Omega^*(\mathsf{FM}_n(U)). \tag{3.4.24}$$

The CDGA $\mathsf{Graphs}_n(U)$ is spanned by graphs similar to those that span $\mathrm{GC}_n^{\geq 3}$, except they now have two kinds of vertices. The first kind, called internal vertices, are indistinguishable among themselves and behave similarly to the vertices of graphs in $\mathrm{GC}_n^{\geq 3}$. The second kind, called external vertices, are equipped with a bijection to U and are seen as "fixed." The degree of a graph Γ is $(n-1)\#E_\Gamma - n\#I_\Gamma$, where E_Γ is the set of edges and I_Γ is the set of internal vertices. Graphs are not necessarily connected anymore, but we mod out by graphs where a connected component contains only internal vertices (also known as internal components).

The differential is defined analogously to that of $\mathrm{GC}_n^{\geq 3}$, except that edges between two external vertices are not contracted, and an edge between an external and an internal vertex results in an external vertex with the same label as the one of the original external vertex – see Fig. 3.6 for an example. Finally, the product consists in gluing graphs together at the external vertices, and the cooperad structure is given by subgraph contraction.

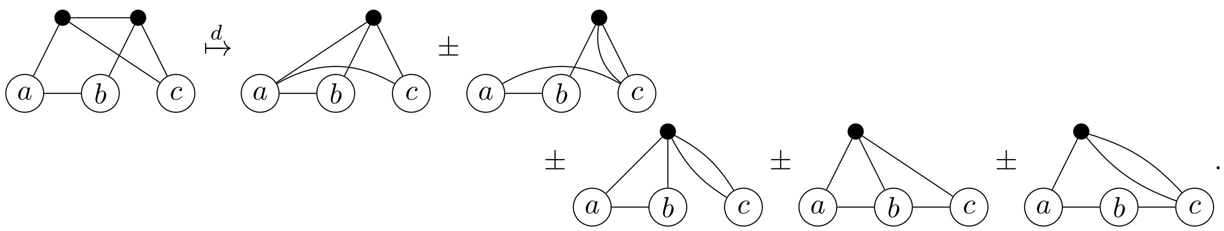


Figure 3.6: The differential in $\mathsf{Graphs}_n(\{a, b, c\})$. The graphs containing double edges are actually modded out in the definition of the CDGA.

Remark 3.4.25. The CDGA $\mathsf{Graphs}_n(U)$ is quasi-free, i.e., free as a CGA. Its generators are given by "internally connected graphs," i.e., graphs that remain connected when internal vertices are removed (but dangling edges are kept).

The left map in Equation (3.4.24) is merely the quotient map that mods out the graphs containing internal vertices, and which sends an edge e_{uv} between $u, v \in U$ to the generator $\omega_{uv} \in H^{n-1}(\mathrm{Conf}_{\mathbb{R}^n}(U))$. Proving that this map is well defined, that it commutes with the cooperad structure maps, and that it is a quasi-isomorphism, is nontrivial but purely algebraic.

The right map in Equation (3.4.24) is much more difficult to construct. If $\Gamma \in \mathsf{Graphs}_n(U)$ is a graph, we can temporarily index its internal vertices by a set I. If $\mathrm{vol}_{n-1} \in \Omega^{n-1}(S^{n-1})$ is a volume form, then the take the wedge product over all the edges of Γ:

$$I'(\Gamma) \coloneqq \bigwedge_{e \in E_G} \theta^*_{s(e)t(e)}(\mathrm{vol}_{n-1}) \in \Omega^{(n-1)\#E_\Gamma}(\mathsf{FM}_n(U \sqcup I)). \tag{3.4.26}$$

Then, we perform the integration along the fibers of the projection map $\pi \colon \mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)$:

$$I(\Gamma) \coloneqq \pi_*(I'(\Gamma)) = \int_{\mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)} I'(\Gamma) \in \Omega^{\deg(\Gamma)}(\mathsf{FM}_n(U)). \tag{3.4.27}$$

Then the Stokes formula, the description of the fiberwise boundary of π, and computations of special cases of integrals, show that $I(d\Gamma) = dI(\Gamma)$. A key part of the argument is checking that I vanishes on graphs with internal components, which can be produced by the cooperad structure (if one contracts a disconnected subgraph). While for $n = 2$ this computation is done by ad-hoc means (the Kontsevich symmetry trick [104, Lem. 2.1]), for $n \geq 3$ this follows from a degree counting argument. This proves that integrating away an internal component is the same thing as applying the Maurer–Cartan element z from Rem. 3.4.13.

Similar tools are used to prove that I is a CDGA map, and that it commutes with the cooperad structure. Finally, every generator of the cohomology is clearly hit by I, as they are images of graphs with no internal vertices and exactly one edge. Since Graphs_n was previously shown to have the correct cohomology, this ends the proof.

The devil is, of course, in the details. One of the most difficult parts of the proof is to define the integration map π_* properly, and check that it satisfies various properties expected of integration along the fibers of a bundle. Such a construction is lacking for piecewise linear forms. It exists for de Rham forms and submersions; however, as previously stated, the projections $\mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)$ are, in general, *not* submersions. These projections are, however, semi-algebraic bundles. Hardt–Lambrechts–Turchin–Volić [89], based on insights of Kontsevich–Soibelman [107], developed in depth the homotopy theory of semi-algebraic sets, piecewise semi-algebraic forms, and integration along the fibers of semi-algebraic bundles. This allowed Lambrechts–Volić [112] to complete the proof whose steps we outlined above.

* * *

We view this proof as a template to find models of configuration-space-like collections $C(M)$ of some manifold M. The steps of that template are essentially the following:

1. Define suitable compactifications $C[M]$ of $C(M)$, to ensure that integrals converge and to define operadic structures. Prove that the projections $C_{k+r}(M) \to C_k(M)$ extend to fiber bundle maps on the compactifications.
2. Find a candidate operadic model $G(M)$ for $C(M)$. For example, $H^*(\mathsf{D}_n)$, G_A, $\tilde{\mathsf{G}}_P$, $\mathsf{G}^{\mathrm{fr}}_{S_g}$ from Sec. 2.3.
3. Define an "algebraic" resolution $R^0(M)$ of $G(M)$ using graph-like complexes and check (algebraically) that $R^0(M)$ is quasi-isomorphic to $G(M)$ as an operadic object.
4. Internal components define a graph complex $\mathrm{GC}_{C,M}$, and integrating away these internal components defines a "transcendental" Maurer–Cartan element $z \in \mathrm{GC}_{C,M}$. Define a "transcendental" graph-like resolution $R^z(M)$, obtained by twisting $R^0(M)$ by z.
5. Define integral maps $I\colon R^z(M) \to \Omega^*(C[M])$ compatible with operadic structures.
6. Prove that the homology of $\mathrm{GC}_{C,M}$ vanishes in just the right place to deduce that the transcendental Maurer–Cartan element z is trivial up to homotopy (i.e., gauge equivalent to 0), thus proving that $R^0(M)$ and $R^z(M)$ are equivalent as operadic objects.

This template, or parts of it, was applied to get the results summarized by Th. 3.3.13.

Remark 3.4.28. The algebraic side of this template (and much more) has now been developed in depth by Willwacher [172].

Remark 3.4.29. For D_n, steps 4 and 6 are missing, as integrating away internal components already gives the same Maurer–Cartan element as the algebraic resolution. However, in general, these steps are necessary and highly nontrivial.

Remark 3.4.30. As evidenced by the large and complicated zeroth homology of GC_2^3, implementing step 6 is often hard. There is now a large literature on the computation of graph homology, including a variety of techniques – too large and quickly moving to reproduce here. Let us perhaps mention, at the risk of being less than exhaustive, the results of Khoroshkin–Willwacher–Živković [99] and its sequels, Felder–Naef–Willwacher [58], Bar-Natan–McKay [21]...

Computations of graph homology can prove invaluable. Their uses include the computation homotopy invariants of the spaces of long knots [13, 70, 71], or the cohomology of moduli spaces of curves [35].

As an example of the type of question that we would like to see answered as part of this program, let us give an example for step 1 that we have found arduous:

Question 3.4.31. *How to define an analogue of the Fulton–MacPherson compactification for the non-k-equal configuration spaces of Eq. (2.1.3)?*

3.5 Swiss-Cheese operads

Let us conclude this part with some results that are purely operadic in nature. These results concern the Swiss-Cheese operads SC_n and its variants. The Swiss-Cheese operads were introduced by Voronov [167] based on the use of configuration spaces of the upper half-plane by Kontsevich [106] and the study of open-closed string theory by Zwiebach [173]. The operad SC_n, for $n \geq 1$, is a colored operad that encodes a structure on a pair of objects (A, B), namely, the data of a D_n-structure on A, a D_{n-1}-structure on B, and (heuristically) a morphism of D_n-algebras from A to the "center" of B.

We will not define colored operads in general. Let us just say that colored operads are to operads what categories are to monoids. We will only define a special kind of colored operads that fit the case of SC_n, namely, relative operads, also known as Swiss-Cheese type operads.

Definition 3.5.1. Let P be an operad in a symmetric monoidal category $\mathcal{C}$. A relative P-operad is an operad in the category of right P-modules.

As this definition is quite compact, let us unpack it. A relative P-operad is a bisymmetric collection, i.e., a functor $\mathsf{Q}\colon \mathsf{FB}^{\mathrm{op}} \times \mathsf{FB}^{\mathrm{op}} \to \mathcal{C}$. For an element $q \in \mathsf{Q}(U, V)$, the first set of inputs are called the open inputs, while the second set of inputs are called the closed inputs. It is understood that the output of an element of Q is open, while the input of an element of P is closed, and only matching inputs/outputs can be composed. More precisely, there are structure maps, for every finite sets $W \subseteq U$, $R \subseteq S$, T (see Fig. 3.7):

$$\circ_W \colon \mathsf{Q}(U, S/R) \otimes \mathsf{P}(R) \to \mathsf{Q}(U, S); \tag{3.5.2}$$

$$\circ_{W,T} \colon \mathsf{Q}(U/W, S) \otimes \mathsf{Q}(W, T) \to \mathsf{Q}(U, S \sqcup T); \tag{3.5.3}$$

as well as actions of the symmetric groups and a unit in $\mathsf{Q}(1, 0)$ which satisfies the obvious axioms.

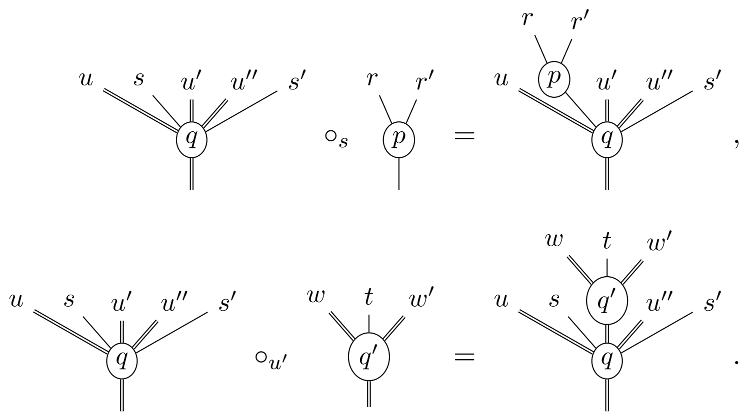


Figure 3.7: Composition in a relative operad. Closed edges are normal, open edges are double.

Example 3.5.4. Let X, Y be objects in some symmetric monoidal category $\mathcal{C}$. The prototypical example of relative operad is the relative End_X-operad $\mathrm{End}^2_{X,Y}$ defined by:

$$\mathrm{End}^2_{X,Y}(U, S) := \mathcal{C}(Y^{\otimes U} \otimes X^{\otimes S}, Y). \tag{3.5.5}$$

Definition 3.5.6. An algebra over a relative P-operad Q is a couple (A, B) where A is a P-algebra and the couple is equipped with structure maps:

$$\mathsf{Q}(U, S) \otimes B^{\otimes U} \otimes A^{\otimes S} \to B, \tag{3.5.7}$$

satisfying obvious axioms.

Remark 3.5.8. If the category $\mathcal{C}$ is closed, then this is the same thing as a morphism of relative operads $(\mathsf{P}, \mathsf{Q}) \to (\mathrm{End}_A, \mathrm{End}^2_{A,B})$ for the enriched version of the endomorphism operads.

Let us now define the Swiss-Cheese operad SC_n for $n \geq 1$. We write $\sigma\colon \mathbb{D}^n \to \mathbb{D}^n$ for the reflection along $\mathbb{D}^{n-1} = \mathbb{D}^{n-1} \times \{0\}$, and we let:

$$\mathbb{D}^n_+ := \mathbb{D}^n \cap \mathbb{R}^{n-1} \times \mathbb{R}_+. \tag{3.5.9}$$

Definition 3.5.10. For finite sets U, T, we let T' be a disjoint copy of T, and for $t \in T$ we let $t' \in T'$ be the matching element. The space of operations $\mathsf{SC}_n(U, S)$ is defined by:

$$\mathsf{SC}_n(U, S) := \left\{ c \in \mathsf{D}_n(U \sqcup S \sqcup S') \;\middle|\; \begin{array}{l} \forall u \in U, \mathrm{im}(c(\_, u)) = \sigma(\mathrm{im}(c(\_, u))); \\ \forall t \in T, \mathrm{im}(c(\_, t)) = \sigma(\mathrm{im}(c(\_, t'))) \subseteq \mathbb{D}^n_+. \end{array} \right\}. \tag{3.5.11}$$

In plain words, the disks indexed by U are self-symmetric with respect to D^{n-1}, while the disks indexed by S are symmetric to the ones indexed by S' and contained in the upper half-disk. The operad structure is obtained by restricting that of D_n.

When drawing an element of $\mathsf{SC}_n(U, S)$, it is customary to only draw the content of the upper half-disk, since the content of the lower half-disk is uniquely determined by that. See Fig. 3.8 for an example.

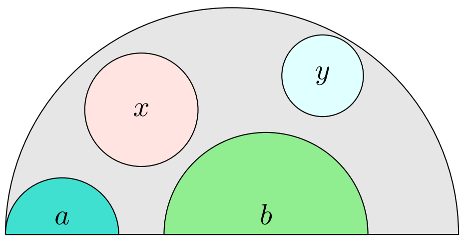


Figure 3.8: An element of $\mathsf{SC}_2(\{a, b\}, \{x, y\})$.

Remark 3.5.12. Again, we are looking at unital operads: $\mathsf{SC}_n(\varnothing, \varnothing)$ is a singleton. Moreover, $\mathsf{SC}_n(\varnothing, S)$ is nonempty for all S, i.e., SC_n encodes operations of the form $A^{\times S} \to B$ for an algebra (A, B).

* * *

Recall (Th. 3.2.13) that an algebra over $H_*(\mathsf{D}_n)$ (for $n \geq 2$) is an n-Poisson algebra, while an $H_*(\mathsf{D}_1)$-algebra is an associative algebra.

Definition 3.5.13. The center of an $H_*(\mathsf{D}_n)$-algebra is the commutative algebra:

$$Z(A) := \{a \in A \mid \forall b \in A,\ [a, b] = 0\}, \tag{3.5.14}$$

where $[a, b]$ is either the n-Lie bracket of a and b (for $n \geq 2$) or the commutator of a and b (for $n = 1$).

Theorem 3.5.15 (Consequence of Voronov [167, Thm. 3.3], noted in Hoefel–Livernet [93, Thm. 6.1.1])**.** *Let $n \geq 2$. An algebra over SC_n is a triple (A, B, f) where:*

- *A is an $H_*(\mathsf{D}_n)$-algebra;*
- *B is an $H_*(\mathsf{D}_{n-1})$-algebra;*
- *$f\colon A \to Z(B)$ is a central morphism of commutative algebras.*

There is also an analogue of the recognition principle.

Definition 3.5.16. Let $x_0 \in A \subseteq X$ be a pair of based spaces and let $n \geq 1$. The n-fold iterated relative loop space of (X, A) is the space:

$$\Omega^n(X, A) := \left\{ \gamma\colon [0,1] \to X \;\middle|\; \begin{array}{l} \gamma([0,1] \times \{0\}^{n-1}) \subseteq A, \\ \gamma(\partial[0,1]^n \cap (0,1] \times [0,1]^n) = \{x_0\} \end{array} \right\}. \tag{3.5.17}$$

Remark 3.5.18. The space $\Omega^n(X, A)$ is the homotopy fiber of the inclusion $\Omega^{n-1}A \to \Omega^{n-1}X$.

Theorem 3.5.19 (Hoefel–Livernet–Stasheff [94] for $n = 1$, Vieira [165] for $n \geq 3$)**.** *Let $n = 1$ or $n \geq 3$. Let (Y, B) be an SC_n-algebra. Suppose either that Y is a group-like D_1-algebra (for $n = 1$) or that Y is $(n-1)$-connected and B is $(n-2)$-connected (for $n \geq 3$). Then there exists a pair of based spaces (X, A) and a zigzag of weak equivalence of SC_n-algebras $(Y, B) \simeq (\Omega^n X, \Omega^n(X, A))$.*

* * *

The following result explains why we wrote that results about SC_n are "purely operadic in nature." From a topological point of view, the homotopy types of the components of SC_n are completely determined by what happens to D_n:

Proposition 3.5.20. *For all $n \geq 1$ and U, S finite sets, there is a homotopy equivalence:*

$$\mathsf{SC}_n(U, S) \simeq \mathsf{D}_{n-1}(U) \times \mathsf{D}_n(S) \simeq \mathrm{Conf}_{\mathbb{R}^{n-1}}(U) \times \mathrm{Conf}_{\mathbb{R}^n}(S). \tag{3.5.21}$$

Since all the components of SC_n are formal, and its structure maps are restriction of formal maps, one could potentially expect that SC_n is itself formal. This was shown to be false for $n \geq 2$ (the case $n = 1$ is trivial):

Theorem 3.5.22 (Livernet [115] and Willwacher [170])**.** *The operad SC_n is not formal over any field for $n \geq 2$.*

The approach of Livernet [115] is based on the theory of operadic Massey products, which are generalizations of classical Massey products for dg-algebras. The proof of Willwacher [170] reduces the (non)-formality of SC_n to the (non)-formality of the inclusion of operads $\mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$, which was shown to be non-formal by Turchin–Willwacher [162].

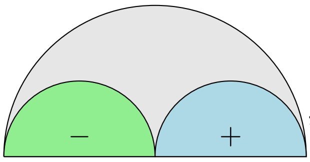
Both proofs, however, crucially use elements in $\mathsf{SC}_n(\varnothing, \{*\})$. Voronov's original definition of the Swiss-Cheese operad actually did not allow such elements. The operad considered in [167] is a sub-operad $\mathsf{SC}_n^{\mathrm{vor}} \subseteq \mathsf{SC}_n$ defined by:

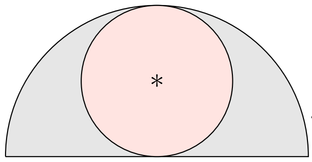
$$\mathsf{SC}_n^{\mathrm{vor}}(U, S) := \begin{cases} \mathsf{SC}_n(U, S), & \text{if } U \neq \varnothing; \\ \varnothing, & \text{if } U = \varnothing. \end{cases} \tag{3.5.23}$$

Formality of SC_n would have implied formality of $\mathsf{SC}_n^{\mathrm{vor}}$, but Th. 3.5.22 does not imply that $\mathsf{SC}_n^{\mathrm{vor}}$ is non-formal. This question is now settled, by Vieira [164] and in joint work with Vieira:

Theorem 3.5.24 (Vieira [164] for $n = 2$, [9] for all n)**.** *Voronov's Swiss-Cheese operad $\mathsf{SC}_n^{\mathrm{vor}}$ is not formal for any $n \geq 2$.*

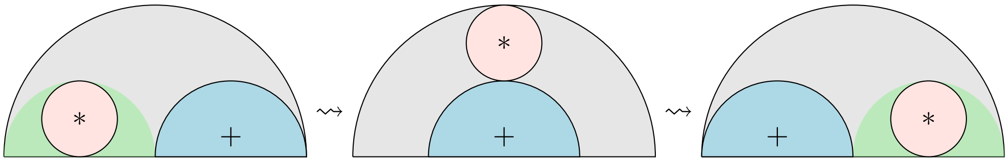
The proof of this theorem is inspired by the construction of the nontrivial Massey product in [115] for $n = 2$ (which is not exactly analogous to the one for $n \geq 3$). Consider the following elements (which are part of a set of generators of $H_*(\mathsf{SC}_2)$):

(3.5.25) $\mu =$ 

, $f =$ 

.

Then there is a homotopy η_1 from $\mu \circ_- f$ and $(\mu \cdot \tau) \circ_- f$ (where $\tau \colon \{\pm\} \to \{\pm\}$ is the transposition) given by letting the disk go "above" the half-disk:

(3.5.26) 

This proves that $\langle \mu + \mu \cdot \tau; f, f \rangle$ defines a Massey product: one has $(\mu + \mu \cdot \tau) \circ_+ f = (\mu + \mu \cdot \tau) \cdot_- f = 0$ thanks to the homotopy η. Computing this Massey product gives $f \circ \lambda$, where $\lambda \in H_1(\mathsf{D}_2(2))$ is a generator. This element is not in the ideal generated by μ on the left and f on the right, so this is a nonzero Massey product and thus an obstruction to formality.

This proof cannot work for $\mathsf{SC}_2^{\mathrm{vor}}$, as it uses the element $f \in \mathsf{SC}_2(\varnothing, 1)$. Vieira [164] found another nonzero Massey product in $\mathsf{SC}_2^{\mathrm{vor}}$. The construction of half of the Massey product is summarized by Fig. 3.9; note that it is more convenient to work with little cubes rather than little disks for these constructions.

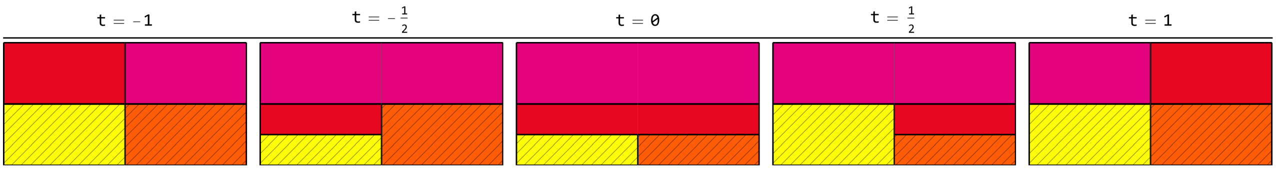


Figure 3.9: Half of the Massey product in $\mathsf{SC}_2^{\mathrm{vor}}$. Reproduced from [9].

This construction was generalized in [9] in joint work with Vieira. While the construction for $\mathsf{SC}_2^{\mathrm{vor}}$ only uses operations of arity $(2, 2)$ at most, the one for $\mathsf{SC}_n^{\mathrm{vor}}$ ends up using operations of arity $(2^n, 2)$. Moreover, the construction for $\mathsf{SC}_2^{\mathrm{vor}}$ is obtained by gluing only 8 "basic" paths (obtained by composing elementary paths together); the construction for $\mathsf{SC}_n^{\mathrm{vor}}$ uses, in total, 2^{n+1} such elementary paths. We refer to Fig. 3.10 for an illustration of 1/16th of the Massey product in $\mathsf{SC}_3^{\mathrm{vor}}$.

One interesting note we have is that the obstruction to formality is not an operadic Massey product, strictly speaking. The issue lies in the action of the symmetric group: the permutation of the two open colors occurs after all the compositions have been performed.

Question 3.5.27. *Can we give a more general definition of an operadic "Massey-like product" that covers the obstruction to the formality of $\mathsf{SC}_n^{\mathrm{vor}}$ found in [9]?*

We are also lead to the following natural question: what is the largest arity up to which $\mathsf{SC}_n^{\mathrm{vor}}$ is formal? More precisely,

Definition 3.5.28. For $r \geq 0$, let $\mathsf{FB}_{\leq r}$ be the category of sets of cardinality $\leq r$ and bijections. An r-truncated symmetric collection is a functor on $\mathsf{FB}_{\leq r}^{\mathrm{op}}$. For $r, s \geq 0$, an (r, s)-truncated bisymmetric collection is a bifunctor on $\mathsf{FB}_{\leq r}^{\mathrm{op}} \times \mathsf{FB}_{\leq s}^{\mathrm{op}}$.

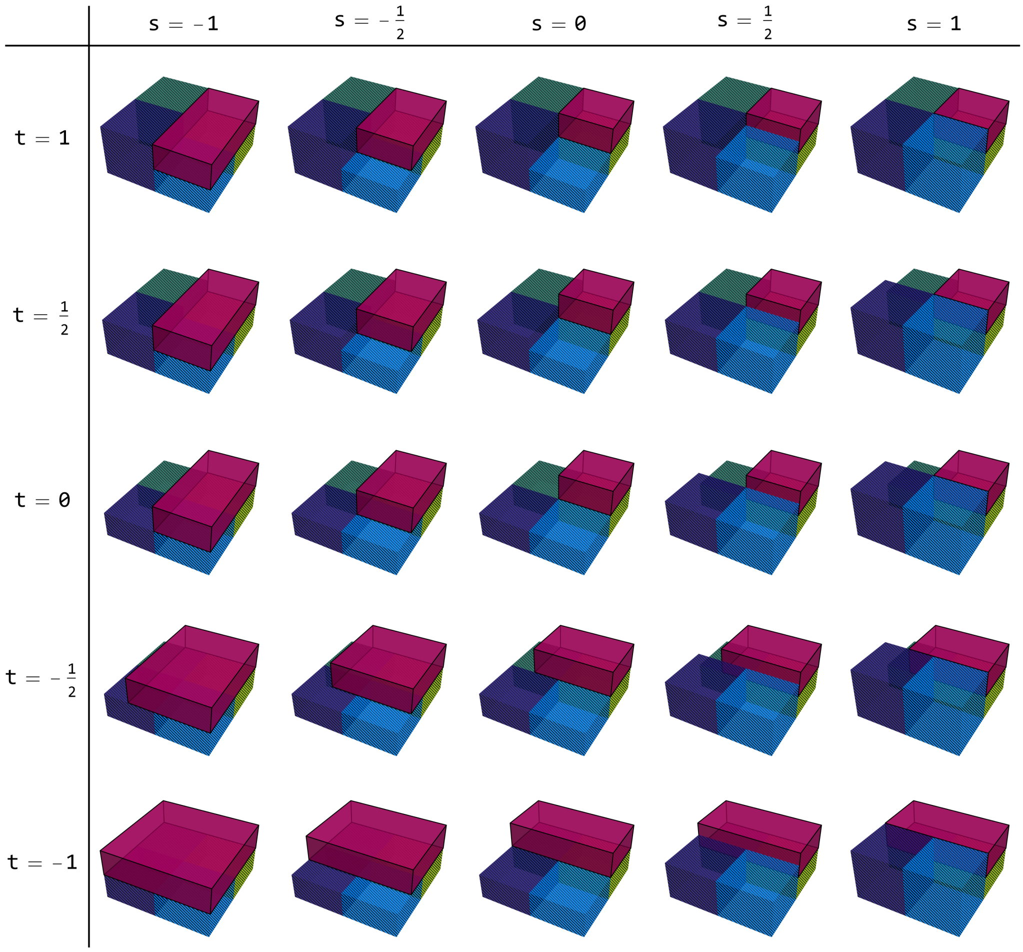


Figure 3.10: One sixteenth of the Massey product in $\mathsf{SC}_3^{\text{vor}}$. Reproduced from [9].

Definition 3.5.29. For $r \geq 0$, an r-truncated operad is an r-truncated symmetric collection P equipped with structure maps $\circ_W : \mathsf{P}(U/W) \otimes \mathsf{P}(W) \to \mathsf{P}(U)$ for $\#U, \#W, \#(U/W) \leq r$, as well as a unit if $r \geq 1$, satisfying the same axioms as operads (when arities make sense). Truncated (bi)-modules and truncated relative operads are defined analogously.

Remark 3.5.30. Truncated operads and modules appear in the computation of finite degree polynomial approximations in embedding calculus (see Sec. 2.4).

Remark 3.5.31. An operad (resp., module, bimodule, relative operad) defines a truncated operad (resp., module, bimodule, relative operad) by restriction.

Of course, if an operad is formal, then its truncation to any arity is formal. The converse is of course not true. For example, the little disks operads D_n are clearly not formal over a field of positive characteristic: the action of Σ_p on $\mathsf{D}_n(p)$ is non-formal, even if we forget about the operadic structure.

Remark 3.5.32. Even if we forget the action of the symmetric group, it is known that D_2 is not formal over $\mathbb{F}_2$ [145].

Nevertheless, there are formality results for the truncations of D_n:

Theorem 3.5.33 (Cirici–Horel [41] for $n = 2$, Boavida de Brito–Horel [29] for $n \geq 3$)**.** *The truncation of the dg-operad $C_*(\mathsf{D}_n; \mathbb{F}_p)$ to arity $\leq p - 1$ is formal.*

This thus raises the following question:

Question 3.5.34. *What are the maximal couples (r, s) such that the (r, s)-truncation of $\mathsf{SC}_n^{\mathrm{vor}}$ is formal (over $\mathbb{Q}$ or $\mathbb{F}_p$)?*

The author would be satisfied with any partial order on $\mathbb{N}^2$ to define "maximal," e.g., either one of the two lexicographic (total) orders, or the (partial) order induced by $(r, s) \mapsto r + s$. This question is subsumed by this more general question:

Question 3.5.35. *What is the homotopy Hopf cooperad structure transferred onto $H^*(\mathsf{SC}_n)$ or $H^*(\mathsf{SC}_n^{\mathrm{vor}})$?*

Indeed, once the transferred structure is computed, one could perhaps "see" that the obstruction to the non-formality of $\mathsf{SC}_n^{\mathrm{vor}}$ is in higher arity than that of SC_n. Conversely, if the transferred structure is expressed in terms of integrals indexed by graphs using the results of Willwacher [169], this could give a hint that some of these integrals vanish for low numbers of vertices.

* * *

Let us now discuss a formality result regarding another variant of the Swiss-Cheese operad. Philosophically speaking, the Swiss-Cheese operad encodes actions of D_n-algebras on D_{n-1}-algebras via central morphisms. The operad we introduce next encodes actions of D_n-algebras on D_m-algebras (for $1 \leq m < n$) via central derivations. For convenience, we consider $\mathbb{D}^m \subseteq \mathbb{D}^n$ as the intersection $\mathbb{D}^n \cap \mathbb{R}^m \times \{0\}$.

Definition 3.5.36. Let $1 \leq m < n$ be integers and U, V be finite sets. The space $\mathsf{CD}_{mn}(U, V)$ is the subspace of elements $c \in \mathsf{D}_n(U \sqcup V)$ such that (i) for $u \in U$, $c(0, u) \in \mathbb{D}^m$; for $v \in V$, $\mathrm{im}(c(\_, v)) \cap \mathbb{D}^m = \varnothing$. The operad structure of D_n induces a relative D_n-operad structure on CD_{mn}.

Roughly speaking, an element of $\mathsf{CD}_{mn}(U, V)$ is a configuration of little n-disks in the unit n-disk of two kinds: (i) "terrestrial disks," indexed by U and constrained to be centered on a point of $\mathbb{D}^m$; (ii) "aerial disks," indexed by V and constrained to never touch $\mathbb{D}^m$. This operad is equivalent to the operad $\mathsf{Disk}^{\mathrm{fr}}_{m \subseteq n}$ that is useful in factorization homology to study knot complements [20, Sec. 4.3].

Remark 3.5.37. This operad is not equivalent to the extended Swiss-Cheese operad ESC_{mn} considered by Willwacher [170]. He proved that formality of ESC_{mn} is equivalent to the formality of the inclusion $\mathsf{D}_m \subseteq \mathsf{D}_n$, which occurs if and only if $n - m \neq 1$ by the results of Turchin–Willwacher [162].

Recall that an algebra over $H_*(\mathsf{D}_n)$ is an associative algebra for $n = 1$, or an n-Poisson algebra when $n \geq 2$ (Th. 3.2.13). The following computation is performed by adapting and extending the methods of Cohen [42] and Hoefel–Livernet [93]:

Proposition 3.5.38 ([6]). *An algebra over $H_*(\mathsf{CD}_{mn})$ is the data of:*
- *an $H_*(\mathsf{D}_n)$-algebra A;*
- *an $H_*(\mathsf{D}_m)$-algebra B;*
- *a central morphism $f + \delta\epsilon \colon A \to B[\epsilon]$, where $B[\epsilon] = B \oplus B\epsilon$, $\deg \epsilon = n - m - 1$, and with the Lie bracket of degree $n - 1$:*

$$[x + y\epsilon, x' + y'\epsilon] \coloneqq [x, x'] + \epsilon([x, y'] \pm [x', y]). \tag{3.5.39}$$

By adapting the template presented at the end of Sec. 3.4, we proved:

Theorem 3.5.40 ([6]). *The operad CD_{mn} is formal in characteristic zero.*

Remark 3.5.41. Strictly speaking, in [6], we are proving that the Hopf cooperad $\Omega^*(\mathsf{CD}_{mn}) \otimes_{\mathbb{Q}} \mathbb{R}$ is formal. It is known that formality over $\mathbb{R}$ descends to formality over $\mathbb{Q}$ for dg-operads, thanks to the results of Guillén Santos–Navarro–Pascual–Roig [87]. Thus $C_*(\mathsf{CD}_{mn}; \mathbb{Q})$ is also formal. We could not find an answer to the following question in the literature, although a positive answer seems likely.

Question 3.5.42 (Asked on MO [96]). *Does formality of (colored) Hopf cooperads satisfy the descent property?*

Let us conclude with a couple of questions. First, we wonder if the recognition principle (Th. 3.2.15, 3.5.19) generalizes to CD_{mn}.

Conjecture 3.5.43. *The operad CD_{mn} recognizes pairs $(\Omega^n(X), \Omega^{m,n}(A, X))$, where (X, A, x_0) is a pair of based spaces (possibly under some connectivity assumptions), and*

$$\Omega^{m,n}(A, X) \coloneqq \{\gamma \colon \mathbb{D}^n \to X \mid \gamma(\partial\mathbb{D}^n) = \{x_0\} \text{ and } \gamma(\mathbb{D}^m) \subseteq A\}. \tag{3.5.44}$$

Note that this recognition principle is part of our motivation for considering CD_{mn} rather than the extended Swiss-Cheese operad ESC_{mn} considered by Willwacher [170], as $(\Omega X, \Omega^{m,n}(A, X))$ is not an algebra over ESC_{mn} in general.

Question 3.5.45. *Is the operad CD_{mn} intrinsically formal (in the sense of [72])?* (The techniques of [56] may prove useful.)

The operad CD_{mn} is inspired by the theory of factorization algebras (see e.g., [81, 20, 48] and Def. 4.3.30). More specifically, CD_{mn} encodes locally constant prefactorization algebras on the stratified space $\{\mathbb{R}^m \subseteq \mathbb{R}^n\}$.

Question 3.5.46. *Let F be a stratification of $\mathbb{R}^n$ by (semi)-affine subspaces. Let D_F be the topological colored operad that encodes locally constant prefactorization algebras on that stratified space. When is D_F formal?*

An obviously necessary condition is that F does not contain two strata $F_i \subset F_j$ such that $\dim F_j - \dim F_i = 1$; otherwise, D_F would contain the (non-formal) Swiss-Cheese operad. It seems doubtful that this condition is sufficient. The most basic nontrivial case to study would be, for integers $a, b, c, d \geq 2$, the space $\mathbb{R}^{a+b+c+d}$ stratified by its subspaces $\mathbb{R}^{a+b} \times \{0\}^{c+d}$ and $\{0\}^a \times \mathbb{R}^{b+c} \times \{0\}^d$.

4 Resolutions

In this last section, we explain what tools we use to perform concrete computations from the results obtained in the previous sections. We first introduce the bar and cobar constructions, and explain their relationship to the Boardman–Vogt $\mathcal{W}$ construction. We then briefly review Koszul duality, its extension to operads, and our contribution to that. We then show an application to homotopy prefactorization algebras.

4.1 Bar, cobar, and $\mathcal{W}$ constructions

The computation of homotopy invariants of algebraic objects often requires to find a *resolution* of these objects, that is, a quasi-(co)free, weakly equivalent replacement of the initial object. Examples include some invariants that already appeared in this memoir, e.g., Harrison cohomology of commutative algebras to compute rational homotopy groups out of a Sullivan model, or (operadic) Massey products. Other examples include various kinds of algebraic homology theories (Hochschild, Lie, Poisson. . .), or derived tensor products as appearing in e.g., factorization homology.

One key issue is to compute *efficient* resolutions, which are not too big but sufficiently well-behaved so that applying a derived functor to that resolution gives a tractable answer. Indeed, it is generally easy to define generic resolutions, which tend to be large.

* * *

One example of canonical resolution is the bar-cobar resolution of an associative algebra. Recall (Sec. 1.2) that we always use a cohomological grading, and that the shift of a graded vector space is defined by $(V[i])^k = V^{k+i}$. We will just write "algebra" or "coalgebra" instead of "dg-algebra" or "dg-coalgebra."

Notation 4.1.1. Let V be a graded vector space. The free algebra on V is denoted by $T(V)$. The cofree conilpotent coalgebra on V is denoted $T^c(V)$.

Definition 4.1.2. An augmented algebra is an algebra A equipped with a morphism $\varepsilon_A \colon A \to \mathbb{K}$. Its augmentation ideal is denoted $\bar{A} = \ker(\varepsilon_A)$. A coaugmented coalgebra is a coalgebra C equipped with a morphism $\eta_C \colon \mathbb{K} \to C$. Its coaugmentation coideal is $\bar{C} = \operatorname{coker}(\eta_C)$.

Definition 4.1.3. Let A be an augmented algebra. The bar construction $\mathcal{B}A$ of A is:

$$\mathcal{B}A := (T^c(\bar{A}[1]), d_B), \tag{4.1.4}$$

where $d_{\mathcal{B}}$ is the unique coderivation whose projection onto cogenerators is $a \mapsto da$, $a \otimes b \mapsto ab$, and zero on tensors of length ≥ 3.

Definition 4.1.5. Let C be a coaugmented coalgebra. The cobar construction ΩC of C is:

$$\Omega C := (T(\bar{C}[-1]), d_\Omega), \tag{4.1.6}$$

where d_Ω is the unique derivation whose restriction to generators is $c \mapsto dc + \Delta(c)$.

Theorem 4.1.7. *Let A be an augmented algebra. The natural morphism of algebras $\Omega\mathcal{B}A \to A$, given on generators by the projection $\mathcal{B}A \to \bar{A}[1]$, is a resolution of A.*

This resolution is always well-defined and functorial. However, it is very large in general:

Example 4.1.8. Let $A_n = T(V_n)$ where $\dim V_n = n$ is concentrated in degree zero. Let $A_n^{(k)} \subseteq A$ be the subspace of elements of weight k, that is, linear combinations of products of k generators; then $\dim A_n^{(k)} = n^k$. If we let $(\Omega\mathcal{B}A_n)^{(k)} \subseteq \Omega\mathcal{B}A_n$ be the subspace of elements of weight k (that is, the total number of elements of V_n involved is k, no matter the degree), then $\dim(\Omega\mathcal{B}A_n)^{(k)} = 3^{k-1}n^k$ is much larger than $\dim A_n^{(k)}$, even though A_n is already a resolution of itself.

There exists a different bar construction, which is a special case of the two-sided bar construction.

Definition 4.1.9. Let A be an algebra, M be a right A-module, and N be a left A-module. The two-sided bar construction $\mathcal{B}(M, A, N)$ is the chain complex:

$$\mathcal{B}(M,A,N) \coloneqq \Bigl(\bigoplus_{n\geq 0} M \otimes (A[1])^{\otimes n} \otimes N, d\Bigr), \tag{4.1.10}$$

where $d\colon M \otimes (A[1])^{\otimes n} \otimes N \to M \otimes (A[1])^{\otimes(n-1)} \otimes N$ is given by the signed sum $\sum_{i=0}^{n}(-1)^i d_i$, and, for $a_0 \in M$, $a_{n+1} \in N$, and $a_1, \dots, a_n \in A$:

$$d_i(a_0 \otimes \dots \otimes a_{n+1}) \coloneqq (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}. \tag{4.1.11}$$

Note that d_0 uses the module structure of M, d_n uses the module structure of N, and d_i $(0 < i < n)$ uses the algebra structure of A.

The two-sided cobar construction $\Omega(M, C, N)$, for a coalgebra C, a right C-comodule M, and a left C-comodule N, is defined analogously.

Definition 4.1.12. Let A be an augmented algebra, so that $\mathbb{K}$ is naturally an (A, A)-bimodule. The reduced bar construction of A is the chain complex:

$$\mathcal{B}'A \coloneqq \mathcal{B}(\mathbb{K}, A, \mathbb{K}) = (\bigoplus_{n\geq 0}(A[1])^{\otimes n}, d). \tag{4.1.13}$$

* * *

Generalizations of the bar and cobar constructions for (algebraic) operads exist. However, there are several such generalizations, depending on how one views operads. Livernet [114] summarizes two constructions, one by Ginzburg–Kapranov [82], the other by Rezk [140], Shnider–Van Osdol [153], and Fresse [64], who were proved to be equivalent by Fresse [64]; and constructs a third one denoted $\mathcal{B}$, and proves that it is equivalent to the other two using a levelization morphism. We will focus on the first two constructions.

The first point of view is to see an operad as a monoid for the plethysm (Rem. 3.1.20). Let P be an operad. Recall from Rem. 3.1.35 that an operadic right P-module can be viewed as a right module (in the classical sense) over P seen as a monoid in the category of symmetric collections. That is, a right P-module is a symmetric collection M endowed with a structure map $\mathsf{M} \circ \mathsf{P} \to \mathsf{M}$ (where $\circ$ stands for the plethysm (3.1.22)) satisfying certain associativity axioms. Dually, a left P-module (Def. 3.1.41) is a symmetric collection N endowed with a structure map $\mathsf{P} \circ \mathsf{N} \to \mathsf{N}$ satisfying some axioms. This yields the following construction [140, 153, 64].

Definition 4.1.14. Let P be an operad, M be a right P-module, and N be a left P-module. The two-sided bar construction $\mathcal{B}(\mathsf{M}, \mathsf{P}, \mathsf{N})$ is given by the direct sum:

$$\mathcal{B}(\mathsf{M},\mathsf{P},\mathsf{N}) \coloneqq \Bigl(\bigoplus_{n\geq 0} \mathsf{M} \circ (\mathsf{P}[1])^{\circ n} \circ \mathsf{N}, d\Bigr), \tag{4.1.15}$$

where d is defined analogously to the one in Def. 4.1.9.

The two-sided cobar construction $\Omega(\mathsf{M}, \mathsf{C}, \mathsf{N})$, for a coalgebra C, a right C-comodule M, and a left C-comodule N, is defined analogously

Remark 4.1.16. Strictly speaking, a cooperad is not a comonoid for the plethysm (Eq. (3.1.22)) but rather for the restricted plethysm, which is defined by an end rather than a coend:

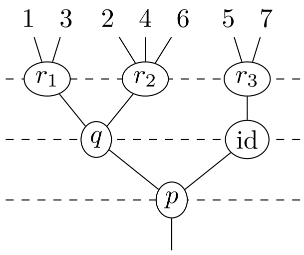
$$(\mathsf{C} \,\tilde{\circ}\, \mathsf{D})(U) := \int_{V \in \mathsf{FB}} \Big(\mathsf{C}(V) \otimes \big(\bigoplus_{f \colon U \to V} \bigotimes_{v \in V} \mathsf{D}(f^{-1}(v)) \big) \Big). \tag{4.1.17}$$

In characteristic zero, plethysm and restricted plethysm are isomorphic. Since we mostly work in characteristic zero, we will remain imprecise.

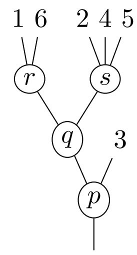
Definition 4.1.18. Let P be an augmented operad, so that I naturally forms a (P, P)-bimodule. The leveled bar construction of P is:

$$\mathcal{B}^\circ \mathsf{P} := \mathcal{B}(\mathsf{I}, \mathsf{P}, \mathsf{I}) = \big(\bigoplus_{n \geq 0} (\mathsf{P}[1])^{\circ n}, d\big). \tag{4.1.19}$$

An element of $(\mathcal{B}^\circ \mathsf{P})(U)$ can be viewed as a rooted tree with levels whose vertices are decorated by elements of P (with the appropriate arity) and whose leaves are in bijection with U. See Fig. 4.1 for an example. The differential is the signed sum of the "contraction" of two consecutive levels. Note that $\mathcal{B}^\circ \mathsf{P}$ does not naturally define a cooperad.



(a) An element of $\mathcal{B}^\circ \mathsf{P}(7)$.



(b) An element of $\mathcal{B}\mathsf{P}(5)$.

Figure 4.1: Elements of the two bar constructions for an operad P.

The other bar construction $\mathcal{B}\mathsf{P}$ is built differently, as we now explain.

Notation 4.1.20. The free operad on a symmetric sequence V is denoted $\mathcal{T}(\mathsf{V})$. The cofree (conilpotent) cooperad on V is denoted $\mathcal{T}^c(\mathsf{V})$.

Definition 4.1.21. The bar construction $\mathcal{B}\mathsf{P}$ of an augmented operad P is the cofree cooperad on the suspension of the augmentation ideal of P:

$$\mathcal{B}\mathsf{P} := (\mathcal{T}^c(\bar{\mathsf{P}}[1]), d), \tag{4.1.22}$$

with the differential d being the unique coderivation whose projection on cogenerators maps a tree with two vertices of the form $[p] \circ_i [q]$ to $[p \circ_i q]$, and vanishes on other trees. The cobar construction $\Omega\mathsf{C}$ of a cooperad is defined analogously.

The main advantage of $\mathcal{B}\mathsf{P}$ over $\mathcal{B}^\circ \mathsf{P}$ is that it actually defines a cooperad.

Theorem 4.1.23. *For an operad P, the natural morphism $\Omega\mathcal{B}\mathsf{P} \to \mathsf{P}$, given on generators by the projection $\mathcal{B}\mathsf{P} \twoheadrightarrow \bar{\mathsf{P}}[1]$, is a quasi-isomorphism.*

Just like the bar-cobar resolution of an algebra, this resolution is often wasteful.

Remark 4.1.24. If P is an augmented operad concentrated in arity 1, i.e., an augmented algebra, then $\mathcal{B}\mathsf{P}$ coincides with the bar construction for algebras, and $\Omega\mathcal{B}\mathsf{P}$ coincides with the bar-cobar resolution for algebras.

* * *

In [1], with Campos and Ducoulombier, we study the bar and cobar constructions for Hopf cooperads and comodules (Def. 3.3.3). One of our primary goals is to define fibrant resolutions for cobimodules.

Remark 4.1.25. In [1], we also study the case of operads and bimodules in spectra, where the bar construction was defined by Salvatore [143] and Ching [36]. For brevity, we will only discuss the dual case of Hopf cooperads and cobimodules here, but all the results below have dual results.

Given cooperads C, D and a (C, D)-cobimodule M, one could be tempted to define a fibrant resolution by:

$$\Omega(\mathsf{C},\mathsf{C},\mathsf{M}) \circ^{\mathsf{M}} \Omega(\mathsf{M},\mathsf{D},\mathsf{D}) \coloneqq \mathrm{eq}\big(\Omega(\mathsf{C},\mathsf{C},\mathsf{M}) \circ \Omega(\mathsf{M},\mathsf{D},\mathsf{D}) \rightrightarrows \Omega(\mathsf{C},\mathsf{C},\mathsf{M}) \circ \mathsf{M} \circ \Omega(\mathsf{M},\mathsf{D},\mathsf{D})\big). \tag{4.1.26}$$

However, this does not define a cobimodule, in general. The issue lies in the "levels" in the two-sided cobar constructions (cf. Fig. 4.1). Because of these levels, the natural candidate for the cocomposition does not satisfy the associativity axioms required of cooperads and cobimodules. Depending on the order in which cocomposition is performed, the same node of a tree could land in different levels.

It is easy to adapt Def. 4.1.21, which uses the definition of operads with partial compositions, to define a fibrant resolution of a right comodule over a cooperad. One should simply replace the decoration of the root with a decoration from the considered comodule in the definition of the cobar construction. Everything then falls into place, because a right comodule can equivalently be defined in terms of total cocomposition or partial cocomposition. The crux of the issue is in the *left* comodule structure; total cocomposition and partial cocomposition define different structures.

To alleviate this issue, we define in [1] a leveled cobar construction for 1-reduced cooperads and cobimodules (where 1-reduced means that arity zero contains just a point and arity one just the identity). This construction uses leveled trees, just like the two-sided cobar construction. However, we also allow some levels to be permuted (or rather, we consider linear combinations of trees invariant under permutation of some levels). Two consecutive levels are called "permutable" if every edge between the two has either a bivalent source or a bivalent target, see Fig. 4.2.

Remark 4.1.27. In our definition of 1-reduced, the component of arity zero is a point (or one-dimensional). Whenever we speak of (co)fibrant resolutions below, we place ourselves in the Reedy model structure [68].

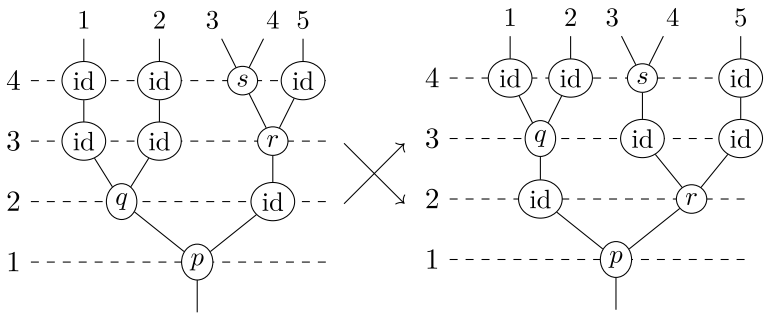


Figure 4.2: Permutable levels. Levels (2,3) are permutable in the first tree to get the second one. Levels (3,4) are not permutable in the first tree, but become permutable in the second.

Theorem 4.1.28 ([1])**.** *The leveled (co)bar constructions give functors:*

$$\Omega_l \colon \{\text{1-red. dg-coop.}\} \leftrightarrows \{\text{1-red. dg-op.}\} \colon B_l, \tag{4.1.29}$$

which are isomorphic to the usual (co)bar constructions. Moreover, if P, Q *are dg-(co)operads and* M *is a* (P, Q)*-(co)bimodule, then the leveled two-sided (co)bar construction* $B_l(\mathsf{P}, \mathsf{M}, \mathsf{Q})$ *(resp.,* $\Omega_l(\mathsf{P}, \mathsf{M}, \mathsf{Q})$*)*

is a well-defined dg-(B_lP, B_lQ)-cobimodule (resp., dg-(Ω_lP, Ω_lQ)-bimodule). The bar-cobar construction of a bimodule is a cofibrant resolution (in the category of bimodules over the bar-cobar constructions of the relevant operads) of the original bimodule.

* * *

Finally, in [1], we consider another type of resolution, the Boardman–Vogt $\mathcal{W}$ construction. It is easiest to explain this construction for topological operads, which we do now. Let $I = [0,1]$ denote the unit interval.

Definition 4.1.30. Let P be a topological operad. The Boardman–Vogt construction of P, denoted $\mathcal{W}\mathsf{P}$, is the operad defined as follows. Let $\mathcal{T}_r$ be the set of rooted planar trees with r numbered leaves. For $T \in \mathcal{T}_r$, let $\mathcal{W}\mathsf{P}(T)$ be the space consisting of decorations of T, where internal vertices of arity $i \in \mathbb{N}$ are decorated by elements of $\mathsf{P}(i)$, and inner edges are decorated by elements of I. We then set $\mathcal{W}\mathsf{P}(r) = \left(\bigsqcup_{T\in\mathcal{T}_r} \mathcal{W}\mathsf{P}(T)\right)/\sim$. The equivalence relation $\sim$ is generated by the following relations, for $T \in \mathcal{T}_r$ and $x \in \mathcal{W}\mathsf{P}(T)$ (see Fig. 4.3):

- if the decoration of an inner edge e is zero, then let T/e be the tree obtained from T by contracting e, let p, q be the decorations of the endpoints of e, and let $x' \in \mathcal{W}\mathsf{P}(T/e)$ be the decoration defined like x, except the vertex that used to be e before the contraction is now decorated by $p \circ_i q$ (where i is the number of e in the children of the vertex decorated by p); then we have $x \sim x'$;
- if the decoration of an internal vertex v is $\mathrm{id}_{\mathsf{P}} \in \mathsf{P}(1)$, then let $T \setminus v$ be the tree obtained from T' by erasing v (which is bivalent), and let $x' \in \mathcal{W}\mathsf{P}(T \setminus v)$ be defined like x, except the decoration of the new edge is $s + t - st$, where $s, t \in I$ were the decorations of the edges incident to v; then we have $x \sim x'$.

The operad structure is defined by tree grafting, with new edges decorated by $1 \in I$.

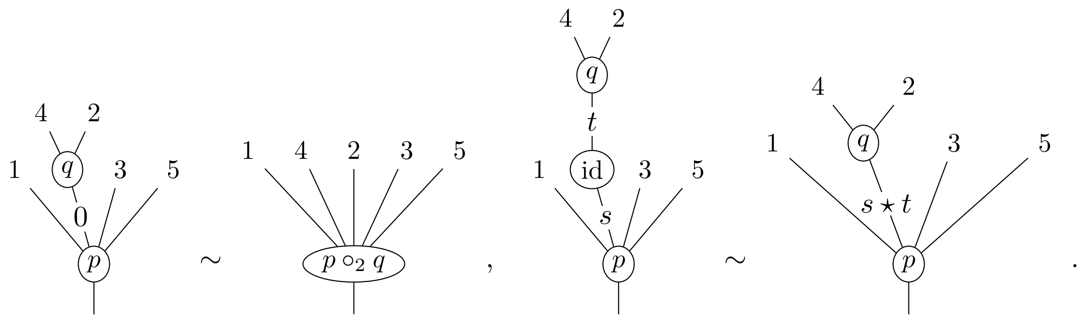


Figure 4.3: Identifications in the Boardman–Vogt $\mathcal{W}$ construction, where $p \in \mathsf{P}(4)$, $q \in \mathsf{P}(2)$, $s, t \in I$, and $s \star t = s + t - st$.

Remark 4.1.31. The $\mathcal{W}$ construction is, of course, connected to the story of configuration spaces. Let us mention for example the result of Salvatore [146], who proved that $\mathcal{W}\mathsf{FM}_n^{nu} = \mathsf{FM}_n^{nu}$, where FM_n^{nu} is the sub-operad of the Fulton–MacPherson operad FM_n (Def. 3.4.18) obtained by removing the operation of arity zero. In the presence of this operation of arity zero, FM_n is no longer cofibrant, but an explicit cofibrant replacement can be found in, e.g., [161].

This construction defines a cofibrant resolution [166] (in a suitable model structure on reduced operads, see [24]). It was extended to any category equipped with a suitable interval object by Berger–Moerdijk [25]. Ching [36, 37] studied these constructions in the case of the category of spectra. The construction was modified to deal with Hopf cooperads by Fresse–Turchin–Willwacher [70].

In [1], we define a leveled version of the $\mathcal{W}$-construction, and extend to (co)bimodules. As above, we only state the result for Hopf cooperads and cobimodules.

Theorem 4.1.32 ([1])**.** *Let C be a 1-reduced Hopf cooperad. The leveled Boardman–Vogt construction $W_l\mathsf{C}$ is a fibrant resolution of C which is isomorphic to the usual Boardman–Vogt construction.*

Moreover, if C, D are 1-reduced Hopf cooperads and M is a 1-reduced (C, D)-cobimodule, then the leveled bar construction $W_l\mathsf{M}$ is a fibrant resolution of M as a Hopf $(W_l\mathsf{C}, W_l\mathsf{D})$-cobimodule.

Finally, we identify the primitive elements of the $\mathcal{W}$-construction. Intuitively, these elements are the ones where the decorations of the edges are not equal to 1 (although a little care is needed to properly define it, as we work with an interval object, not an actual interval).

Theorem 4.1.33 ([1])**.** *Let C be a 1-reduced Hopf cooperad. The subspace of primitive elements* $\mathrm{Prim}\, W_l\mathsf{C}$ *forms a 1-reduced dg-operad which, and $W_l\mathsf{C}$ is quasi-isomorphic to the bar construction on the suspension of that dg-operad.*

Moreover, if C, D are 1-reduced Hopf cooperads and M is a 1-reduced (C, D)-cobimodule, the primitive subspace $\mathrm{Prim}\, W_l\mathsf{M}$ *forms a 1-reduced dg-bimodule over* $(\mathrm{Prim}\, W_l\mathsf{C}, \mathrm{Prim}\, W_l\mathsf{D})$*, and $W_l\mathsf{M}$ is isomorphic to the bar construction on the suspension of that bimodule.*

4.2 Koszul duality

Knowing something about an algebra (or an operad) often enables one to produce a resolution much more efficiently than the bar-cobar resolution. One particularly useful example is Koszul duality. Initially developed by Priddy [137] for inhomogeneous quadratic algebras, Koszul duality has since proved to be useful in a number of settings, including (without pretension of being exhaustive): in the curved algebraic setting [135, 133], for quadratic operads ([82, 79] for binary ones, [78] in general), inhomogeneous quadratic operads [77], for properads [163], in the curved (pr)operadic setting [91, 92], for quadratic algebras over certain operads [129], in the curved setting [7], etc.

As is evident from this list, the study of Koszul duality is quite fruitful. The development of operadic Koszul duality greatly renewed interest for the theory of operads in the 1990's, see [116]. In this section, we will only review the very basics of Koszul duality. We then present our contribution to curved Koszul duality for algebras over binary operads [7].

Definition 4.2.1. A quadratic algebra is an algebra of the form $A = T(V, R) \coloneqq T(V)/(R)$, where V is a graded vector space and $R \subseteq V \otimes V$ is a set of monomials of weight 2 generating an ideal (R).

Remark 4.2.2*.* A quadratic algebra $T(V, R)$ is a initial among algebra equipped with a morphism from $T(V)$ which vanishes on R.

Definition 4.2.3. A quadratic coalgebra is a coalgebra of the form $T^c(V, R)$, where V is a graded vector space, $R \subseteq V \otimes V$, and $T^c(V, R)$ is universal among sub-coalgebras $C \hookrightarrow T^c(V)$ such that the composite $C \hookrightarrow T^c(V) \to (V \otimes V)/R$ vanishes.

While this definition is dual to that of quadratic algebras, intuition is often sharper for algebras than for coalgebras, so let us briefly illustrate how quadratic coalgebras work.

Proposition 4.2.4. *Let V, R be as above. The coalgebra $T^c(V, R)$ decomposes as $\bigoplus_{n \geq 0} T^c(V, R)^{(n)}$, where:*

$$T^c(V, R)^{(n)} = \bigcap_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \subseteq V^{\otimes n}, \tag{4.2.5}$$

with the conventions that $T^c(V, R)^{(0)} = \mathbb{K}$ and $T^c(V, R)^{(1)} = V$.

Remark 4.2.6*.* This echoes the classical fact that $T(V, R) = \bigoplus_{n \geq 0} T(V, R)^{(n)}$, where

$$T(V, R)^{(n)} = V^{\otimes n} / \sum_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}. \tag{4.2.7}$$

Example 4.2.8. Let $V = \langle x, y, z\rangle$ be a vector space of dimension 3 concentrated in degree 0. Let $R \subseteq V \otimes V$ be spanned by $xy - yx$, $xz - zx$, $yz - zy$ (we omit tensors for brevity). Clearly, the quadratic algebra $T(V, R)$ is isomorphic to the polynomial algebra $S(V) = \mathbb{K}[x, y, z]$.

On the coalgebra side, we have $T^c(V, R)^{(0)} = \mathbb{K}$, $T^c(V, R)^{(1)} = V$, and $T^c(V, R)^{(2)} = R$. In weight 3, we look for tensors that can be written at the same time $r \otimes v$ and $w \otimes s$, for $v, w \in V$ and $r, s \in R$. A little linear algebra shows that solutions are multiples of $xyz - xzy - yxz + yzx + zxy - zyx$. Moreover, $T^c(V, R)^{(n)} = 0$ for $n \geq 4$. We obtain that $T^c(V, R) = \Lambda^c(V)$ is the exterior coalgebra on V.

Example 4.2.9. Let $V = \langle x, y, z\rangle$ and R be spanned by xx, yy, zz, $xy + yx$, $xz + zx$, and $yz + zy$. Then $T(V, R) = \Lambda(V)$ is the exterior algebra on V, while $T^c(V, R)$ is the polynomial coalgebra $S^c(V)$.

Definition 4.2.10. Let $A = T(V, R)$ be a quadratic algebra. The Koszul dual coalgebra is:

$$A^{¡} \coloneqq T^c(V[1], R[2]) \tag{4.2.11}$$

The Koszul dual algebra of a quadratic coalgebra $C = T^c(V, R)$ is:

$$C^{¡} \coloneqq T(V[-1], R[-2]). \tag{4.2.12}$$

Remark 4.2.13. Clearly, $(A^{¡})^{¡} = A$ for any quadratic presentation of A.

Remark 4.2.14. It is often easier to think about the Koszul dual algebra of a quadratic algebra A:

$$A^! \coloneqq T(V^\vee, R^\perp), \tag{4.2.15}$$

where $V^\vee$ is the linear (graded) dual of V and $R^\perp \subseteq V^\vee \otimes V^\vee$ is the annihilator of R. Note that $(A^!)^{(n)} = ((A^{¡})^{(n)})^\vee[-n]$ and that $(A^!)^! = A$ if V is finite dimensional.

Example 4.2.16. If $A = T(V)$ is free then $A^! = \mathbb{K} \oplus V$ is the trivial algebra. If $A = S(V)$ is the (graded commutative) polynomial algebra on V, then $A^! = \Lambda(V^*)$ is the (graded skew-commutative) exterior algebra on V.

Remark 4.2.17. For a quadratic algebra A, the Koszul dual $A^{¡}$ equipped with the zero differential is a sub-dg-coalgebra of $\mathcal{B}A$.

Definition 4.2.18. Let $A = T(V, R)$ be a quadratic algebra. Let κ be the linear map defined by:

$$\kappa \colon A^{¡}[1] \twoheadrightarrow V \hookrightarrow A. \tag{4.2.19}$$

There is a morphism of algebras $f_\kappa \colon \Omega A^{¡} \to A$ given on generators by κ.

Definition 4.2.20. A quadratic algebra A is Koszul if f_κ is a quasi-isomorphism.

Remark 4.2.21. This definition seems to depend on the choice of quadratic presentation. However, there exist intrinsic definitions of the Koszul property. One can for example ask for the Ext-algebra $\mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$ (with the Yoneda product) to be generated by its elements of weight one [118, 74].

Remark 4.2.22. For a quadratic algebra $A = T(V, R)$, the (right) Koszul complex of A is the dg-module given by:

$$K(A) = (A \otimes A^{¡}, d), \quad d \colon A \otimes A^{¡} \xrightarrow{1\otimes\Delta} A \otimes A^{¡} \otimes A^{¡} \xrightarrow{1\otimes\kappa\otimes 1} A \otimes A \otimes A^{¡} \xrightarrow{\mu\otimes 1} A \otimes A^{¡}. \tag{4.2.23}$$

The algebra A is Koszul if and only if the Koszul complex of A is acyclic.

Remark 4.2.24. If a quadratic algebra A is Koszul, then so is $A^!$.

Example 4.2.25. Free algebras, symmetric algebras, and exterior algebras are Koszul.

* * *

The above version of Koszul duality only applies to quadratic algebras. However, not all algebras admit quadratic presentations.

Example 4.2.26. Any quadratic algebra admits an augmentation. This is not the case of the $\mathbb{R}$-algebra $\mathbb{C} = T(i)/(i^2+1)$.

Example 4.2.27. The $\mathbb{R}$-algebra $A = T(x)/(x^2 - x)$ is augmented but does not admit a quadratic presentation.

Priddy [137] introduced Koszul duality for inhomogeneous quadratic algebras, in which relations are allowed to contain linear terms (i.e., elements of the generating space V). Positsel'skii [135] and Polishchuk–Positselski [133] introduced curved Koszul duality to deal with presentations in which relations can contain quadratic, linear, and constant terms. To avoid repeating ourselves, we will only present this latter version of Koszul duality. See also [136] for a survey.

Definition 4.2.28. A quadratic-linear-constant (QLC) algebra is one of the form $T(V, R)$ where V is a graded vector space and $R \subseteq V^{\otimes 2} \oplus V \oplus \mathbb{K}$. Such a presentation is called good if it satisfies:
1. the space of generators is minimal: $R \cap (V \oplus \mathbb{K}) = 0$;
2. the space of relations is maximal: if $(R) \subseteq T(V)$ is the two-sided ideal generated by R, then $(R) \cap (V^{\otimes 2} \oplus V \oplus \mathbb{K}) = R$.

If R is contained in $V^{\otimes 2} \oplus V$, then we will call such a presentation a quadratic-linear (QL) presentation.

Remark 4.2.29. Checking the second condition can be difficult. Consider $A = T(x, y, z)/(xy - y, x^2 - z)$. The obvious choice $R = \langle xy - y, x^2 - z\rangle$ does not satisfy the second condition, as we have:

$$xy - zy = (xy - xxy) + (xxy - zy) = x(y - xy) + (xx - z)y \in (R) \cap (V^{\otimes 2} \oplus V \oplus \mathbb{K}).$$

In other words, we can deduce a new quadratic relations from the existing one (by going through cubical monomials).

Example 4.2.30. Any A admits a good QLC presentation with $V = A$ and $R = \langle a \otimes b - ab \mid a, b \in A\rangle$.

Definition 4.2.31. Let $A = T(V, R)$ be an algebra equipped with a QLC presentation. The quadratic part of A is the quadratic algebra $qA = T(V, qR)$ where qR is the projection of R to $V^{\otimes 2}$.

Lemma 4.2.32. *Let $A = T(V, R)$ be an algebra equipped with a good QLC presentation. There exists linear maps $\varphi_1 \colon qR \to V$ and $\varphi_0 \colon qR \to \mathbb{K}$ such that $R = \{r + \varphi_1(r) + \varphi_0(r) \mid r \in qR\}$.*

Definition 4.2.33. A curved dg-coalgebra is a triple (C, d, θ) where C is a coalgebra, $d \colon C \to C$ is a coderivation of degree 1, and $\theta \colon C \to \mathbb{K}$ is a linear map of degree 2 called the curvature, satisfying the equation $d^2 = (\theta \otimes 1 + 1 \otimes \theta)\Delta$.

Remark 4.2.34. The coderivation d in the definition above may not be a differential. A sufficient condition is $\theta = 0$, in which case we may view (C, d) as a plain dg-coalgebra. It is not necessary, though. For example, let $C = \mathbb{F}_2\langle 1, x\rangle$ with $\deg(x) = -2$, $\Delta(x) = 1 \otimes x + x \otimes 1$, and $\theta(x) = 1$.

Definition 4.2.35. Let $A = T(V, R)$ be an algebra equipped with a good QLC presentation. The Koszul dual of A is the curved dg-coalgebra $A^{¡} = (qA^{¡}, d, \theta)$ where:
- the coderivation d is the unique one whose projection on cogenerators is:

$$d|^{V[1]} \colon qA^{¡} \twoheadrightarrow (qA^{¡})^{(2)} = qR[2] \xrightarrow{\varphi_1} V[2]; \tag{4.2.36}$$

- the curvature is the composite:

$$\theta \colon qA^{¡} \twoheadrightarrow (qA^{¡})^{(2)} = qR[2] \xrightarrow{\varphi_0} \mathbb{K}[2]. \tag{4.2.37}$$

Example 4.2.38. For the QL presentation of Ex. 4.2.30, the Koszul dual dg-coalgebra is the bar construction.

Example 4.2.39. Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ has a good QL presentation:

$$U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}). \tag{4.2.40}$$

The quadratic part of $U(\mathfrak{g})$ is simply the symmetric algebra $S(\mathfrak{g})$. The Koszul dual of $U(\mathfrak{g})$ is $(S^c(\mathfrak{g}[-1]), d, \theta = 0)$ where d is the coderivation given by:

$$d(x_1 \dots x_n) = \sum_{i=1}^{n-1} (-1)^{i-1} x_1 \dots [x_i, x_{i+1}] \dots x_n. \tag{4.2.41}$$

We thus recognize the Chevalley–Eilenberg complex $C_*^{\mathrm{CE}}(\mathfrak{g})$.

Definition 4.2.42. A semi-augmented dg-algebra is a dg-algebra (A, d) equipped with a linear map $\varepsilon \colon A \to \mathbb{K}$. Given such a semi-augmented dg-algebra, we let $\bar{A} = \ker(\varepsilon)$.

The unit and the semi-augmentation define a linear isomorphism $A \cong \bar{A} \oplus \mathbb{K}$. We let $\bar{d} \colon \bar{A} \to \bar{A}$ and $\bar{\mu} \colon \bar{A} \otimes \bar{A} \to \bar{A}$ be the linear maps deduced from the differential and the product of A (by pre/post-composing with the injection/projection on the semi-augmentation kernel).

Definition 4.2.43. Let (C, d, θ) be a curved dg-coalgebra. The curved cobar construction of C is the semi-augmented dg-algebra $\Omega C = (T(C[-1]), d_0 + d_1 + d_2)$, where the summands of the differential are given on generators by the following maps:

$$d_0|_{C[-1]} \colon C[-1] \xrightarrow{\theta} \mathbb{K}[1] \subseteq T(C[-1])[1]; \tag{4.2.44}$$

$$d_1|_{C[-1]} \colon C[-1] \xrightarrow{d} C \subseteq T(C[-1])[1]; \tag{4.2.45}$$

$$d_2|_{C[-1]} \colon C[-1] \xrightarrow{\Delta} (C \otimes C)[-1] \subseteq T(C[1])[1]. \tag{4.2.46}$$

The semi-augmentation $\varepsilon \colon \Omega C \to \mathbb{K}$ is the projection onto $T(C[-1])^{(0)} = \mathbb{K}$.

Remark 4.2.47. The curved cobar construction admits a right adjoint. The curved bar construction of a semi-augmented dg-algebra (A, d, ε) is the curved dg-coalgebra $\mathcal{B}A = (T^c(\bar{A}[1]), d_1 + d_2, \theta)$, where:

- the coderivations d_1, d_2 of $T^c(A[1])$ are respectively given onto cogenerators by:

$$d_2|^{\bar{A}[1]} \colon T^c(\bar{A}[1]) \twoheadrightarrow \bar{A}[1] \otimes \bar{A}[1] \xrightarrow{\bar{\mu}} \bar{A}[1]; \tag{4.2.48}$$

$$d_1|^{\bar{A}[1]} \colon T^c(\bar{A}[1]) \twoheadrightarrow \bar{A}[1] \xrightarrow{\bar{d}} \bar{A}[1]. \tag{4.2.49}$$

- the curvature is defined by $\theta(a) = \varepsilon(da)$ on tensors of length 1, $\theta(a \otimes b) = \varepsilon(\bar{\mu}(a, b))$ on tensors of length 2, and θ vanishes on tensors of length ≥ 3.

Theorem 4.2.50. *Let $A = T(V, R)$ be an algebra equipped with a good QLC presentation. If qA is Koszul, then the canonical morphism $\Omega(A^{\text{i}}) \to A$ is a quasi-isomorphism.*

Remark 4.2.51. Unlike the quadratic case, the application of the previous theorem depends on the chosen presentation. For the QL presentation of Ex. 4.2.30, the Koszul dual of A is the bar construction, and $\Omega \mathcal{B}A \to A$ is always a quasi-isomorphism.

* * *

As mentioned in the introduction of this section, Koszul duality of associative algebras generalizes to operads [82, 79]. Recall that we denote the free operad (resp., cooperad) on a symmetric sequence E by $\mathcal{T}(\mathsf{E})$ (resp., $\mathcal{T}^c(\mathsf{E})$)

Remark 4.2.52. The symmetric sequence $\mathcal{T}(\mathsf{E})$ (resp., $\mathcal{T}^c(\mathsf{E})$) decomposes as $\bigoplus_{n\geq 0} \mathcal{T}(\mathsf{E})^{(n)}$ (resp., $\bigoplus_{n\geq 0} \mathcal{T}^c(\mathsf{E})^{(n)}$), where the superscript (n) indicates the weight, i.e., the number of (co)generators (co)composed together.

Definition 4.2.53. A quadratic operad is one of the form $\mathsf{P} = \mathcal{T}(\mathsf{E},\mathsf{R}) \coloneqq \mathcal{T}(\mathsf{E})/(\mathsf{R})$, where E is a symmetric sequence and $\mathsf{R} \subseteq \mathcal{T}(\mathsf{E})^{(2)}$ is a symmetric subsequence.

Remark 4.2.54. With the above notation, $\mathcal{T}(\mathsf{E},\mathsf{R})$ is initial among operads equipped with a morphism from $\mathcal{T}(\mathsf{E})$ which vanishes on R.

Definition 4.2.55. A quadratic cooperad is one of the form $\mathsf{C} = \mathcal{T}^c(\mathsf{E},\mathsf{R})$, where E is a symmetric sequence and $\mathsf{R} \subseteq \mathcal{T}(\mathsf{E})^{(2)}$ is a symmetric subsequence, and $\mathcal{T}^c(\mathsf{E},\mathsf{R})$ is terminal among cooperads equipped with a morphism to $\mathcal{T}^c(\mathsf{E})$ whose postcomposition with the projection to $\mathcal{T}^c(\mathsf{E})^{(2)}/\mathsf{R}$ vanishes.

Remark 4.2.56. Quadratic (co)operads admit explicit descriptions similar to that of quadratic (co)algebras.

Example 4.2.57. The operads Ass, Com, and Pois_n (which are the non-unital versions of the operads introduced in Sec. 3) are all quadratic.

Definition 4.2.58. Let $\mathsf{P} = \mathcal{T}(\mathsf{E},\mathsf{R})$ be a quadratic operad. The Koszul dual cooperad is:

$$\mathsf{P}^{\mathsf{i}} \coloneqq \mathcal{T}^c(\mathsf{E}[1],\mathsf{R}[2]), \tag{4.2.59}$$

where degree shifts are performed arity-wise. The Koszul dual operad of a quadratic cooperad $\mathsf{C} = \mathcal{T}^c(\mathsf{E},\mathsf{R})$ is:

$$\mathsf{C}^{\mathsf{i}} \coloneqq \mathcal{T}(\mathsf{E}[-1],\mathsf{R}[-2]). \tag{4.2.60}$$

Remark 4.2.61. Just like for algebras, it is often easier to think about the Koszul dual operad $\mathsf{P}^! \coloneqq \mathcal{T}(\mathcal{S}^{-1}\mathsf{E}^\vee[1],\mathsf{R}^\perp)$ of a quadratic operad $\mathsf{P} = \mathcal{T}(\mathsf{E},\mathsf{R})$, where $\mathsf{E}^\vee$ is the arity-wise linear dual of E, $\mathsf{R}^\perp$ is the annihilator of R, and for a symmetric sequence F and $n \geq 0$, we have as a representation of Σ_n:

$$(\mathcal{S}^{-1}\mathsf{F})(n) \coloneqq \mathsf{F}(n) \otimes \mathrm{Hom}(\mathbb{K}[-1]^{\otimes n}, \mathbb{K}[-1]). \tag{4.2.62}$$

Example 4.2.63. We have that $\mathsf{Ass}^! = \mathsf{Ass}$ is auto-dual, $\mathsf{Com}^! = \mathsf{Lie}$ encodes Lie algebras, and $\mathsf{Pois}_n^! = \mathcal{S}^{-n}\mathsf{Pois}_n$ encodes Poisson n-algebras shifted by $n-1$ (i.e., $A[1-n]$ is a $\mathsf{Pois}_n^!$-algebra if and only if A is a Pois_n-algebra).

Remark 4.2.64. From the previous example, we can see that the operad $H_*(\mathsf{D}_n)$ is auto-dual up to degree shifts [82]. This is also true at the chain level over the rationals [79] and over the integers [66]. At the level of spectra, full self-duality is established in [38] (see also earlier results [120, 19] for categories of (co)algebras). Interestingly, the right module given by configuration spaces of a framed manifold is also Koszul self-dual in some sense [122].

Definition 4.2.65. Let P be an operad and C be a cooperad. A twisting morphism $\alpha \colon \mathsf{C}[1] \to \mathsf{P}$ is a morphism of symmetric collections of degree 1 satisfying the Maurer–Cartan equation $\partial\alpha + \alpha \star \alpha = 0$, where $\partial\alpha = d \circ \alpha - \alpha \circ d$ and $\alpha \star \alpha$ is the convolution product [117, Sec. 6.4.2].

Remark 4.2.66. The Maurer–Cartan equation implies that $f_\alpha \colon \Omega\mathsf{C} \to \mathsf{P}$ defined on generators by α preserves the differential.

Definition 4.2.67. Let $\mathsf{P} = \mathcal{T}(\mathsf{E},\mathsf{R})$ be quadratic algebra. There is a canonical twisting morphism $\kappa \colon \mathsf{C} \to \mathsf{P}$ given by:

$$\kappa \colon \mathsf{P}^{\mathsf{i}}[-1] \twoheadrightarrow \mathsf{E} \hookrightarrow \mathsf{P}. \tag{4.2.68}$$

This defines a canonical morphism of operads $f_\kappa \colon \Omega\mathsf{P}^{\mathsf{i}} \to \mathsf{P}$.

Definition 4.2.69. A quadratic operad P is Koszul if f_κ is a quasi-isomorphism.

Example 4.2.70. The operads Ass, Com, Lie, Pois_n are all Koszul.

Koszul duality of quadratic operads has been generalized greatly to cover the case of inhomogeneous quadratic operads by Gálvez-Carrillo–Tonks–Vallette [77], and to the curved case by Hirsh–Millès [91]. For the sake of brevity, we will not expand on these generalizations. Let us simply mention that they have been used e.g., to produce a resolution of the BV operad [77] (i.e., the homology of the framed little 2-disks operad, see Def. 3.2.25) and the operad encoding *unital* associative algebras [91]. See also [141] for the theory in the non-conilpotent completed setting.

* * *

Millès [129] developed a Koszul duality theory for certain types of *algebras* over quadratic Koszul operads. This theory specializes to classical Koszul duality when applied to the operad Ass.

Definition 4.2.71. A symmetric sequence M is reduced if $\mathsf{M}(0) = 0$.

Definition 4.2.72. Let $\mathsf{P} = \mathcal{T}(\mathsf{E}, \mathsf{R})$ be a quadratic reduced operad. A monogenic P-algebra is an algebra of the form $\mathsf{P}(V, S) := \mathsf{P}(V)/(S)$, where V is a graded vector space and S is a subspace of $\mathsf{E}(V)$.

Example 4.2.73. When $\mathsf{P} = \mathsf{Ass}$ with the usual quadratic presentation, a monogenic algebra is the same thing as a quadratic algebra.

Remark 4.2.74. The algebra $\mathsf{P}(V, S)$ is initial among algebras A equipped with a morphism from $\mathsf{P}(V)$ which vanishes on $\mathsf{E}(S)$.

Definition 4.2.75. Let $\mathsf{C} = \mathcal{T}^c(\mathsf{E}, \mathsf{R})$ be a quadratic reduced cooperad. A monogenic C-coalgebra is an coalgebra of the form $\mathsf{C}(V, S)$, where $\mathsf{C}(V, S)$ is terminal among coalgebras equipped with a morphism to $\mathsf{C}(V)$ whose postcomposition with the projection to $\mathsf{E}(S)$ vanishes.

Recall the construction $\mathcal{S}^{-1}$ from Rem. 4.2.61.

Definition 4.2.76. Let $A = \mathsf{P}(V, S)$ be a monogenic algebra over a quadratic reduced operad $\mathsf{P} = \mathcal{T}(\mathsf{E}, \mathsf{R})$. The Koszul dual of A is the monogenic $\mathcal{S}^{-1}\mathsf{P}^{\mathrm{i}}$-coalgebra given by $A^{\mathrm{i}} = \mathsf{P}^{\mathrm{i}}(V, S[1])[-1]$.

The Koszul duality between P and P^{i} defines a bar-cobar adjunction $\Omega_\kappa \dashv B_\kappa$ between the category of P^{i}-coalgebras and P-algebras [79]. This specializes, when $\mathsf{P} = \mathsf{Ass}$, to a non-unital version of the usual bar-cobar adjunction.

Proposition 4.2.77. *Let P be a quadratic reduced operad and A be a monogenic P-algebra. There is a canonical morphism of P-algebras $f_\varkappa \colon \Omega_\kappa A^{\mathrm{i}} \to A$.*

Definition 4.2.78. With the previous notation, A is called Koszul if P is Koszul and $f_\varkappa$ is a quasi-isomorphism.

Remark 4.2.79. Millès [129, Th. 4.9] proves that, when P and A are concentrated in degree 0, A is Koszul if and only if its associated Koszul complex (analogous to the one mentioned in Rem. 4.2.22) is acyclic.

* * *

In [7], we combine the results and ideas of Millès [129] and Hirsh–Millès [91] to obtain a curved Koszul duality theory for algebras over unital versions of binary quadratic operads.

Definition 4.2.80. Let $\mathsf{P} = \mathcal{T}(\mathsf{E}, \mathsf{R})$ be a binary (i.e., $\mathsf{E}(r) = 0$ for $r \neq 2$) quadratic operad. A unital version of P is an operad of the form $\mathsf{uP} = \mathcal{T}(\mathsf{E} \oplus \mathbb{1}, \mathsf{R} \oplus \mathsf{R}')$, where $\mathbb{1}$ is a generator of arity 0 and degree 0, and where:

1. the additional relations R' are quadratic-linear, i.e., $\mathsf{R}' \subseteq \mathcal{T}(\mathsf{E} \oplus \mathbb{K}^{\dagger})^{(2)} \oplus \mathbb{K}\,\mathrm{id}$;
2. the presentation of uP satisfies conditions analogous to those of Def. 4.2.28;
3. the inclusion $E \hookrightarrow E \oplus \mathbb{K}^{\dagger}$ induces an injective morphism $\mathsf{P} \hookrightarrow \mathsf{uP}$;
4. the quadratic part $q\mathsf{uP}$ of P is isomorphic to the coproduct of operads $\mathsf{P} \oplus \mathbb{K}^{\dagger}$.

Until the end of the section, let us fix a binary quadratic operad P and a unital version uP (with the above notation for the spaces of generators and relations), as well as a coaugmented conilpotent cooperad C and a twisting morphism $\alpha \colon \mathsf{C} \to \mathsf{P}$. We moreover assume that α is zero on $\mathsf{C}(r)$ for $r \neq 2$.

Definition 4.2.81. Let C be a C-coalgebra and $\theta \colon C[2] \to \mathbb{K}$ a map of degree 2. The α-star product of θ is the map:

$$\star_\alpha(\theta) \colon C \xrightarrow{\Delta} \mathsf{C}(C) \xrightarrow{\alpha \circ' \Theta} \mathsf{uP}(\mathsf{uP}(C)) \xrightarrow{\gamma} \mathsf{uP}(C), \tag{4.2.82}$$

where Δ is the structure map of C, γ is the structure map of the free uP-algebra $\mathsf{uP}(C)$, Θ is the composite of θ and the inclusion of $\mathbb{K} = \mathbb{K}^{\dagger}$ in $\mathsf{uP}(C)$, and:

$$\forall c \in \mathsf{C}(r), \forall c_i \in C, \quad (\alpha \circ' \Theta)(x(c_1, \dots, c_r)) \coloneqq \sum_{i=1}^{n} (-1)^{i-1} \alpha(x)(c_1, \dots, c_{i-1}, \Theta(c_i), c_{i+1} \dots, c_r). \tag{4.2.83}$$

Remark 4.2.84. Thanks to our hypotheses on uP and α, the image of $\star_\alpha(\theta)$ is actually contained in $C \subseteq \mathsf{uP}(C)$.

Definition 4.2.85. An α-curved C-coalgebra is a triple (C, d, θ) where C is a C-coalgebra, $d \colon C \to C$ is a derivation of degree 1, $\theta \colon C \to \mathbb{K}$ is a map of degree 2, satisfying the two equations:

$$d^2 = \star_\alpha(\theta), \quad \theta \circ d = 0. \tag{4.2.86}$$

Example 4.2.87. Let $\mathsf{P} = \mathsf{Ass}$, uP be the operad encoding unital algebras, $\mathsf{C} = \mathsf{Ass}^{\mathsf{i}}$, and $\kappa \colon \mathsf{C} \to \mathsf{P}$ the Koszul twisting morphism. Then a κ-curved C-coalgebra is a (shifted) coalgebra C endowed with a coderivation d and a linear map $\theta \colon C \to K$ of degree 2 satisfying $\theta d = 0$ and:

$$d^2 = (\theta \otimes \mathrm{id} - \mathrm{id} \otimes \theta)\Delta. \tag{4.2.88}$$

We recognize the dual of the usual notion of curved dg-algebra [135].

Definition 4.2.89 (Cf. Def. 4.2.42)**.** A semi-augmented uP-algebra is a uP-algebra A endowed with a linear map $\varepsilon \colon A \to \mathbb{K}$.

Proposition 4.2.90 ([7, Prop. 2.14])**.** *There is a bar-cobar adjunction $\Omega_\alpha \dashv B_\alpha$ between the category of α-curved C-coalgebras and the category of semi-augmented uP-algebras.*

Definition 4.2.91. A uP-algebra with QLC relations is a uP-algebra of the form $(A = \mathsf{uP}(V)/I, d = 0)$ where:
- the ideal I is generated by $S = I \cap (\mathbb{K}^{\dagger} \oplus V \oplus \mathsf{E}(V))$;
- the generating relations all contain quadratic terms, i.e., $S \cap (\mathbb{K}^{\dagger} \oplus V) = 0$.

Definition 4.2.92. Let $A = \mathsf{uP}(V)/(S)$ be a uP-algebra with QLC relations. Its quadratic part is the P-algebra $qA \coloneqq \mathsf{P}(V)/(qS)$, where qS is the projection of S to $E(V)$.

Remark 4.2.93. Note that qA is a monogenic algebra (Def. 4.2.72).

Remark 4.2.94. Thanks to the hypotheses on the presentation, there exist linear maps $\varphi_1 \colon qS \to V$ and $\varphi_0 \colon qS \to \mathbb{K}^{\dagger}$ such that $S = \{x + \varphi_1(x) + \varphi_0(x) \mid x \in qS\}$.

Definition 4.2.95. Let $A = \mathsf{uP}(V)/(S)$ be a uP-algebra with QLC relations. The Koszul dual of A is the α-curved C-coalgebra $(qA^{\text{¡}}, d, \theta)$, where $qA^{\text{¡}}$ is the Koszul dual of qA in the sense of Def. 4.2.76, the coderivation d is the unique coderivation whose projection to cogenerators is $qA^{\text{¡}} \twoheadrightarrow qS \xrightarrow{\varphi_1} V$, and $\theta \colon qA^{\text{¡}} \twoheadrightarrow qS \xrightarrow{\varphi_0} \mathbb{K}\text{↾}$.

There is a canonical morphism of uP-algebras $f_\varkappa \colon \Omega_\kappa qA^{\text{¡}} \to A$.

Definition 4.2.96. Let $A = \mathsf{uP}(V)/(S)$ be a uP-algebra with QLC relations. It is called Koszul if $f_\varkappa$ is a quasi-isomorphism.

Theorem 4.2.97 ([7])**.** *With the above notation, A is Koszul if and only if qA is Koszul in the sense of [129].*

Question 4.2.98. *Can the previous theory be adapted to the case where P is not binary?*

4.3 Applications

To perform the computations of Ex. 2.4.36, we used a particular nice kind of basis for quadratic algebras: a Poincaré–Birkhoff–Witt (PBW) basis. The existence of such a basis can be used to prove that a quadratic algebra is Koszul. Let us now briefly explain what these bases are and give an example; we refer to [137, 133, 117] for details.

In what follows, let us fix a vector space V with a totally ordered finite basis $(v_1, \dots, v_k)$, and a quadratic algebra $A = T(V, R)$ for $R \subseteq V \otimes V$. Let us also write $I = \{1, \dots, k\}$ for the set of indices of the basis of V.

Definition 4.3.1. We let $I^* \coloneqq \bigsqcup_{n \geq 0} I^n$ be the set of multi-indices in I, equipped with the lexicographic order (also known as dictionary order).

Lemma 4.3.2. *The free algebra $T(V)$ admits a totally ordered basis $(v_{\bar\imath})_{i \in I^*}$, where:*

$$\forall \bar\imath = (i_1, \dots, i_n) \in I^*,\ v_{\bar\imath} \coloneqq v_{i_1} \dots v_{i_n}. \tag{4.3.3}$$

Definition 4.3.4. Let $o \colon \mathbb{N} \to I^*$ be the unique increasing bijection. For $p \in \mathbb{N}$, let $F_p A \subseteq A$ be the subspace of A spanned by $\{v_{o(k)} \mid k \leq p\}$. The associated graded algebra $\operatorname{gr} A$ is the one associated to the increasing filtration $F_\bullet A$, that is,

$$\operatorname{gr} A \coloneqq \bigoplus_{p \in \mathbb{N}} F_p A / F_{p-1} A. \tag{4.3.5}$$

Definition 4.3.6. The leading space of relations R_{lead} is the kernel of the morphism of algebras $T(V) \to \operatorname{gr} A$ which sends a generator to its class in $\operatorname{gr} A$.

Concretely, an element of $r \in R$ can always be written (up to a scalar) as:

$$r = v_i v_j - \sum_{(k,l) < (i,j) \in I^*} \lambda_{k,l} v_k v_l. \tag{4.3.7}$$

The summand $v_i v_j$ is called the leading term of the relator r. The space $R_{\text{lead}} \subseteq V \otimes V$ is spanned by the leading terms of the relators.

Definition 4.3.8. Let $\bar L^{(2)} \subseteq I^2$ be the set of multi-indices that appear in R_{lead}. Let $L^{(2)} \coloneqq I^2 \setminus \bar L^{(2)}$ be its complement. Let $L = \bigsqcup_{n \geq 0} L^{(n)}$ and $\bar L = \bigsqcup_{n \geq 0} \bar L^{(n)}$, where:

$$\begin{aligned} L^{(n)} &= \{(i_1, \dots, i_n) \in I^n \mid \forall 1 \leq k < n,\ (i_k, i_{k+1}) \in L^{(2)}\}; \\ \bar L^{(n)} &= \{(i_1, \dots, i_n) \in I^n \mid \forall 1 \leq k < n,\ (i_k, i_{k+1}) \in \bar L^{(2)}\}. \end{aligned} \tag{4.3.9}$$

Remark 4.3.10. The above conditions are vacuous when $n \leq 1$, so that $L^{(0)} = \bar{L}^{(0)} = I^0 = \{*\}$ and $L^{(1)} = \bar{L}^{(1)} = I = \{1, \dots, k\}$.

Let us now explain the idea behind these definitions. Let r be a relator as in Equation (4.3.7). Intuitively, we can "rewrite" the term $v_i v_j$ (indexed by an element of $\bar{L}$) into a sum of terms indexed by smaller multi-indices. Since we mod out by the ideal generated by R, this is true of any monomial that contains $v_i v_j$. We can then repeat this process iteratively to end up with a sum of monomials that do not contain any leading term, i.e., monomials indexed by elements of L. By induction, this process (or its dual for the Koszul dual coalgebra) proves that:

Proposition 4.3.11. *The family $(v_{\bar{\imath}})_{\bar{\imath} \in L}$ spans A, and the family $(s^{|\bar{\imath}|} v_{\bar{\imath}})_{\bar{\imath} \in \bar{L}}$ spans $A^{\text{!`}}$.*

However, these families may be linearly dependent. The issue is that in monomials of degree ≥ 3, there may be more than one way to start rewriting terms. Choosing different branches may lead to different terms, leading to new relations.

Example 4.3.12. Let $k = 2$ and let R be spanned by $v_2 v_2 - v_1 v_1$. Then R_{lead} is spanned by $(v_2 v_2)$. In particular, $(1, 1, 2)$ and $(2, 1, 1)$ belong to $L^{(3)}$. However, the corresponding elements $v_1 v_1 v_2$ and $v_2 v_1 v_1$ of A are equal, because we can rewrite the monomial $v_2 v_2 v_2$ in two different ways to get them.

Definition 4.3.13. If the family $(v_{\bar{\imath}})_{\bar{\imath} \in L}$ is a basis of A, then it is called a Poincaré–Birkhoff–Witt (PBW) basis of A.

Since we are only dealing with quadratic relations, we get the following useful criterion:

Theorem 4.3.14 (Diamond lemma)**.** *Let $A = T(V, R)$ be a quadratic algebra with the same notation as above. If the family $(v_{\bar{\imath}})_{\bar{\imath} \in L^{(3)}}$ is linearly independent in A, then $(v_{\bar{\imath}})_{\bar{\imath} \in L}$ is a PBW basis of A.*

The existence of a PBW basis has important consequences.

Theorem 4.3.15. *If a quadratic algebra A admits a PBW basis, then it is Koszul.*

We presented the theory of PBW bases in the non-necessarily-commutative case for simplicity. However, an analogue of the theory exists in the commutative realm, see [133, Chap. V, Sec. 8]. Commutative PBW bases are defined analogously to (plain) PBW bases, with the main difference being that arbitrary monomials are replaced with commutative monomials, i.e., there is no difference between xy and yx for generators $x, y \in V$. The definitions of L and $\bar{L}$ are slightly different: instead of checking that consecutive elements are (or are not) leading terms, one must check that for any sub-monomial of a given commutative monomial. The analogue of the diamond lemma, and the implication of the Koszul property by the existence of a PBW basis, remain true for commutative PBW bases [97].

Example 4.3.16. Let us now apply the above theory to the algebra $\mathsf{G}_{A_g}(r)$ from Ex. 2.4.36. Let us order the set of generators of that algebra as follows:

1. First, the generators $\alpha_1^1, \beta_1^1, \alpha_2^1, \beta_2^1, \dots, \alpha_r^1, \beta_r^1, \dots, \alpha_r^g, \beta_r^g$;
2. Then, the generators ω_{ij} with $i < j$, with the lexicographic order on (i, j).

This gives rise to these relations, which we write with the leading term first and where $\gamma \in \{\alpha, \beta\}$:

$$\begin{gathered}
\gamma_j^u \omega_{ij} - \gamma_i^u \omega_{ij}, \ \forall u, i, j; \\
\alpha_i^u \beta_i^u - \alpha_i^1 \beta_i^1, \ \forall u > 1, i; \quad \alpha_i^u \beta_i^v, \ \forall u \neq v, i; \\
\alpha_i^u \alpha_i^v, \ \forall u \neq v, i; \quad \beta_i^u \beta_i^v, \ \forall u \neq v, i; \\
\omega_{ik} \omega_{jk} - \omega_{ij} \omega_{jk} + \omega_{ij} \omega_{ik}, \ \forall i < j < k.
\end{gathered}$$

It is then a little exercise to check that the diamond lemma applies, so that this choice of ordered basis for the generators gives rise to a commutative PBW basis for the algebra. The basis is given as follows, where $\nu_i = \alpha_i^1 \beta_i^1$:

$$\left\{ \gamma_{k_1} \dots \gamma_{k_a} \omega_{i_1 j_1} \dots \omega_{i_b j_b} \;\middle|\; \begin{array}{l} \gamma = \nu \text{ or } \gamma \in \{\alpha^u, \beta^u \mid 1 \le u \le g\}; \\ k_1 < \dots < k_a;\ j_1 < \dots < j_b;\ i_l < j_l,\ \forall l; \\ \{k_1, \dots, k_a\} \cap \{j_1, \dots, j_b\} = \varnothing. \end{array} \right\}. \tag{4.3.17}$$

* * *

One of the motivations for our study of configuration was the computation of factorization homology as defined by Ayala–Francis–Tanaka [20]. Let us now briefly explain their definition, and how one can recover (special cases of) a theorem of Knudsen [102] from our results.

Definition 4.3.18. Let P be an operad, M be a right P-module, and N be a left P-module. The tensor product of M and N over P is the coequalizer:

$$\mathsf{M} \circ_{\mathsf{P}} \mathsf{N} \coloneqq \operatorname{coeq}(\mathsf{M} \circ \mathsf{P} \circ \mathsf{N} \rightrightarrows \mathsf{M} \circ \mathsf{N}). \tag{4.3.19}$$

Remark 4.3.20. If M is a right P-module and A is a P-algebra, then we can view A as a left P-module concentrated in arity zero and we have $(\mathsf{M} \circ_{\mathsf{P}} A)(0) = S_{\mathsf{M}}(A)$ (Def. 3.1.39). By abuse of notation, we will write $\mathsf{M} \circ_{\mathsf{P}} A = (\mathsf{M} \circ_{\mathsf{P}} A)(0)$, and similarly for the derived tensor product.

Definition 4.3.21. Let $\mathsf{P}, \mathsf{M}, \mathsf{N}$ be as above. The derived tensor product of M and N over P is the homotopy coequalizer:

$$\mathsf{M} \circ^{\mathbb{L}}_{\mathsf{P}} \mathsf{N} \coloneqq \operatorname{hocoeq}(\mathsf{M} \circ \mathsf{P} \circ \mathsf{N} \rightrightarrows \mathsf{M} \circ \mathsf{N}). \tag{4.3.22}$$

Remark 4.3.23. The tensor product is a left Quillen functor in each variable [65], so the derived tensor product can be computed by taking a cofibrant resolution of either the left or the right module.

Let M be a framed n-manifold. Recall the Fulton–MacPherson operad FM_n and the right module FM_M from Sec. 3.4.

Definition 4.3.24. Let A be an FM_n-algebra. The factorization homology of M with coefficients in A is $\int_M A \coloneqq \mathsf{FM}_M \circ^{\mathbb{L}}_{\mathsf{FM}_n} A$.

Intuitively, an element of (the underived version of) $\int_M A$ is a configuration of points of M, each decorated by an element of A. When several points collide, their decoration are multiplied together using the FM_n-algebra structure of A. This is reminiscent of the configuration spaces with summable labels of Salvatore [144].

Remark 4.3.25. If we considered the non-unital version of FM_n, then FM_M would be cofibrant [161, Lem. 2.3] and $\int_M A$ would be given by the plain tensor product rather than the derived one.

Let M be a simply connected framed closed manifold of dimension ≥ 4, let P be a Poincaré duality model of M, and let A be an FM_n-algebra. Using the formality of FM_n (Th. 3.3.7), the category of $C_*(\mathsf{FM}_n)$-algebras and the category of Pois_n-algebras are Quillen equivalent. Thus, there exists a Poisson n-algebra B such that $C_*(A) \simeq B$ as $C_*(\mathsf{FM}_n)$-algebras. Using Th. 2.4.18, Th. 3.3.13, and results on the tensor product [65], we then get that:

$$C_*\left(\int_M A\right) \simeq \mathsf{G}_P^\vee \circ^{\mathbb{L}}_{\mathsf{Pois}_n} B, \tag{4.3.26}$$

where $\mathsf{G}_P^\vee$ is the (arity-wise) linear dual of G_P, which forms a right Pois_n-module.

Suppose now that $B = S(\mathfrak{g}[1-n])$ is the symmetric algebra on a shifted Lie algebra, which has a natural n-Poisson structure (the Lie bracket is extended as a biderivation). This is an analogue of the higher enveloping algebra of Knudsen [102]. By reinterpreting a computation of Félix–Thomas [62], we obtain:

Proposition 4.3.27 (Special case of [100, Th. 3.16], [5, Prop. 81])**.** *Let M and $B = S(\mathfrak{g}[1-n])$ be as above. Then $C_*(\int_M B)$ is quasi-isomorphic to the Chevalley–Eilenberg complex of the Lie algebra $P \otimes \mathfrak{g}$ (with homological grading).*

Using curved Koszul duality (Th. 4.2.97), we generalized this result to symplectic Poisson n-algebras. Let $n \geq 1$ and $D \geq 0$. The Poisson n-algebra $A_{n;D}$ is the algebra of polynomial functions on the standard shifted symplectic space $T^*\mathbb{R}^D[1-n]$. Concretely, $A_{n;D} = S(x_1, \dots, x_D, \xi_1, \dots, \xi_D)$ is the unital graded symmetric algebra on $2D$ variables, with $\deg x_i = 0$ and $\deg \xi_i = 1-n$. The Lie bracket is extended as a biderivation from the following formulas on generators:

$$\forall i, j, \quad \{x_i, x_j\} = 0, \quad \{\xi_i, \xi_j\} = 0, \quad \{x_i, \xi_j\} = \delta_{ij} 1. \tag{4.3.28}$$

We prove that factorization homology with coefficients in $A_{n;D}$ can be computed by a unital version of the Chevalley–Eilenberg chain complex. Using a perfect pairing on the generating Lie algebra of $A_{n;D}$ tensored with P, we deduce that:

Proposition 4.3.29 ([7, Prop. 5.17])**.** *Let M be a simply connected framed manifold of dimension ≥ 4 and A a symplectic Poisson n-algebra. Then $\int_M A$ is acyclic, i.e., $H_*(\int_M A) = \mathbb{R}$.*

* * *

To conclude this section, we present our application of Koszul duality to prefactorization algebras (joint work with Rabinovich [8]). We refer again to Costello–Gwilliam [48, 49] for background on (pre)factorization algebras.

Definition 4.3.30. Let M be a topological space. A prefactorization algebra $\mathcal{F}$ on M is the data of:

- for each open set $U \subseteq M$, of a dg-module $\mathcal{F}(U)$;
- for each collection of disjoint open subsets $U_1, \dots, U_k$ contained in some open set V, of maps:

$$\mu^V_{U_1,\dots,U_k} \colon \bigotimes_{i=1}^{k} \mathcal{F}(U_i) \to \mathcal{F}(V); \tag{4.3.31}$$

satisfying associativity and equivariance constraints.

Example 4.3.32. A prefactorization algebra on $\varnothing$ is a commutative algebra.

Example 4.3.33. A prefactorization algebra on $\{*\}$ is the data of a commutative algebra and a module over it.

Example 4.3.34. Let A be a unital associative algebra. There is a prefactorization algebra $\mathcal{F}_A$ on $\mathbb{R}$ defined as follows. Any open subset $U \subseteq \mathbb{R}$ can be written as a (potentially infinite) disjoint union of nonempty intervals, $U = \bigsqcup_i J_i$. Let $\mathcal{F}(U) \coloneqq \bigotimes_i^{\mathrm{res}} A$ be the restricted tensor product, which is the subspace of the tensor product spanned by elementary tensors of the form $\bigotimes_i a_i$ such that all but finitely many a_i is equal to 1. Given a collection of disjoint open subsets $U_1 \sqcup \dots \sqcup U_k \subseteq V$, the structure map $\mu^V_{U_1,\dots,U_k}$ is defined using the multiplication of A.

Remark 4.3.35. The prefactorization algebra $\mathcal{F}$ of the previous example satisfies a particular property: if $U \subseteq V$ is an inclusion of open subsets which is a homotopy equivalence, then the structure map $\mu^V_U \colon \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism. Such a prefactorization algebra is called a locally constant prefactorization algebra. Any locally constant prefactorization algebra on $\mathbb{R}$ is actually of the form $\mathcal{F}_A$ for some A.

Prefactorization algebras on M are algebras over a certain colored operad:

Definition 4.3.36. Let Disj_M (or simply Disj when M is clear) be the colored operad whose colors are the open subsets of M, and such that:

$$\mathsf{Disj}(U_1, \dots, U_k; V) = \begin{cases} \mathbb{R}\{\mu^V_{U_1,\dots,U_k}\}, & \text{if the } U_i \text{ are pairwise disjoint and contained in } V; \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.37}$$

Definition 4.3.38. Let Opens be the sub-operad of Disj spanned by unary operations. Let Tens be the sub-operad of Disj spanned by those $\mu^V_{U_1,\dots,U_k}$ such that V is the (disjoint) union of the U_i.

Remark 4.3.39. An algebra over Opens is a precosheaf on M. The operad Opens admits a presentation with generators μ^V_U (for $U \subsetneq V \subseteq M$ open) and relations $\mu^W_V \mu^V_U = \mu^W_U$ (for $U \subsetneq V \subsetneq W \subseteq M$ open).

Proposition 4.3.40. *The operad* Disj *is isomorphic to the composition product* $\mathsf{Opens} \circ \mathsf{Tens}$ *for a certain distributive law (see [117, Sec. 8.6]).*

Theorem 4.3.41 ([8])**.** *The operad* Tens *is quadratic Koszul. The presentation of* Opens *from Rem. 4.3.39 is inhomogeneous Koszul. Consequently, the operad* Disj *is (inhomogeneous) Koszul.*

The proof of the Koszul property goes through an analysis of the Koszul complex of $q\mathsf{Opens}$, which is easy to deal with; and the Koszul complex of Tens, which is similar to a colored version of the Koszul complex of Com.

This theorem allows us to define an operad $\mathsf{hoDisj} = \Omega(\mathsf{Disj}^{\mathsf{i}})$ whose algebras are homotopy prefactorization algebras. The explicit description of hoDisj-algebras [8, Prop. 4.1] is lengthy due to complicated sums and signs. Briefly, a hoDisj algebra (on M) is the data of:

- for every open subset $U \subseteq M$, a dg-module $\mathcal{F}(U)$;
- for every sequence of towers of inclusions $\mathcal{U} = (U_{11} \subseteq \dots \subseteq U_{1s_1}, \dots, U_{k1} \subseteq \dots \subseteq U_{ks_k})$ of open subsets, an operation of degree $2-k$:

$$\mu_{\mathcal{U}} \colon \mathcal{F}(U_{11}) \otimes \dots \otimes \mathcal{F}(U_{k1}) \to \mathcal{F}(U_{1s_1} \sqcup \dots \sqcup U_{ks_k}). \tag{4.3.42}$$

The operations $\mu_{\mathcal{U}}$ vanish on signed shuffles, and the differential is setup in such a way that the operations $\mu_{\mathcal{U}}$ provide homotopies for the precosheaf relation $\mu^W_V \mu^V_U = \mu^W_U$ and the associativity relation of $\mu^{U \sqcup V}_{U,V}$.

Example 4.3.43. A homotopy prefactorization algebra on $\varnothing$ is a C_∞-algebra, that is, an algebra over $\Omega(\mathsf{Com}^{\mathsf{i}})$.

Example 4.3.44. A homotopy prefactorization algebra on $\{1\}$ is a triple (A, M, η) where A is a C_∞-algebra, M is a C_∞-A-module, and $\eta \colon A \to M$ is an ∞-morphism of modules.

Example 4.3.45. A homotopy prefactorization algebra $\mathcal{F}$ on the Sierpiński space $\{1,2\}$ (whose only nontrivial open set is $\{1\}$) is the data of:

- a C_∞-algebra $A = \mathcal{F}(\varnothing)$;
- a C_∞-A-module $M = \mathcal{F}(\{1\})$;
- an ∞-morphism $\eta_M \colon A \to M$;
- a C_∞-A-module $N = \mathcal{F}(\{1,2\})$;
- an ∞-morphism $\eta_M \colon A \to N$;
- an ∞-morphism $f \colon M \to N$;
- a C_∞-homotopy h between $f \circ \eta_M$ and η_N.

Example 4.3.46. We consider the prefactorization algebra $\mathcal{F}_{\mathfrak{g}}$ on $\mathbb{R}$ out of the Chevalley–Eilenberg complex of a Lie algebra $\mathfrak{g}$ tensored with the algebra of differential forms with compact support on $\mathbb{R}$. Costello–Gwilliam [48, Sec. 3.4] proved that the cohomology of $\mathcal{F}_{\mathfrak{g}}$ is the universal enveloping algebra $U(\mathfrak{g})$ (viewed as a prefactorization algebra on $\mathbb{R}$ as in Ex. 4.3.34). We actually construct an ∞-quasi-isomorphism of hoDisj-algebras of $\mathcal{F}_{\mathfrak{g}}$ with its cohomology, i.e., we prove that it is formal as a Disj-algebra.

5 Appendix: Computation of Massey products in the configuration spaces of a surface

The objective of this appendix is to compute explicitly a Massey product in the cohomology of the configuration space of $r = 3$ points in a surface of genus $g = 1$, that is, $\mathrm{Conf}_{S^1 \times S^1}(3)$. We proceed in several steps:

1. We first compute a graded PBW basis for the Lambrechts–Stanley model $G_A(3)$ of the configuration space, where $A = H^*(S^1 \times S^1)$ is a model of the surface (which is formal).
2. We compute the matrices of the differential $d : G_A(3)^k \to G_A(3)^{k+1}$ and the product $\mu : G_A(3)^k \otimes G_A(3)^l \to G_A(3)^{k+l}$ in this graded basis.
3. We compute representatives for bases of the cohomology groups $H^k(G_A(3)) = Z^k / B^k$, where Z^k is the kernel of the differential and B^k its image in degree k.
4. Using these representatives, we find a choice of deformation retract of cochain complexes of $G_A(3)$ onto its cohomology.
5. This deformation retract leads to explicit formulas for the transferred A_∞-structure on $H^*(G_A(3))$. We deal with the binary product $m_2 : H^k \otimes H^l \to H^{k+l}$ and the ternary product $m_3 : H^k \otimes H^l \otimes H^n \to H^{k+l+n-1}$.
6. We end with a search for explicit classes α, β, γ such that $m_3(\alpha, \beta, \gamma) \neq 0$ modulo the ideal generated by (α, γ), i.e., a nonzero obstruction to formality.

Remark : nothing in our code is specific to the case $r = 3$, $g = 1$. We could change the definitions of r and g at the beginning to compute the cohomology and transferred structure for any number of points in a surface of any genus, at the cost of increased computation time. Our choice is simply the smallest one that yields a non-formal configuration space.

Produced with Mathematica `14.1.0 for Microsoft Windows (64-bit) (July 16, 2024)`.

Technical concerns

We define error messages to be used when the functions defined below are used in incorrect degrees.

```
In[1]:= General::neg = "Degree `1` cannot be negative.";
General::neg2 = "Degrees `1` and `2` cannot be negative.";
General::neg3 = "Degrees `1`, `2`, and `3` cannot be negative.";
General::top = "Degree `1` cannot be more than `2`.";
General::top2 = "The sum of degrees `1` and `2`cannot be more than `3`.";
General::top3 = "The sum of degrees `1`, `2`, and `3` cannot be more than `4`.";
```

Skew-commutative product and relations

Basic definitions

These definitions make the wedge product bilinear:

```
In[7]:= ClearAll[Wedge]
        (a___) ∧ (q_)?NumericQ ∧ (b___) := q * a ∧ b
        (a___) ∧ ((q_)?NumericQ * (b_)) ∧ (c___) := q * a ∧ b ∧ c
        (a___) ∧ (b_Plus) ∧ (c___) := (a ∧ #1 ∧ c &) /@ b
        Wedge[a_] := a;
        Wedge[] = 1;
        SetAttributes[Wedge, {Flat, Listable}]
```

Generators

Formatting of the generators of the cohomology of the surface:

```
In[14]:= MakeBoxes[α[u_, i_], StandardForm] ^:=
          TemplateBox[MakeBoxes /@ {u, i}, "α", DisplayFunction → (SubsuperscriptBox["α", #2, #1] &)]
         MakeBoxes[β[u_, i_], StandardForm] ^:=
          TemplateBox[MakeBoxes /@ {u, i}, "β", DisplayFunction → (SubsuperscriptBox["β", #2, #1] &)]
         MakeBoxes[ω[i_, j_], StandardForm] ^:= TemplateBox[MakeBoxes /@ {i, j},
           "ω", DisplayFunction → (SubscriptBox["ω", RowBox[{#1, ",", #2}]] &)]
```

Let us consider the case of $r = 3$ points in a surface of genus $g = 1$ as our main example:

```
In[17]:= g = 1; r = 3;
```

The list of generators. The order is important for the PBW basis later.

```
In[18]:= generators = Catenate[
           {SortBy[Flatten[Table[γ[u, i], {γ, {α, β}}, {u, g}, {i, r}]], {Apply[List], Head}],
            Flatten[Table[ω[i, j], {i, r}, {j, i + 1, r}]]}]
Out[18]=
```

$\{\alpha_1^1, \beta_1^1, \alpha_2^1, \beta_2^1, \alpha_3^1, \beta_3^1, \omega_{1,2}, \omega_{1,3}, \omega_{2,3}\}$

Useful definition: the top-degree class of the surface, in position i.

```
In[19]:= ν[i_] := α[1, i] ∧ β[1, i];

In[20]:= positions = First /@ PositionIndex[generators]
Out[20]=
```

$\langle| \alpha_1^1 \to 1, \beta_1^1 \to 2, \alpha_2^1 \to 3, \beta_2^1 \to 4, \alpha_3^1 \to 5, \beta_3^1 \to 6, \omega_{1,2} \to 7, \omega_{1,3} \to 8, \omega_{2,3} \to 9 |\rangle$

Twisting & square zero

These definitions make it so that $u \wedge u = 0$ and $u \wedge v = \pm v \wedge u$, including if these terms appear in the middle of a monomial.

```
In[21]:= Wedge[OrderlessPatternSequence[u_, u_, ___]] := 0

In[22]:= (x___) ∧ (u_) ∧ (v_) ∧ (y___) /; positions[u] > positions[v] := -(x ∧ v ∧ u ∧ y)
```

Relations

The relations defining the Lambrechts–Stanley model.

```
In[23]:= (x___) ∧ (γ : α | β)[u_, j_] ∧ (y___) ∧ ω[i_, j_] ∧ (z___) := x ∧ γ[u, i] ∧ y ∧ ω[i, j] ∧ z

In[24]:= (x___) ∧ α[u_, i_] ∧ (y___) ∧ β[u_, i_] ∧ (z___) /; u > 1 := x ∧ α[1, i] ∧ y ∧ β[1, i] ∧ z

In[25]:= Wedge[OrderlessPatternSequence[α[u_, i_], β[v_, i_], ___]] /; u ≠ v := 0

In[26]:= Wedge[OrderlessPatternSequence[α[_, i_], α[_, i_], ___]] := 0
```

```
In[27]:= Wedge[OrderlessPatternSequence[β[_, i_], β[_, i_], ___]] := 0
```

```
In[28]:= (x___) ∧ ω[i_, k_] ∧ (y___) ∧ ω[j_, k_] ∧ (z___) /; i < j < k :=
          x ∧ ω[i, j] ∧ y ∧ ω[j, k] ∧ z - x ∧ ω[i, j] ∧ y ∧ ω[i, k] ∧ z
```

Basis for the CDGA

Produces the PBW basis of the CDGA:

- For a generator *x*, **delPatt[*x*]** returns the list of generators that cannot appear together with *x* in a monomial (i.e., terms that would be rewritten)
- Then we “fold” the lists by successively adding new generators (but only wedging them with monomials that are not forbidden by **delPatt**).

For this to work correctly, we need **generators** to be in the correct order.

```
In[29]:= delPatt[(α | β)[1, i_]] := (α | β)[v_ /; v > 1, i] | ω[_, i];
         delPatt[(α | β)[u_, i_]] /; u > 1 := (α | β)[v_ /; v ≥ u, i] | ω[_, i];
         delPatt[ω[i_, j_]] := ω[_, j];
```

```
In[32]:= combineMonomials[l_, x_] := Join[l, x ∧ Select[l, FreeQ[delPatt[x]]]]
```

```
In[33]:= basis = Fold[combineMonomials, {1}, Reverse[generators]];
```

```
In[34]:= Length[basis]
Out[34]=
         120
```

Computes the degree of a monomial and groups elements of the basis according to their degree.

```
In[35]:= deg[x_Wedge] := deg[x] = Length[x]
         deg[_α] = 1; deg[_β] = 1; deg[_ω] = 1;
         deg[(x_)?NumberQ] = 0;
```

```
In[38]:= gradedBasis = GroupBy[basis, deg];
```

```
In[39]:= topDegree = Max[Keys[gradedBasis]]
Out[39]=
         6
```

Computes the dimension of the piece of the model of degree *d*.

```
In[40]:= dim[d_Integer] := Length[gradedBasis[d]]
```

```
In[41]:= TableForm[Table[{d, dim[d], Short[gradedBasis[d]]}, {d, 0, topDegree}],
          TableHeadings → {None, {"Degree", "Dimension", "Basis"}}]
```

Out[41]//TableForm=

Degree	Dimension	Basis
0	1	$\{1\}$
1	9	$\{\omega_{2,3}, \omega_{1,3}, \omega_{1,2}, \beta_3^1, \alpha_3^1, \beta_2^1, \alpha_2^1, \beta_1^1, \alpha_1^1\}$
2	29	$\{\omega_{1,2} \wedge \omega_{2,3}, \omega_{1,2} \wedge \omega_{1,3}, \beta_3^1 \wedge \omega_{1,2}, \alpha_3^1 \wedge \omega_{1,2}, \alpha_3^1 \wedge \beta_3^1, \beta_2^1 \wedge \omega_{2,3}, \beta_2^1 \wedge \omega_{1,3}, \beta_2^1 \wedge \beta_3^1, \beta_2^1 \wedge \alpha_3^1$
3	42	$\{\alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}, \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \alpha_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{2,3}, \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{1,3}, \alpha_2^1 \wedge \beta_2^1 \wedge \beta_3^1, \alpha_2^1 \wedge \beta$
4	29	$\{\alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \beta_1^1 \wedge \alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}, \beta_1^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \beta_1^1 \wedge \alpha_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{2,3},$
5	9	$\{\beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \alpha_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \alpha_1^1 \wedge \beta_1^1 \wedge \alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}, \alpha_1^1 \wedge \beta_1^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1,$
6	1	$\{\alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1\}$

Differential

Defines the differential as a linear map satisfying the Leibniz rule, with prescribed behavior on generators.

```
In[42]:= δ[(x_)?NumericQ] = 0;

In[43]:= δ[sum_Plus] := δ /@ sum

In[44]:= δ[(x_)?NumericQ * (w_)] := x * δ[w]

In[45]:= δ[_α] = 0; δ[_β] = 0;

In[46]:= δ[ω[i_, j_]] := ν[i] + ν[j] - Sum[α[u, i] ∧ β[u, j] + α[u, j] ∧ β[u, i], {u, g}]

In[47]:= δ[m : (x_) ∧ (y__)] := δ[m] = δ[x] ∧ y - x ∧ δ[Wedge[y]]
```

Example

```
In[48]:= δ[ω[1, 2]]
```

Out[48]=

$\alpha_1^1 \wedge \beta_1^1 - \alpha_1^1 \wedge \beta_2^1 + \alpha_2^1 \wedge \beta_2^1 + \beta_1^1 \wedge \alpha_2^1$

```
In[49]:= δ[ν[2] ∧ ω[1, 2]]
```

Out[49]=

$\alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1$

Matrix of the differential and of the product

Coefficients

Gets the coefficients of a (sum of) monomials in the graded basis.

```
In[50]:= basisCoeff[0][k_] := {k};

In[51]:= basisCoeff[d_Integer][_] /; d > topDegree = {0};

In[52]:= basisCoeff[d_Integer][x_] :=
          CoefficientList[ExpandAll[x] /. Thread[gradedBasis[d] → x^Range[0, dim[d] - 1]], x, dim[d]]
```

Example:

```
In[53]:= basisCoeff[2][ω[1, 3] ∧ ω[2, 3]]
```

Out[53]=

```
{1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Sanity check:

```
In[54]:= Reduce[Table[
           With[{u = Array[a, dim[d]]}, basisCoeff[d][u . gradedBasis[d]] == u], {d, 0, topDegree}]]
```

Out[54]=

True

The matrix of $\_ \wedge \_ : C^{d_1} \otimes C^{d_2} \to C^{d_1+d_2}$

Remark: this computation (and the next one) can be quite slow for high values of g, r.

```
In[55]:= ClearAll[wedgeMatrix]
         wedgeMatrix[d1_Integer, d2_Integer] /; d2 < d1 := Transpose[wedgeMatrix[d2, d1], {1, 3, 2}]
         wedgeMatrix[d1_Integer, d2_Integer] :=
          (wedgeMatrix[d1, d2] = Transpose[SparseArray[Table[basisCoeff[d1 + d2][x ∧ y],
                 {x, gradedBasis[d1]}, {y, gradedBasis[d2]}], {}], {2, 3, 1}]) /;
           ((d2 ≥ d1 && (d1 ≥ 0 && d2 ≥ 0)) || Message[wedgeMatrix::neg2, d1, d2]) &&
            (d1 + d2 ≤ topDegree || Message[wedgeMatrix::top2, d1, d2, topDegree])
```

Example:

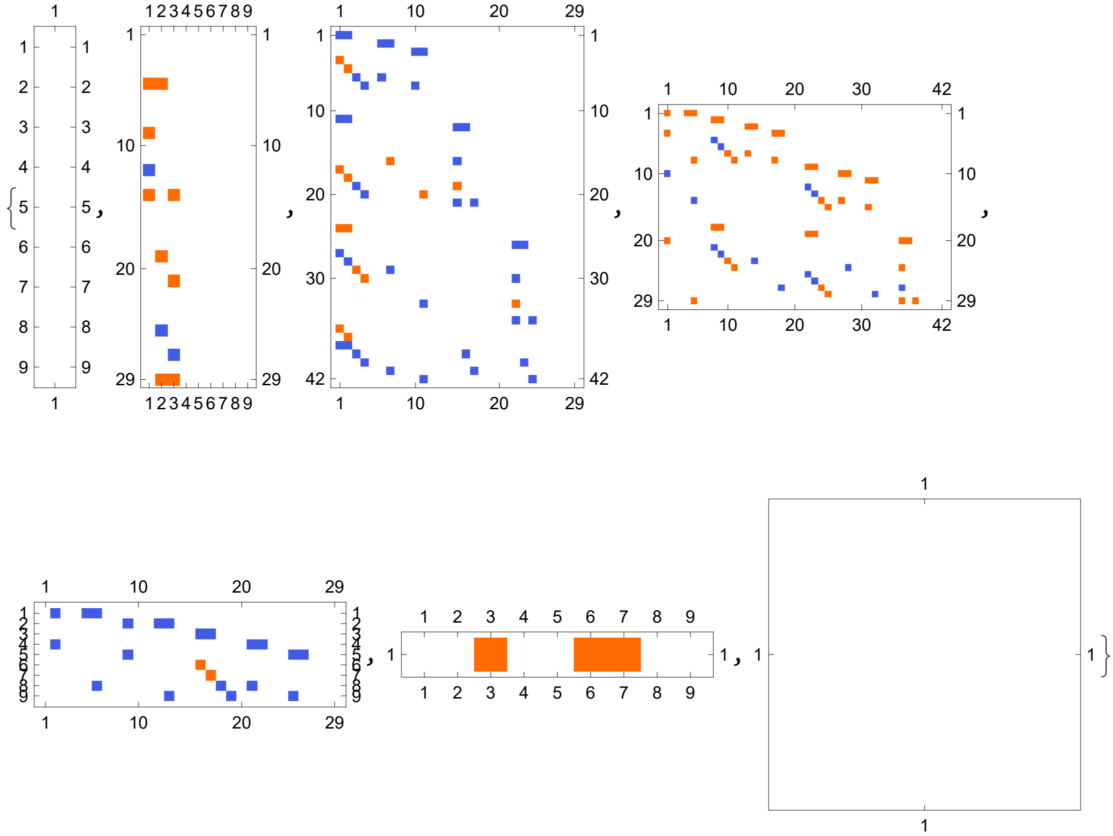
```
In[58]:= wedgeMatrix[2, 1] . UnitVector[dim[1], 5] . UnitVector[dim[2], 2] . gradedBasis[3] ==
          gradedBasis[2][[2]] ∧ gradedBasis[1][[5]]
Out[58]=
         True
```

The matrix of $\delta : C^d \to C^{d+1}$

```
In[59]:= ClearAll[δMatrix]
         δMatrix[-1] = 0;
         δMatrix[d_Integer] :=
          (δMatrix[d] = Transpose[SparseArray[basisCoeff[d + 1] @* δ /@ gradedBasis[d]]]) /;
          (d ≥ 0 || Message[δMatrix::neg, d]) && (d ≤ topDegree || Message[δMatrix::top, d, topDegree])
```

“Plots” of the matrices (colors indicate nonzero entries, negative is blue, positive is orange).

```
In[62]:= Table[MatrixPlot[δMatrix[d]], {d, 0, topDegree}]
Out[62]=
```



Sanity check: the matrix faithfully encodes δ

```
In[63]:= δFromMatrix[x_] /; deg[x] + 1 > topDegree := 0
```

```
In[64]:= δFromMatrix[x_] := With[{d = deg[x]}, gradedBasis[d + 1] . δMatrix[d] . basisCoeff[d][x]]
```

In[65]:=
```
Comap[{First, Equal}][{δFromMatrix[ν[2] ∧ ω[1, 2]], δ[ν[2] ∧ ω[1, 2]]}]
```

Out[65]=

$\{\alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1, \text{True}\}$

In[66]:=
```
AllTrue[basis, δ[#1] == δFromMatrix[#1] & ]
```

Out[66]=

True

Computing the cohomology

Goal: decompose $C^d = Z^d \oplus Q^d = \tilde{Q}^{d-1} \oplus H^d \oplus Q^d$, where $Z^d = \ker(\delta)$ and $\delta : Q^d \to \tilde{Q}^d$ is an isomorphism

Basis of $Z^d = \ker(\delta : C^d \to C^{d+1}) \subset C^d$

In[67]:=
```
ClearAll[ker]
ker[d_Integer] := (ker[d] = SparseArray[RowReduce[NullSpace[δMatrix[d]]]]) /;
  (d ≥ 0 || Message[ker::neg, d]) && (d ≤ topDegree || Message[ker::top, d, topDegree])
```

In[69]:=
```
TableForm[ Table[{d, Length[ker[d]], Short[ker[d] . gradedBasis[d]]}, {d, 0, 6}],
 TableHeadings → {None, {"d", "dim[ker[d]]", "ker[d]"}}]
```

Out[69]//TableForm=

d	dim[ker[d]]	ker[d]
0	1	$\{1\}$
1	6	$\{\beta_3^1, \alpha_3^1, \beta_2^1, \alpha_2^1, \beta_1^1, \alpha_1^1\}$
2	17	$\{-(\beta_1^1 \wedge \omega_{1,2}) - \beta_1^1 \wedge \omega_{1,3} + \beta_1^1 \wedge \omega_{2,3} + \beta_2^1 \wedge \omega_{1,3} - \beta_2^1 \wedge \omega_{2,3} + \beta_3^1 \wedge \omega_{1,2}, -(\alpha_1^1 \wedge \omega_{1,2}) - \alpha_1^1 \wedge \omega_{1,3}$ -
3	26	$\{-(\alpha_1^1 \wedge \beta_1^1 \wedge \omega_{1,2}) - 3\,\alpha_1^1 \wedge \beta_1^1 \wedge \omega_{1,3} + 2\,\alpha_1^1 \wedge \beta_1^1 \wedge \omega_{2,3} + 2\,\alpha_1^1 \wedge \beta_2^1 \wedge \omega_{1,3} - \alpha_1^1 \wedge \beta_2^1 \wedge \omega_{2,3} - \alpha_2^1 \wedge \beta_2^1$ ∧
4	21	$\{\alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, -2\,\alpha_1^1 \wedge \beta_1^1 \wedge \beta_2^1 \wedge \omega_{1,3} + \alpha_1^1 \wedge \beta_1^1 \wedge \beta_2^1 \wedge \omega_{2,3} - \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{1,3} + \beta_1^1 \wedge \alpha_3^1 \wedge \beta_3^1 \wedge$ (
5	8	$\{\beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, \alpha_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1, -(\alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{1,3}) + \alpha_1^1 \wedge \beta_1^1 \wedge \alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}$
6	1	$\{\alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \alpha_3^1 \wedge \beta_3^1\}$

Basis for $Q^d \subset C^d$, the complement of $\ker(\delta : C^d \to C^{d+1})$

The elements of **ker[*d*]** are row vectors, so its nullspace is its the orthogonal complement (an arbitrary choice of complementary subspace).

In[70]:=
```
ClearAll[complKer]
complKer[0] = {};
complKer[d_Integer] := (complKer[d] =
    With[{null = NullSpace[ker[d]]}, If[null == {}, {}, SparseArray[RowReduce[null]]]]) /;
  (d > 0 || Message[complKer::neg, d]) && (d ≤ topDegree || Message[complKer::top, d, topDegree])
```

In[73]:=
```
TableForm[Table[{d, Length[complKer[d]], If[complKer[d] != {}, Short[complKer[d] . gradedBasis[d
  TableHeadings -> {None, {"d", "dim[complKer[d]", "complKer[d]"}}]
```

Out[73]//TableForm=

d	dim[complKer[d]	complKer[d]
0	0	$\emptyset$
1	3	$\{\omega_{2,3}, \omega_{1,3}, \omega_{1,2}\}$
2	12	$\{\omega_{1,2} \wedge \omega_{2,3}, \omega_{1,2} \wedge \omega_{1,3}, \beta_1^1 \wedge \omega_{1,2} + \beta_3^1 \wedge \omega_{1,2}, \alpha_1^1 \wedge \omega_{1,2} + \alpha_3^1 \wedge \omega_{1,2}, -(\beta_1^1 \wedge \omega_{1,2}) + \beta_2^1 \wedge \omega_2$
3	16	$\{\alpha_1^1 \wedge \beta_1^1 \wedge \omega_{1,3} + \alpha_1^1 \wedge \beta_1^1 \wedge \omega_{2,3} + \alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}, -(\alpha_1^1 \wedge \beta_1^1 \wedge \omega_{1,2}) - \alpha_1^1 \wedge \beta_1^1 \wedge \omega_{1,3} - 2\,\alpha_1^1 \wedge \beta_1^1$
4	8	$\{\alpha_1^1 \wedge \beta_1^1 \wedge \beta_2^1 \wedge \omega_{1,3} + \alpha_1^1 \wedge \beta_1^1 \wedge \beta_2^1 \wedge \omega_{2,3} + \beta_1^1 \wedge \alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}, -(\alpha_1^1 \wedge \beta_1^1 \wedge \beta_2^1 \wedge \omega_{1,3}) - 2\,\alpha_1^1 \wedge$
5	1	$\{\alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{1,3} + \alpha_1^1 \wedge \beta_1^1 \wedge \alpha_2^1 \wedge \beta_2^1 \wedge \omega_{2,3} + \alpha_1^1 \wedge \beta_1^1 \wedge \alpha_3^1 \wedge \beta_3^1 \wedge \omega_{1,2}\}$
6	0	$\emptyset$

Basis for $\delta(Q^d) = \tilde{Q}^d \subset C^{d+1}$

In[74]:=
```
ClearAll[imComplKer]
imComplKer[-1] = {};
imComplKer[0] = {};
imComplKer[d_Integer] := (imComplKer[d] = Transpose[δMatrix[d].Transpose[complKer[d]]]) /;
  (d > 0 || Message[imComplKer::neg, d]) &&
   (d ≤ topDegree - 1 || Message[imComplKer::top, d, topDegree - 1])
```

In[78]:=
```
TableForm[Table[{d, Length[imComplKer[d]],
    If[imComplKer[d] ≠ {}, Short[imComplKer[d] . gradedBasis[d + 1]], "∅"]},
  {d, 0, topDegree - 1}], TableHeadings → {None, {"d", "dim[imComplKer[d]]", "imComplKer[d]"}}]
```

Out[78]//TableForm=

d	dim[imComplKer[d]]	imComplKer[d]
0	0	∅
1	3	$\{\alpha_2^1\wedge\beta_2^1-\alpha_2^1\wedge\beta_3^1+\alpha_3^1\wedge\beta_3^1+\beta_2^1\wedge\alpha_3^1,\ \alpha_1^1\wedge\beta_1^1-\alpha_1^1\wedge\beta_3^1+\alpha_3^1\wedge\beta_3^1+\beta_1^1\wedge\alpha_3^1,\ \alpha_1^1\wedge\beta_1^1-\alpha_1^1\wedge$
2	12	$\{-(\alpha_1^1\wedge\beta_1^1\wedge\omega_{1,2})+\alpha_1^1\wedge\beta_1^1\wedge\omega_{2,3}-\alpha_1^1\wedge\beta_2^1\wedge\omega_{2,3}+\alpha_1^1\wedge\beta_3^1\wedge\omega_{1,2}+\alpha_2^1\wedge\beta_2^1\wedge\omega_{2,3}-\alpha_3^1\wedge$
3	16	$\{\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1-\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_3^1+3\,\alpha_1^1\wedge\beta_1^1\wedge\alpha_3^1\wedge\beta_3^1+\alpha_1^1\wedge\beta_1^1\wedge\beta_2^1\wedge\alpha_3^1-\alpha_1^1\wedge\beta_2^1\wedge\alpha_3^1$
4	8	$\{-(\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1\wedge\beta_3^1)-3\,\alpha_1^1\wedge\beta_1^1\wedge\beta_2^1\wedge\alpha_3^1\wedge\beta_3^1-\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1\wedge\alpha_3^1\wedge\beta_3^1,\ \ll 6\gg,\ \alpha_1^1\wedge$
5	1	$\{3\,\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1\wedge\alpha_3^1\wedge\beta_3^1\}$

Sanity check: $\dim(Q^d) = \dim(\tilde{Q}^d)$ (except for the top dimension)

In[79]:=
```
Reduce[Table[Length[imComplKer[d]] == Length[complKer[d]], {d, 0, topDegree - 1}]]
```

Out[79]=
```
True
```

Basis for $\tilde{D}^d \subset Z^{d+1}$ expressed in terms of the basis of Z^d found before

In[80]:=
```
ClearAll[restrImComplKer]
restrImComplKer[-1] = {};
restrImComplKer[0] = {};
restrImComplKer[d_Integer] := (restrImComplKer[d] =
    Transpose[SparseArray[LinearSolve[Transpose[ker[d + 1]], Transpose[imComplKer[d]]]]]) /;
  (d > 0 || Message[restrImComplKer::neg, d]) &&
   (d ≤ topDegree - 1 || Message[restrImComplKer::top, d, topDegree - 1])
```

In[84]:=
```
TableForm[Table[{d, Length[restrImComplKer[d]], If[restrImComplKer[d] != {}, Short[restrImComplKe
  TableHeadings -> {None, {"d", "dim[restrImComplKer[d]", "restrImComplKer[d]"}}]
```

Out[84]//TableForm=

d	dim[restrImComplKer[d]	restrImComplKer[d]
0	0	∅
1	3	$\{\alpha_2^1\wedge\beta_2^1-\alpha_2^1\wedge\beta_3^1+\alpha_3^1\wedge\beta_3^1+\beta_2^1\wedge\alpha_3^1,\ \alpha_1^1\wedge\beta_1^1-\alpha_1^1\wedge\beta_3^1+\alpha_3^1\wedge\beta_3^1+\beta_1^1\wedge\alpha_3^1,\ \alpha_1^1\wedge\beta_1^1-$
2	12	$\{-(\alpha_1^1\wedge\beta_1^1\wedge\omega_{1,2})+\alpha_1^1\wedge\beta_1^1\wedge\omega_{2,3}-\alpha_1^1\wedge\beta_2^1\wedge\omega_{2,3}+\alpha_1^1\wedge\beta_3^1\wedge\omega_{1,2}+\alpha_2^1\wedge\beta_2^1\wedge\omega_{2,3}-$
3	16	$\{\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1-\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_3^1+3\,\alpha_1^1\wedge\beta_1^1\wedge\alpha_3^1\wedge\beta_3^1+\alpha_1^1\wedge\beta_1^1\wedge\beta_2^1\wedge\alpha_3^1-\alpha_1^1\wedge\beta_2^1$
4	8	$\{-(\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1\wedge\beta_3^1)-3\,\alpha_1^1\wedge\beta_1^1\wedge\beta_2^1\wedge\alpha_3^1\wedge\beta_3^1-\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1\wedge\alpha_3^1\wedge\beta_3^1,\ \ll 6\gg,$
5	1	$\{3\,\alpha_1^1\wedge\beta_1^1\wedge\alpha_2^1\wedge\beta_2^1\wedge\alpha_3^1\wedge\beta_3^1\}$

In[85]:=
```
Reduce[Table[restrImComplKer[d] . ker[d + 1] . gradedBasis[d + 1] ==
    imComplKer[d] . gradedBasis[d + 1], {d, 1, topDegree - 1}]]
```

Out[85]=
```
True
```

Basis for the homology $H^d \subset Z^d$ and then $H^d \subset C^d$

In[86]:=
```
ClearAll[restrHomology]
restrHomology[d_Integer] := (restrHomology[d] = With[{rk = restrImComplKer[d - 1]},
  If[rk == {}, IdentityMatrix[Length[ker[d]], TargetStructure → "Sparse"],
      With[{null = NullSpace[rk]},
   If[null == {}, {}, SparseArray[RowReduce[null]]]]]]) /;
   (d ≥ 0 || Message[restrHomology::neg, d]) &&
    (d ≤ topDegree || Message[restrHomology::top, d, topDegree])
```

In[88]:=
```
ClearAll[homology]
homology[d_Integer] :=
 (homology[d] = With[{rhom = restrHomology[d]}, If[rhom == {}, {}, rhom.ker[d]]]) /;
   (d ≥ 0 || Message[homology::neg, d]) && (d ≤ topDegree || Message[homology::top, d, topDegree])
```

In[90]:=
```
ClearAll[betti]
betti[d_Integer] := Length[homology[d]] /;
   (d ≥ 0 || Message[betti::neg, d]) && (d ≤ topDegree || Message[betti::top, d, topDegree])
```

In[92]:=
```
TableForm[Table[{d, betti[d], Short[If[betti[d] > 0, homology[d] . gradedBasis[d], "∅"]]}, {d, 0
```

Out[92]//TableForm=

d	betti[d]	homology[d]
0	1	$\{1\}$
1	6	$\{\beta^1_3, \alpha^1_3, \beta^1_2, \alpha^1_2, \beta^1_1, \alpha^1_1\}$
2	14	$\{-(\beta^1_1 \wedge \omega_{1,2}) - \beta^1_1 \wedge \omega_{1,3} + \beta^1_1 \wedge \omega_{2,3} + \beta^1_2 \wedge \omega_{1,3} - \beta^1_2 \wedge \omega_{2,3} + \beta^1_3 \wedge \omega_{1,2}, -(\alpha^1_1 \wedge \omega_{1,2}) - \alpha^1_1 \wedge \omega_{1,3} + \alpha^1_1$
3	14	$\{-2\,\alpha^1_1 \wedge \beta^1_1 \wedge \omega_{1,2} - 4\,\alpha^1_1 \wedge \beta^1_1 \wedge \omega_{1,3} + 3\,\alpha^1_1 \wedge \beta^1_1 \wedge \omega_{2,3} + 3\,\alpha^1_1 \wedge \beta^1_2 \wedge \omega_{1,3} - 2\,\alpha^1_1 \wedge \beta^1_2 \wedge \omega_{2,3} + \alpha^1_1 \wedge \beta^1_3 \wedge \omega_1$
4	5	<<1>>
5	0	∅
6	0	∅

Sanity check: the representatives of homology classes are actually cycles.

In[93]:=
```
Reduce[
 Table[betti[d] == 0 || AllTrue[homology[d] . gradedBasis[d], δ[#1] == 0 &], {d, 0, topDegree}]]
```

Out[93]=
```
True
```

The combined basis of $C^d = \tilde{Q}^{d-1} \oplus H^d \oplus Q^d$

In[94]:=
```
ClearAll[qhqBasis]
qhqBasis[d_Integer] := (qhqBasis[d] = Join[imComplKer[d - 1], homology[d], complKer[d]]) /;
   (d ≥ 0 || Message[qhqBasis::neg, d]) && (d ≤ topDegree || Message[qhqBasis::top, d, topDegree])
```

Sanity check: these are actually bases.

In[96]:=
```
Reduce[Table[Det[qhqBasis[d]] ≠ 0, {d, 0, 6}]]
```

Out[96]=
```
True
```

In[97]:=
```
ClearAll[qhqInverse]
qhqInverse[d_Integer] := (qhqInverse[d] = SparseArray[Inverse[Transpose[qhqBasis[d]]]]) /;
  (d ≥ 0 || Message[qhqInverse::neg, d]) &&
    (d ≤ topDegree || Message[qhqInverse::top, d, topDegree])
```

In[99]:=
```
qhqDecompose[d_Integer][(u_)?VectorQ] := TakeList[qhqInverse[d] . u,
    {Length[imComplKer[d - 1]], Length[homology[d]], Length[complKer[d]]}] /;
  (d ≥ 0 || Message[qhqDecompose::neg, d]) &&
    (d ≤ topDegree || Message[qhqDecompose::top, d, topDegree])
```

Deformation retraction

The goal of this section is to compute the maps that define the deformation retraction of C^d onto H^d.

We define two functions each time: one that computes the matrix, and one that applies that matrix.

Injection $i_d : H^d \to C^d$ (bases: homology[d] for H^d, basis[d] for C^d)

In[100]:=

```
ClearAll[injMatrix]
injMatrix[d_Integer] := (injMatrix[d] = Transpose[homology[d]]) /;
  (d ≥ 0 || Message[injMatrix::neg, d]) && (d ≤ topDegree || Message[injMatrix::top, d])
```

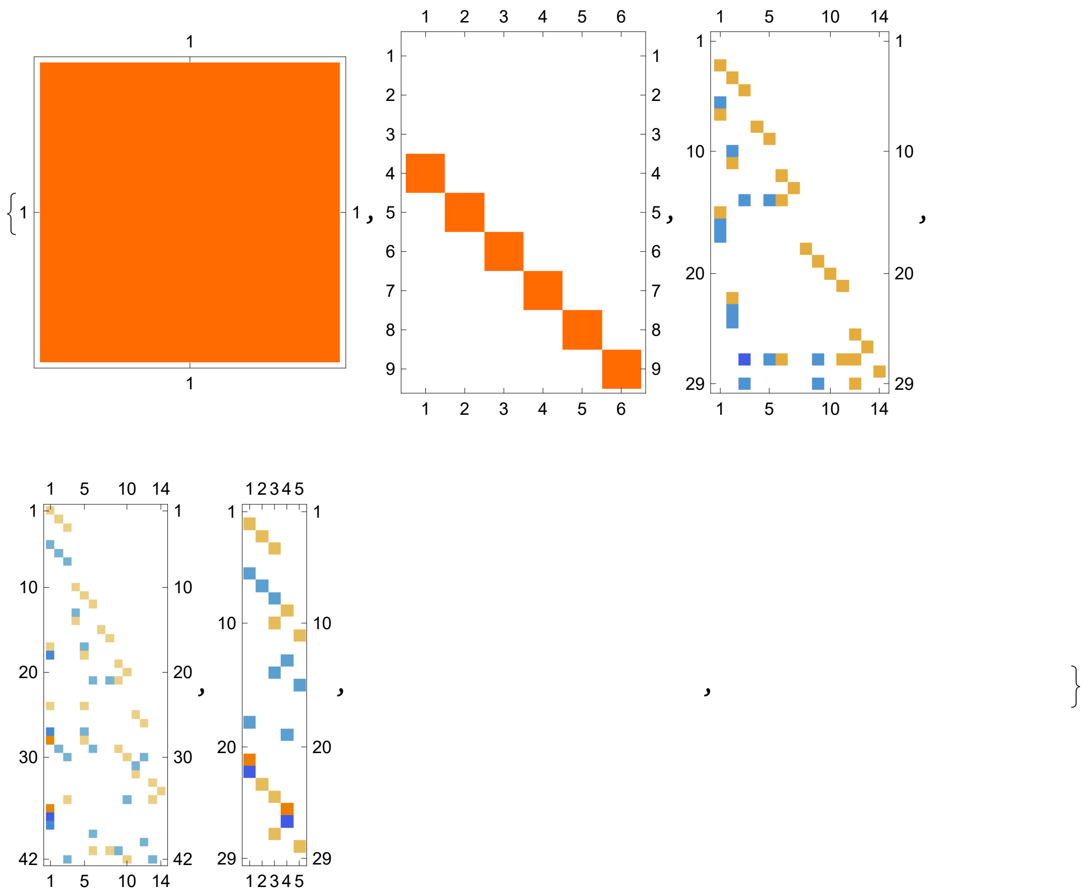
In[102]:=

```
inj[d_Integer][u_] := injMatrix[d].u
```

In[103]:=

```
Table[MatrixPlot[injMatrix[d]], {d, 0, topDegree}]
```

Out[103]=



Examples

In[104]:=

```
UnitVector[betti[2], 1] . homology[2] . gradedBasis[2]
```

Out[104]=

$-(\beta_1^1 \wedge \omega_{1,2}) - \beta_1^1 \wedge \omega_{1,3} + \beta_1^1 \wedge \omega_{2,3} + \beta_2^1 \wedge \omega_{1,3} - \beta_2^1 \wedge \omega_{2,3} + \beta_3^1 \wedge \omega_{1,2}$

In[105]:=

```
inj[2][UnitVector[betti[2], 1]] . gradedBasis[2]
```

Out[105]=

$-(\beta_1^1 \wedge \omega_{1,2}) - \beta_1^1 \wedge \omega_{1,3} + \beta_1^1 \wedge \omega_{2,3} + \beta_2^1 \wedge \omega_{1,3} - \beta_2^1 \wedge \omega_{2,3} + \beta_3^1 \wedge \omega_{1,2}$

In[106]:=

```
UnitVector[betti[2], 2] . homology[2] . gradedBasis[2]
```

Out[106]=

$$-\left(\alpha_1^1 \wedge \omega_{1,2}\right) - \alpha_1^1 \wedge \omega_{1,3} + \alpha_1^1 \wedge \omega_{2,3} + \alpha_2^1 \wedge \omega_{1,3} - \alpha_2^1 \wedge \omega_{2,3} + \alpha_3^1 \wedge \omega_{1,2}$$

In[107]:=

```
inj[2][UnitVector[betti[2], 2]] . gradedBasis[2]
```

Out[107]=

$$-\left(\alpha_1^1 \wedge \omega_{1,2}\right) - \alpha_1^1 \wedge \omega_{1,3} + \alpha_1^1 \wedge \omega_{2,3} + \alpha_2^1 \wedge \omega_{1,3} - \alpha_2^1 \wedge \omega_{2,3} + \alpha_3^1 \wedge \omega_{1,2}$$

Retraction $r_d : C^d \to H^d$ (bases: basis[d] for C^d, homology[d] for H^d)

In[108]:=

```
ClearAll[rtrMatrix]
rtrMatrix[d_Integer] := (rtrMatrix[d] = qhqInverse[d][[
      Length[imComplKer[d - 1]] + 1 ;; Length[imComplKer[d - 1]] + Length[homology[d]]]]) /;
   (d ≥ 0 || Message[rtrMatrix::neg, d]) && (d ≤ topDegree || Message[rtrMatrix::top, d, topDegree])
```

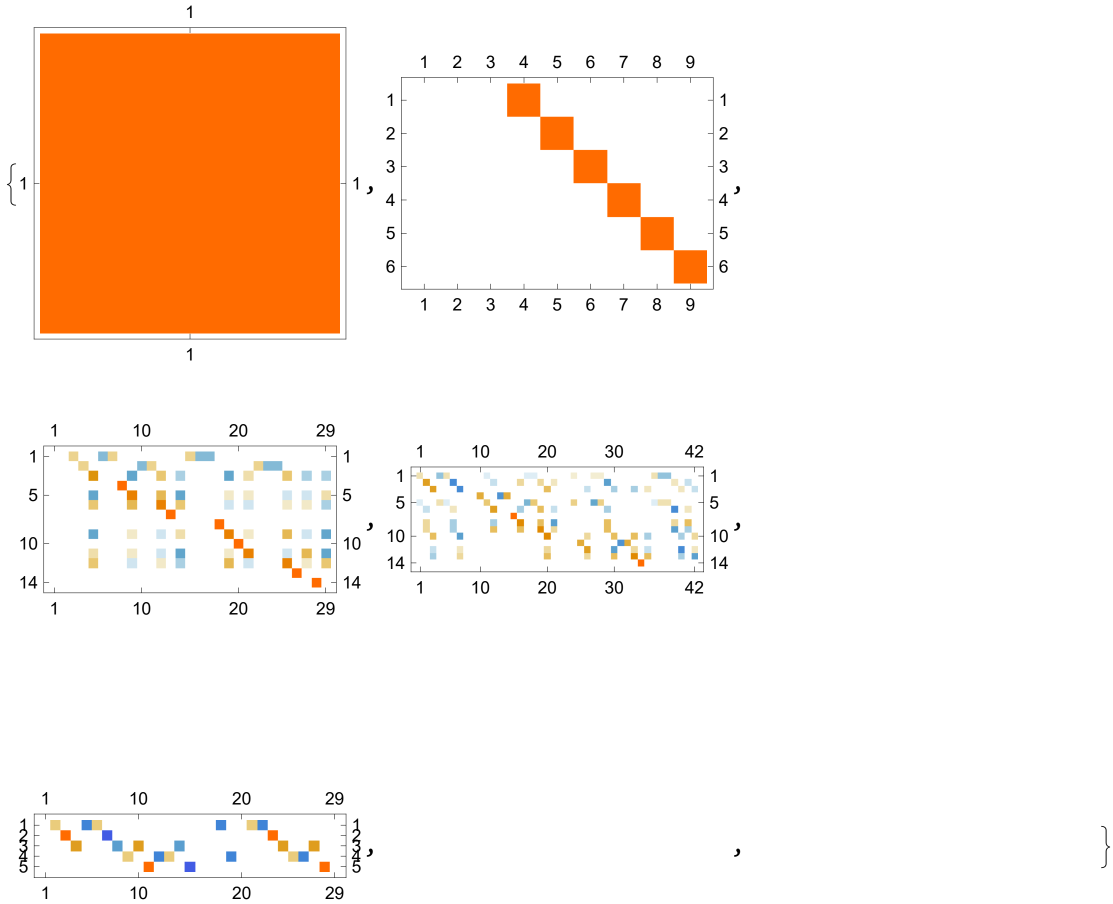
In[110]:=

```
rtr[d_Integer][0 | _?VectorQ] /; rtrMatrix[d] == {} := {}
rtr[d_Integer][u_] := rtrMatrix[d].u
```

In[112]:=

```
Table[MatrixPlot[rtrMatrix[d]], {d, 0, topDegree}]
```

Out[112]=



Examples

In[113]:=

```
UnitVector[dim[2], 3] . gradedBasis[2]
```

Out[113]=

$\beta_3^1 \wedge \omega_{1,2}$

In[114]:=

```
rtr[2][UnitVector[dim[2], 3]] . homology[2] . gradedBasis[2]
```

Out[114]=

$$-\frac{1}{6}\beta_1^1 \wedge \omega_{1,2} - \frac{\beta_1^1 \wedge \omega_{1,3}}{6} + \frac{\beta_1^1 \wedge \omega_{2,3}}{6} + \frac{\beta_2^1 \wedge \omega_{1,3}}{6} - \frac{\beta_2^1 \wedge \omega_{2,3}}{6} + \frac{\beta_3^1 \wedge \omega_{1,2}}{6}$$

In[115]:=

```
δMatrix[2].homology[2]ᵀ.rtr[2][UnitVector[dim[2], 3]]
```

Out[115]=

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Sanity check: $r \circ i = \text{id}$.

In[116]:=

```
Reduce[Table[betti[d] == 0 ||
   rtrMatrix[d].injMatrix[d] == IdentityMatrix[Dimensions[rtrMatrix[d]][[1]]], {d, 0, topDegree}]]
```

Out[116]=

```
True
```

Differential $d_d : C^d \to C^{d+1}$ (bases: basis[_] for source and target)

In[117]:=

```
diff[-1][_] = 0;
```

In[118]:=

```
diff[d_Integer][u_] := δMatrix[d].u
```

Homotopy $h_d : C^{d+1} \to C^d$ (bases: basis[_] for source and target)

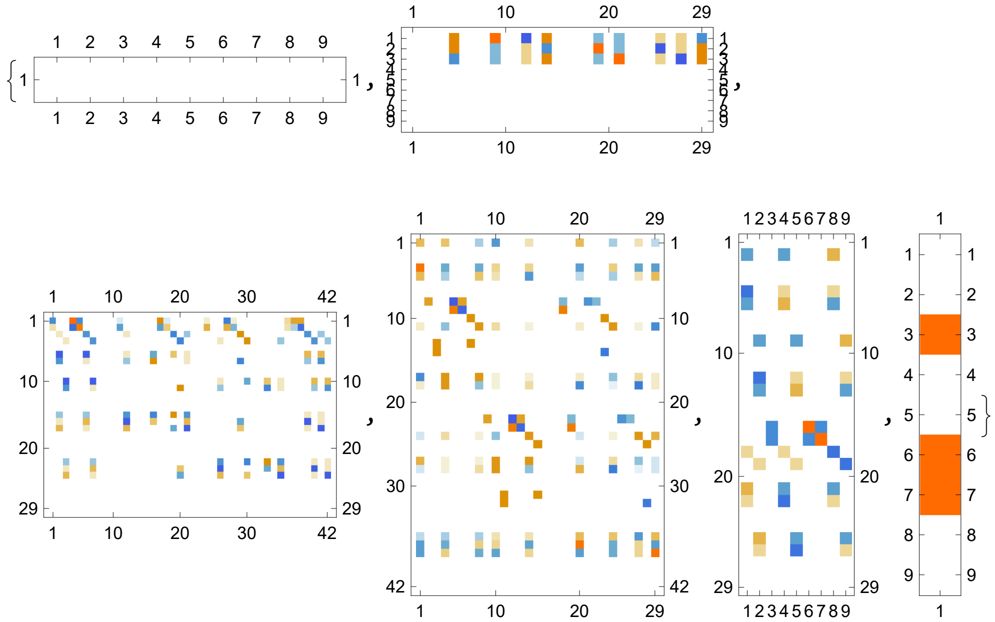
In[119]:=

```
ClearAll[htpMatrix]
htpMatrix[-1] = {{}};
htpMatrix[topDegree] = {{}};
htpMatrix[d_Integer] :=
 (htpMatrix[d] =
    SparseArray[Transpose[qhqBasis[d]] . PadRight[With[{l = Length[imComplKer[d]]},
        PadRight[If[l == 0, {{}}, IdentityMatrix[Length[imComplKer[d]]]],
    {-dim[d], dim[d + 1]}]], {-dim[d], dim[d + 1]}] . qhqInverse[d + 1]]) /;
  (d ≥ 0 || Message[htpMatrix::neg, d]) &&
   (d ≤ topDegree || Message[htpMatrix::top, d, topDegree])
```

In[123]:=

```
Table[MatrixPlot[htpMatrix[d]], {d, 0, topDegree - 1}]
```

Out[123]=



In[124]:=

```
htp[d_Integer][_] /; htpMatrix[d] == {{}} := ConstantArray[0, dim[d]]
htp[d_Integer][u_] := htpMatrix[d].u
```

Sanity check: $\mathrm{id} - i \circ r = d \circ h + h \circ d$

In[126]:=

```
Reduce[Table[With[{u = Array[x, dim[d]]},
   u - inj[d][rtr[d][u]] == diff[d - 1][htp[d - 1][u]] + htp[d][diff[d][u]]], {d, 0, topDegree}]]
```

Out[126]=

```
True
```

Transferred structure

The goal of this section is to compute the transferred A_∞-structure on the cohomology of the configuration space.

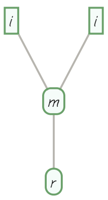
Binary product m_2

Formula for the transferred structure:

In[127]:=

```
ExpressionTree[r[m[i, i]], ImageSize → Tiny, TreeLayout → Bottom]
```

Out[127]=



Computes $m_2(u, v)$ for u, v expressed in the bases **homology[d_1], homology[d_2]**, with result in the basis **homology[$d_1 + d_2$]**.

If **betti[_] = 0**, then the tensor indices get mixed up since some are missing. We do not need to compute the transferred structure in that case, anyway.

In[128]:=

```
ClearAll[mulMatrix]
mulMatrix::zeroBetti = "Transferred structure is trivial (vanishing Betti number).";
mulMatrix[d1_Integer, d2_Integer] :=
 (mulMatrix[d1, d2] = Activate[TensorContract[Inactive[TensorProduct][
       rtrMatrix[d1 + d2], wedgeMatrix[d1, d2], injMatrix[d1], injMatrix[d2]],
   {{2, 3}, {4, 6}, {5, 8}}]]) /; ((d1 ≥ 0 && d2 ≥ 0) || Message[mulMatrix::neg2, d1, d2]) &&
   (d1 + d2 ≤ topDegree || Message[mulMatrix::top2, d1, d2, topDegree]) &&
  ((betti[d1] > 0 && betti[d2] > 0 && betti[d1 + d2] > 0) || Message[mulMatrix::zeroBetti])
```

In[131]:=

```
mul[d1_, d2_][u_, v_] := mulMatrix[d1, d2] . v . u
```

Remark: We cannot plot these “matrices,” which are actually tensors of rank 3. We could choose a basis for the tensor product, but this would be inconveniently large.

Example

This one works:

In[132]:=

```
mulMatrix[1, 1]
```

Out[132]=

SparseArray[Specified elements: 120 Dimensions: {14, 6, 6}]

This one doesn’t:

In[133]:=

```
mulMatrix[3, 3]
```

mulMatrix: Transferred structure is trivial (vanishing Betti number).

Out[133]=

```
mulMatrix[3, 3]
```

In[134]:=

```
ex`v1 = UnitVector[betti[1], 1]; ex`v2 = UnitVector[betti[1], 2];
ex`v1 . homology[1] . gradedBasis[1]
ex`v2 . homology[1] . gradedBasis[1]
```

Out[135]=

β^1_3

Out[136]=

α^1_3

In[137]:=

```
mul[1, 1][ex`v1, ex`v2] . homology[2] . gradedBasis[2]
```

Out[137]=

$$\frac{\alpha^1_1\wedge\beta^1_1}{9}+\frac{\alpha^1_1\wedge\beta^1_2}{9}-\frac{2\,\alpha^1_1\wedge\beta^1_3}{9}+\frac{\alpha^1_2\wedge\beta^1_2}{9}-\frac{2\,\alpha^1_2\wedge\beta^1_3}{9}-\frac{5\,\alpha^1_3\wedge\beta^1_3}{9}-\frac{\beta^1_1\wedge\alpha^1_2}{9}+\frac{2\,\beta^1_1\wedge\alpha^1_3}{9}+\frac{2\,\beta^1_2\wedge\alpha^1_3}{9}$$

Sanity check: $m_2(a, b)$ is cohomologous to $a \wedge b$, i.e., their difference is in the image of δ

In[138]:=

```
ex`mat = LinearSolve[δMatrix[1], mul[1, 1][ex`v1, ex`v2] . homology[2] -
  basisCoeff[2][ex`v1 . homology[1] . gradedBasis[1] ∧ ex`v2 . homology[1] . gradedBasis[1]]]
```

Out[138]=

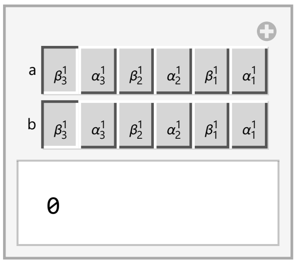
$$\left\{\frac{2}{9}, \frac{2}{9}, -\frac{1}{9}, 0, 0, 0, 0, 0, 0\right\}$$

In[139]:=
```
Simplify[δ[ex`mat . gradedBasis[1]] == mul[1, 1][ex`v1, ex`v2] . homology[2] . gradedBasis[2] -
  ex`v1 . homology[1] . gradedBasis[1] ∧ ex`v2 . homology[1] . gradedBasis[1]]
```
Out[139]=
```
True
```

Examples in degree (1,1)

The cell is interactive in the Mathematica notebook, and the user can choose to view the result of the transferred binary operation of any two elements of the basis of the cohomology.

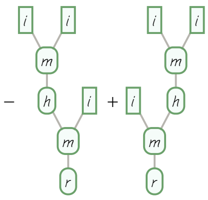
In[140]:=
```
With[{d1 = 1, d2 = 1},
 Manipulate[mul[d1, d2][UnitVector[betti[d1], a], UnitVector[betti[d2], b]] .
   homology[d1 + d2] . gradedBasis[d1 + d2],
  {a, MapIndexed[First[#2] → #1 &, homology[d1] . gradedBasis[d1]], ControlType → SetterBar},
  {b, MapIndexed[First[#2] → #1 &, homology[d2] . gradedBasis[d2]], ControlType → SetterBar}]]
```
Out[140]=



Ternary product m_3

Formula for the transferred structure:

In[141]:=
```
ExpressionTree[r[m[i, h[m[i, i]]]], ImageSize → Tiny, TreeLayout → Bottom] -
 ExpressionTree[r[m[h[m[i, i]], i]], ImageSize → Tiny, TreeLayout → Bottom]
```
Out[141]=



Computes $m_3(u, v, w)$ for u, v, w of degrees d_1, d_2, d_3 expressed in the bases **homology[d_1]**, **homology[d_2]**, **homology[d_3]**, with result in the basis **homology[$d_1 + d_2 + d_3 - 1$]**:

In[142]:=
```
mulMatrix[d1_Integer, d2_Integer, d3_Integer] :=
 (mulMatrix[d1, d2, d3] =
    Activate[TensorContract[Inactive[TensorProduct][rtrMatrix[d1 + d2 + d3 - 1],
        wedgeMatrix[d1, d2 + d3 - 1], injMatrix[d1], htpMatrix[d2 + d3 - 1],
   wedgeMatrix[d2, d3], injMatrix[d2], injMatrix[d3]],
       {{2, 3}, {4, 6}, {5, 8}, {9, 10}, {11, 13}, {12, 15}}] -
  TensorContract[Inactive[TensorProduct][rtrMatrix[d1 + d2 + d3 - 1], wedgeMatrix[d1 + d2 - 1,
         d3], htpMatrix[d1 + d2 - 1], wedgeMatrix[d1, d2], injMatrix[d1], injMatrix[d2],
   injMatrix[d3]], {{2, 3}, {4, 6}, {7, 8}, {9, 11}, {10, 13}, {5, 15}}]]) /;
  ((d1 ≥ 0 && d2 ≥ 0 && d3 ≥ 0) || Message[mulMatrix::neg3, d1, d2, d3]) &&
  (d1 + d2 + d3 ≤ topDegree - 1 || Message[mulMatrix::top3, d1, d2, d3, topDegree - 1]) &&
  ((betti[d1] > 0 && betti[d2] > 0 && betti[d3] > 0 && betti[d1 + d2] > 0 &&
       betti[d2 + d3] > 0 && betti[d1 + d2 + d3 - 1] > 0) || Message[mulMatrix::zeroBetti])
```

In[143]:=
```
mul[d1_, d2_, d3_][u_, v_, w_] := mulMatrix[d1, d2, d3] . w . v . u
```

Example

In[144]:=
```
mulMatrix[1, 1, 1]
```

Out[144]=
```
SparseArray[ Specified elements: 162
             Dimensions: {14, 6, 6, 6} ]
```

This one cannot be done:

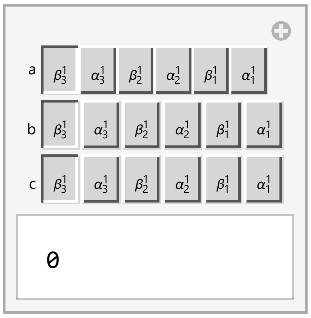
In[145]:=
```
mulMatrix[5, 0, 0]
```

mulMatrix: Transferred structure is trivial (vanishing Betti number).

Out[145]=
```
mulMatrix[5, 0, 0]
```

In[146]:=
```
With[{d1 = 1, d2 = 1, d3 = 1}, Manipulate[mul[d1, d2, d3][
    UnitVector[betti[d1], a], UnitVector[betti[d2], b], UnitVector[betti[d3], c]] .
   homology[d1 + d2 + d3 - 1] . gradedBasis[d1 + d2 + d3 - 1],
   {a, MapIndexed[First[#2] → #1 &, homology[d1] . gradedBasis[d1]],
   ControlType → SetterBar}, {b, MapIndexed[First[#2] → #1 &, homology[d2] . gradedBasis[d2]],
    ControlType → SetterBar, Appearance → "Row"},
  {c, MapIndexed[First[#2] → #1 &, homology[d3] . gradedBasis[d3]],
    ControlType → SetterBar, Appearance → "Row"}]]
```

Out[146]=



Searching for the Massey product

Somehow Solve or FindInstance are unhappy about indexed variables of the form *h*[*i*].

In[147]:=
```
toVars = (h : a | b | c)[i_] :> ToExpression[StringJoin[ToString[h], ToString[i]]];
```
In[148]:=
```
d1 = 1; d2 = 1; d3 = 1;
```
In[149]:=
```
u = Array[a, betti[d1]] /. toVars;
v = Array[b, betti[d2]] /. toVars;
w = Array[c, betti[d3]] /. toVars;
vars = Join[u, v, w];
```
In[153]:=
```
eq1 = Thread[mul[d1, d2][u, v] == 0];
```
In[154]:=
```
eq2 = Thread[mul[d2, d3][v, w] == 0];
```
In[155]:=
```
eq3 = Simplify[{Or @@ Thread[mul[d1, d2, d3][u, v, w] ≠ 0]}];
```
In[156]:=
```
inst = First[FindInstance[Join[eq1, eq2, eq3], vars]]
```
Out[156]=

$$\left\{a1 \to 8,\ a2 \to 6,\ a3 \to -\frac{32}{3},\ a4 \to -8,\ a5 \to \frac{8}{3},\ a6 \to 2,\ b1 \to -1,\ b2 \to 8,\ b3 \to \frac{4}{3},\ b4 \to -\frac{32}{3},\ b5 \to -\frac{1}{3},\ b6 \to \frac{8}{3},\ c1 \to -3,\ c2 \to \frac{9}{2},\ c3 \to 4,\ c4 \to -6,\ c5 \to -1,\ c6 \to \frac{3}{2}\right\}$$

In[157]:=
```
uSol = u /. inst;
vSol = v /. inst;
wSol = w /. inst;
```
In[160]:=
```
uSol . homology[1] . gradedBasis[1]
vSol . homology[1] . gradedBasis[1]
wSol . homology[1] . gradedBasis[1]
```
Out[160]=

$$2\,\alpha_1^1 - 8\,\alpha_2^1 + 6\,\alpha_3^1 + \frac{8\,\beta_1^1}{3} - \frac{32\,\beta_2^1}{3} + 8\,\beta_3^1$$

Out[161]=

$$\frac{8\,\alpha_1^1}{3} - \frac{32\,\alpha_2^1}{3} + 8\,\alpha_3^1 - \frac{\beta_1^1}{3} + \frac{4\,\beta_2^1}{3} - \beta_3^1$$

Out[162]=

$$\frac{3\,\alpha_1^1}{2} - 6\,\alpha_2^1 + \frac{9\,\alpha_3^1}{2} - \beta_1^1 + 4\,\beta_2^1 - 3\,\beta_3^1$$

The solution satisfies $m_2(u, v) = m_2(v, w) = 0$ and $m_3(u, v, w) \neq 0$:

In[163]:=
```
mul[d1, d2][uSol, vSol] . homology[d1 + d2] . gradedBasis[d1 + d2]
```
Out[163]=
```
0
```
In[164]:=
```
mul[d2, d3][vSol, wSol] . homology[d2 + d3] . gradedBasis[d2 + d3]
```
Out[164]=
```
0
```

In[165]:=

```
mul[d1, d2, d3][uSol, vSol, wSol] . homology[d1 + d2 + d3 - 1] . gradedBasis[d1 + d2 + d3 - 1]
```

Out[165]=

$$192\,\alpha_1^1\wedge\omega_{1,2}+192\,\alpha_1^1\wedge\omega_{1,3}-192\,\alpha_1^1\wedge\omega_{2,3}-192\,\alpha_2^1\wedge\omega_{1,3}+192\,\alpha_2^1\wedge\omega_{2,3}-192\,\alpha_3^1\wedge\omega_{1,2}-24\,\beta_1^1\wedge\omega_{1,2}-24\,\beta_1^1\wedge\omega_{1,3}+24\,\beta_1^1\wedge\omega_{2,3}+24\,\beta_2^1\wedge\omega_{1,3}-24\,\beta_2^1\wedge\omega_{2,3}+24\,\beta_3^1\wedge\omega_{1,2}$$

Even better, $m_3(u, v, w)$ is not in the ideal (u, w). This should not find a solution:

In[166]:=

```
idealSol = Simplify[Join[mulMatrix[d1 + d2 - 1, d3] . wSol,
   Activate[TensorContract[Inactive[TensorProduct][mulMatrix[d1, d2 + d3 - 1], uSol],
  {{2, 4}}]], 2]]
```

Out[166]=

```
SparseArray[ Specified elements: 96  Dimensions: {14, 12} ]
```

In[167]:=

```
LinearSolve[idealSol, mul[d1, d2, d3][uSol, vSol, wSol]]
```

LinearSolve: Linear equation encountered that has no solution.

Out[167]=

```
LinearSolve[SparseArray[ Specified elements: 96  Dimensions: {14, 12} ], {24, -192, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}]
```

We could think that we are lucky that the first solution found is such that $m_3(u, v, w)$ not in the ideal generated by (u, w), as this was not included in the equations. However, in this example, the intersection of the ideal and the image of m_3 is trivial.

In[168]:=

```
ideal = Simplify[Join[mulMatrix[d1 + d2 - 1, d3] . w, Activate[
    TensorContract[Inactive[TensorProduct][mulMatrix[d1, d2 + d3 - 1], u], {{2, 4}}]], 2]]
```

Out[168]=

```
SparseArray[ Specified elements: 96  Dimensions: {14, 12} ]
```

Every element of the ideal (u, w) is orthogonal to $m_3(u, v, w)$:

In[169]:=

```
mul[d1, d2, d3][u, v, w] . ideal
```

Out[169]=

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Bibliography

Note that the authors' works presented in this memoir are listed separately in Section 1.1.

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