Odd-primary torsion in the homology of unordered configurations on the torus

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Let $B_k(\Sigma_1)$ denote the unordered configuration space of k points in the torus Σ_1. For every odd prime p, we prove that $H_*(B_k(\Sigma_1);\mathbb{Z})$ has no p-torsion for $k \le 2p-1$. At the threshold $k = 2p$, we prove that $H_{2p-2}(B_{2p}(\Sigma_1);\mathbb{Z})$ has p-torsion if and only if $p \ge 5$. For $p \ge 5$, the class is the image under puncture filling of the unique Bianchi–Stavrou order-p class on the once-punctured torus; for $p = 3$, Napolitano's calculation shows that this punctured class dies after filling. We also prove that $H_{2p}(B_{2p}(\Sigma_1);\mathbb{Z})$ and $H_{2p+1}(B_{2p}(\Sigma_1);\mathbb{Z})$ have no p-torsion for every odd p.

Contents

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Introduction

Let M be a manifold and let $B_k(M)$ denote the unordered configuration space of k points in M. When M is a surface other than S^2 or $\mathbb{RP}^2$, its ordered and unordered configuration spaces are aspherical; hence $B_k(M)$ is a classifying space for the corresponding surface braid group, and the homology of $B_k(M)$ computes the group homology of that braid group [FN62]. For the plane, this homology has been completely understood since the 1970s: Fuks computed the mod-2 cohomology of the classical braid groups [Fuk70], F. Cohen determined the mod-p homology at all primes [CLM76], and Vainshtein described the integral cohomology [Vaĭ78]. For braid groups of closed surfaces, integral results remain far scarcer. In this article, we study the homology of the torus braid groups, or equivalently the homology of the unordered configuration spaces of k points on the torus Σ_1. The group $\pi_1(B_k(\Sigma_1))$ is also called the *elliptic* braid group; its group algebra supplies the braid-group input in constructions of type-A double affine Hecke algebras [Jor09].

With field coefficients, more is known about these spaces. The graded $\mathbb{F}_2$-homology of unordered torus configurations is known from Bödigheimer–Cohen–Taylor [BCT89], and Zhang later described its Dyer–Lashof structure [Zha25]. Rational Betti numbers were computed by Maguire, Schiessl, and Drummond-Cole–Knudsen [Mag16; Sch16; DK17]; Pagaria determined the rational cohomology ring, mixed Hodge structure, and mapping-class-group action [Pag20]. Integrally, Napolitano computed the cohomology of $B_k(\Sigma_1)$ for $k \leq 6$, and in that range all torsion is 2-primary [Nap03, Table 2]. The odd-primary integral homology, by contrast, remains largely unknown: the calculations just cited exhibit no odd-primary torsion class in $H_*(B_k(\Sigma_1); \mathbb{Z})$ and do not decide whether such torsion exists beyond their ranges. The existence odd-primary torsion was in fact predicted in [CZ24, Remark 3.7], but this question remained unsolved.

This scarcity has structural sources. Integral torsion is invisible to rational models, and calculations at odd primes involve genuinely characteristic-p phenomena such as Bocksteins and power operations [CZ24; BS24; Zha25]. Moreover, the most effective geometric tools are specific to open surfaces: there, configurations can be pushed towards the missing boundary, which produces stabilization and scanning maps [McD75] and underlies Bianchi–Stavrou's recent computation of the mod-p homology of configuration spaces of compact orientable surfaces with one boundary component [BS24]. A closed surface admits no such stabilization, and the puncture-filling map comparing the punctured and closed situations can kill torsion classes—and we prove that this actually happens on the torus at the prime 3. Finally, the sought-after torsion lies beyond the reach of homological stability, which fixes the homological degree as k tends to infinity [Ran13]: already for the classical braid groups and an odd prime p, the first p-torsion occurs at weight $2p$ [Vaĭ78], and the torsion class studied here lies in the diagonal regime $k = 2p$, in degree $2p - 2$, with both the weight and the degree growing with p.

Our main results settle this threshold question for the torus. For every prime $p \geq 5$, we show that the first p-torsion in $H_*(B_k(\Sigma_1); \mathbb{Z})$ appears exactly at weight $k = 2p$, where we exhibit it in degree $2p - 2$; for every odd prime, we prove that there is no p-torsion below this threshold weight, nor in degrees $2p$ and $2p + 1$ at the threshold itself. The resulting dichotomy is a genuinely closed-surface phenomenon: the punctured torus carries a unique order-p candidate class for every odd prime, and it is the filling of the puncture that eliminates this class when $p = 3$ while preserving it for all $p \geq 5$.

Beyond the torus case, and in a somewhat different direction, our recent work [IR26a] provides us with explicit point-set chain complexes over $\mathbb{F}_p$ which compute the mod p homology of unordered configuration spaces of parallelizable manifolds (which include the torus). These point-set construction allowed us to perform computations in low values [IR26a, Section 3.4 and Proposition 3.7] using a dedicated computer package [IR26b]. However at the present the computational threshold is easily reached, which explains why we considered different methods in this paper.

Let us also mention that analogous torsion questions are open for graph configuration spaces, whose homology is studied in particular as a model for quantum statistics of particles constrained to networks [MS19]. Ko–Park determine the torsion in first homology of unordered graph configurations [KP12]; torsion-freeness of ordered configuration-space homology is known for restricted graph classes, including trees with loops by Chettih–Lütgehetmann [CL18], while the general assertion remains a folklore conjecture and no odd-primary torsion is presently known in unordered graph-configuration homology [ADK22; ADK26; HKW25].

Main results

Chen–Zhang [CZ24, Theorem 1.1] proved that no p-torsion appears in $H_*(B_k(\Sigma_1);\mathbb{Z})$ for $k \leq p$ and odd p, and explicitly predicted in their Remark 3.7 that the first p-power torsion should occur at weight $2p$. Bianchi–Stavrou's punctured-surface calculation identifies the unique candidate class at that weight [BS24]. We prove that the candidate survives puncture filling for every $p \geq 5$, while $p = 3$ is exceptional by Napolitano's calculation [Nap03]; we also determine a high-degree non-torsion range at weight $2p$. To the best of our knowledge, the surviving classes form the first infinite family of odd-primary torsion classes in unordered configuration spaces on a closed torus.

Theorem A. *For every odd prime p, if $p < k < 2p$, then $H_*(B_k(\Sigma_1);\mathbb{Z})$ has no p-torsion.*

Theorem B. *For every odd prime p, $H_{2p-2}(B_{2p}(\Sigma_1);\mathbb{Z})$ has p-torsion if and only if $p \geq 5$. For $p \geq 5$, puncture filling sends the unique punctured order-p class to a non-zero order-p class on the closed torus.*

Theorem C. *For every odd prime p, $H_{2p}(B_{2p}(\Sigma_1);\mathbb{Z})$ and $H_{2p+1}(B_{2p}(\Sigma_1);\mathbb{Z})$ have no p-torsion.*

Proof strategies

We prove Theorem A using the k-sheeted covering $B_k^{\odot}(\Sigma_1) \to B_k(\Sigma_1)$, where $B_k^{\odot}(\Sigma_1)$ is the space of unordered configurations with a marked point. For $p < k < 2p$, one has $p \nmid k$, so transfer makes the homology of $B_k(\Sigma_1)$ with $\mathbb{Z}_{(p)}$-coefficients a direct summand of that of $B_k^{\odot}(\Sigma_1)$. Translation in the torus gives a homeomorphism

$$B_k^{\odot}(\Sigma_1) \cong \Sigma_1 \times B_{k-1}(\Sigma_{1,1}^{\circ}).$$

Bianchi–Stavrou [BS24] show that the homology of the punctured factor is p-torsion-free in this range.

The proof of Theorem B is the bulk of the article. The key idea is to consider the Gysin sequence associated to the puncture-filling inclusion $j : B_k(\Sigma_{1,1}^{\circ}) \hookrightarrow B_k(\Sigma_1)$. With $\mathbb{Z}_{(p)}$-coefficients, the relevant part is

$$H_{2p-3}(B_{2p-1}(\Sigma_{1,1}^{\circ})) \xrightarrow{\partial} H_{2p-2}(B_{2p}(\Sigma_{1,1}^{\circ})) \xrightarrow{j_*} H_{2p-2}(B_{2p}(\Sigma_1)).$$

This sequence relates the punctured and closed homology groups in the critical weight. Concretely, Bianchi–Stavrou [BS24] identify a unique punctured torsion class $\tau_p \in H_{2p-2}(B_{2p}(\Sigma_{1,1}^{\circ});\mathbb{Z}_{(p)})$, and, for $p \geq 5$, we show that $j_*(\tau_p) \neq 0$.

Equivalently, for $p \geq 5$, we show that $\tau_p \notin \operatorname{im}(\partial)$ by considering the natural group actions on both the source and the target of ∂, with respect to which the map ∂ is equivariant. Let $H = H_1(\Sigma_{1,1};\mathbb{F}_p)$ be the standard $\mathrm{SL}_2(\mathbb{F}_p)$-module and let $\mathcal{T}_{1,1}(p) = \ker(\Gamma_{1,1} \to \mathrm{SL}_2(\mathbb{F}_p))$ be the mod-p Torelli group. The Johnson filtration refers to the residual Torelli action on the two-step nilpotent quotient of the surface group; Bianchi–Stavrou's mod-p version is the homomorphism [BS24, Definition 5.8 and Corollary 5.12]

$$\xi_\tau^p : \mathcal{T}_{1,1}(p) \longrightarrow \mathrm{Hom}_{\mathbb{F}_p}(H, \Lambda^2 H).$$

Its equivariance makes $K^\xi = \ker(\xi_\tau^p)$ normal in $\Gamma_{1,1}$ [BS24, Lemma 5.14], and Bianchi–Stavrou's action calculation shows that K^ξ acts trivially on their mod-p cellular model [BS24, Propositions 4.19 and 5.22]. Consequently, the boundary is equivariant for the finite quotient $\widetilde{G} = \Gamma_{1,1}/K^\xi$, which fits into an exact sequence

$$1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow \mathrm{SL}_2(\mathbb{F}_p) \longrightarrow 1.$$

In the particular case of the torus, $A = \mathrm{im}(\xi_\tau^p)$ is either 0 or isomorphic to H. See Sections 1.3 and 3.2 for details.

We use the same symbol τ_p for its non-zero reduction modulo p in the rest of this introduction. We then apply equivariance to the reduced boundary $\bar{\partial}$. For every odd prime p, the line $\mathbb{F}_p\{\tau_p\}$ is the trivial one-dimensional representation of $\widetilde{G}$. For $p \geq 5$, the source of $\bar{\partial}$ has no trivial composition factor. Hence the reduced class τ_p cannot lie in the image of $\bar{\partial}$, which implies that the integral class does not lie in the image of ∂. For $p = 3$, the source does contain trivial composition factors, so there is no such obstruction; Napolitano's computation [Nap03] shows that the integral class lies in the image of ∂ and that $j_*(\tau_3) = 0$.

Finally, Theorem C is proved using Napolitano's Borel–Moore cell complex [Nap03] and the connecting map between the closed and punctured complexes.

Outlook

We do not produce a complete p-local homology computation of $B_{2p}(\Sigma_1)$: we do not know whether there is further p-torsion in degree $2p-2$ for $p \geq 5$, or whether there is p-torsion in degree $2p-1$.

The proofs suggest extensions to higher weights and higher genus. The marked-point transfer applies only while the covering degree is a p-local unit and the punctured factor lies below the Bianchi–Stavrou threshold.

At larger weights, the source of the filling boundary ∂ contains additional representation-theoretic families, and trivial composition factors may appear. The exceptional prime $p = 3$ already illustrates this limitation at the threshold: the target line remains a trivial $\widetilde{G}$-representation, but the source contains trivial composition factors, so equivariance no longer separates the target class from the image of $\bar{\partial}$. Napolitano's computation determines that the class is in fact hit.

In higher genus, the natural equivariance group retains the mod-p Johnson layer in a finite extension

$$1 \longrightarrow A_g \longrightarrow \widetilde{G}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{F}_p) \longrightarrow 1,$$

where A_g is the image of the mod-p Johnson homomorphism in $\mathrm{Hom}_{\mathbb{F}_p}(H, \Lambda^2 H)$ for $H = H_1(\Sigma_{g,1}; \mathbb{F}_p)$. Unlike in genus one, this image need not vanish or be the standard module; in particular, the $\Lambda^3 H$ part is potentially visible. A higher-genus obstruction must therefore be formulated in terms of $\widetilde{G}_g$-composition factors, with the symplectic calculation appearing only on the associated graded, rather than in terms of $\mathrm{Sp}_{2g}(\mathbb{F}_p)$ alone.

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also drew attention to the issue discussed in Remark 3.11. Every cited source was checked against the original, and all mathematical claims and proofs were independently rederived and verified by the authors. The authors take full responsibility for the manuscript.

1 Background

This section fixes the notation and records the torsion results used throughout the paper. We also recall the two cellular models and the representation-theoretic facts that enter the proofs of Theorems B and C.

1.1 Notation

We establish the coefficient, configuration-space, and surface notation used in the rest of the paper.

Throughout this paper, p is an odd prime, $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at p, and $\mathbb{F}_p = \mathbb{Z}_{(p)}/p$ denotes the field with p elements. A finitely generated abelian group M has no p-torsion if and only if $M \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module. Homology without specified coefficients is integral homology, and $\bar{H}^{\mathrm{BM}}_*$ denotes Borel–Moore homology. For a finitely generated abelian group M, let $\operatorname{Tors}_p M$ denote its p-primary torsion subgroup.

For a space X and an integer k called the *weight*, we set:

$$B_k(X) \coloneqq \{(x_1, \ldots, x_k) \in X^k : x_i \neq x_j \text{ for } i \neq j\} \,/\, \Sigma_k.$$

We let Σ_1 denote the closed torus. We fix throughout a closed disk $D \subset \Sigma_1$ and let $q \in D$ be its center point. We let $\Sigma_{1,1} = \Sigma_1 \setminus \mathring{D}$ denote a torus with one boundary component, and let $\Sigma^\circ_{1,1}$ denote its interior, a once-punctured torus. The puncture-filling inclusion is denoted by

$$j : B_k(\Sigma^\circ_{1,1}) \hookrightarrow B_k(\Sigma_1).$$

1.2 Known torsion results

We collect the known results on p-torsion in the homology of $B_k(\Sigma_1)$ and $B_k(\Sigma^\circ_{1,1})$, notably those of Napolitano [Nap03], Bianchi–Stavrou [BS24], and Chen–Zhang [CZ24]. We then extract the unique integral punctured-torus torsion class at weight $2p$ and explain why the two adjacent mod-p excess classes form a single Bockstein pair.

Proposition 1.1 (Napolitano [Nap03, Table 2])**.** *One has*

$$H^5(B_6(\Sigma_1); \mathbb{Z}) \cong \mathbb{Z}^9 \oplus (\mathbb{Z}/2)^8. \tag{1.2}$$

Consequently, by the universal coefficient theorem, $H_4(B_6(\Sigma_1); \mathbb{Z})$ has no 3-torsion.

Theorem 1.3 (Chen–Zhang [CZ24, Theorem 1.1])**.** *If $k \leq p$, then $H_*(B_k(\Sigma_1); \mathbb{Z})$ has no p-torsion.*

Theorem 1.4 (Consequence of Bianchi–Stavrou [BS24, Corollaries C.5–C.7])**.** *For every $k \leq 2p$,*

$$\operatorname{Tors}_p H_i(B_k(\Sigma^\circ_{1,1}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p, & \text{if } (k,i) = (2p, 2p-2), \\ 0, & \text{otherwise.} \end{cases}$$

Let us write

$$\tau_p \in H_{2p-2}(B_{2p}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)}) \tag{1.5}$$

for the integral order-p generator and

$$\beta_0 = \overline{\tau_p} \in H_{2p-2}(B_{2p}(\Sigma^\circ_{1,1}); \mathbb{F}_p) \tag{1.6}$$

for its non-zero reduction.

We also denote the torsion and excess dimensions of $B_{2p}(\Sigma_{1,1}^\circ)$ by:

$$t_i \coloneqq \dim_{\mathbb{F}_p} \mathrm{Tors}_p\, H_i(B_{2p}(\Sigma_{1,1}^\circ);\mathbb{Z}), \qquad e_i \coloneqq \dim_{\mathbb{F}_p} H_i(B_{2p}(\Sigma_{1,1}^\circ);\mathbb{F}_p) - \dim_{\mathbb{Q}} H_i(B_{2p}(\Sigma_{1,1}^\circ);\mathbb{Q}).$$

Then universal coefficients give $e_i = t_i + t_{i-1}$.

Derivation from Bianchi–Stavrou. We recall the extraction because the bookkeeping is used later. Since $1 \le p-2$ for all odd p, Bianchi–Stavrou's fake-basechange result [BS24, Corollary C.6] applies to genus 1. Its polynomial factor has mod-p generators

$$\alpha_i \in H_{2p^i-1}(B_{2p^i}^\circ;\mathbb{F}_p), \quad i \ge 1, \qquad \beta_i \in H_{2p^{i+1}-2}(B_{2p^{i+1}}^\circ;\mathbb{F}_p), \quad i \ge 0.$$

Thus the only nonconstant mod-p generators of weight $\le 2p$ are β_0, in degree $2p-2$, and α_1, in degree $2p-1$. No nonconstant product occurs in weight $\le 2p$.

For $k < 2p$, the mod-p Betti numbers equal the rational Betti numbers in every degree, so universal coefficients force no p-torsion. At $k = 2p$, the excess satisfies $e_{2p-2} = e_{2p-1} = 1$ and $e_i = 0$ for $i \notin \{2p-2, 2p-1\}$. Bianchi–Stavrou's low-degree no-torsion result [BS24, Corollary C.7] gives $t_i = 0$ for $i < 2p-2$. Hence $t_{2p-2} = 1$, then $t_{2p-1} = 0$, and all higher t_i vanish by the excess equation and the fact that $B_k(\Sigma_{1,1}^\circ)$ has homological dimension k. Their Bockstein theorem [BS24, Corollary C.5] gives exponent p, so the unique torsion group is $\mathbb{Z}/p$. $\square$

The mod-p classes β_0 and α_1 should be read as a Bockstein pair, not as two independent integral features. The integral feature is the single order-p class τ_p in degree $2p-2$. The mod-p class β_0 is its reduction, and the mod-p class α_1 in degree $2p-1$ is the universal-coefficient shadow of the same torsion group. Thus $t_{2p-2} = 1$ and $t_{2p-1} = 0$, even though the mod-p excess appears in both adjacent degrees.

1.3 The Bianchi–Stavrou equivariant action model

We recall the part of Bianchi–Stavrou's cellular model needed for the equivariant obstruction in the proof of Theorem B. For $r \ge 1$, let $\mathcal{R}_r = \bigvee_{i=1}^r S^1$, based at the wedge point. The weighted *unordered Moriyama* algebra $\mathrm{UMor}_\bullet(r)$ is assembled from the reduced cellular chains of the one-point compactifications of $B_n(\mathcal{R}_r \setminus \{*\})$, in weight $-n$ [BS24, Definition 3.4]. Its differential is zero [BS24, Proposition 3.3], and superposition of configurations gives an isomorphism of weighted rings [BS24, Proposition 3.9]

$$\mathrm{UMor}_\bullet(r) \cong \Lambda_{\mathbb{Z}}(x_1,\ldots,x_r) \otimes \Gamma_{\mathbb{Z}}(y_1,\ldots,y_r),$$

where x_i represents one point on the ith open arc and $y_i^{[m]}$ represents $2m$ points there. Thus x_i has weight -1 and $y_i^{[m]}$ has weight $-2m$; their regraded degrees are the same.

We now specialize to the once-bordered torus. Put

$$\Gamma_{1,1} \coloneqq \mathrm{Mod}(\Sigma_{1,1}, \partial\Sigma_{1,1}), \qquad G \coloneqq \pi_1(\Sigma_{1,1}) = \langle \gamma_a, \gamma_b \rangle, \qquad H_{\mathbb{Z}} \coloneqq H_1(\Sigma_{1,1};\mathbb{Z}).$$

Bianchi–Stavrou stratify the one-point compactification of $B_n(\Sigma_{1,1}^\circ)$ by records $t = (b, P, v)$. Such a record describes b vertical bars in a rectangle, with $P_i \ge 1$ points on the ith bar, and $v_a, v_b \ge 0$ points on the two open arcs of a bouquet representing γ_a, γ_b [BS24, Definitions 4.1–4.3 and Proposition 4.4]. The corresponding cell has Borel–Moore dimension $n + b$, where $\sum_i P_i + v_a + v_b = n$. It is placed in weight $-n$ and in regraded homological degree $(n+b) - 2n = b - n$ [BS24, Notation 4.5]. Let

$$\mathcal{C}_{\mathbb{Z}} \coloneqq \bigoplus_{n \ge 0} \widetilde{C}_*^{\mathrm{cell}}\big((B_n(\Sigma_{1,1}^\circ))^+;\mathbb{Z}\big), \tag{1.7}$$

where the displayed chain degree is this regraded degree.

Let

$$\mathcal{A} := \mathrm{UMor}_\bullet(1) = \Lambda_{\mathbb{Z}}(x) \otimes \Gamma_{\mathbb{Z}}(y), \qquad \mathcal{U} := \mathrm{UMor}_\bullet(2) = \Lambda_{\mathbb{Z}}(x_a, x_b) \otimes \Gamma_{\mathbb{Z}}(y_a, y_b).$$

The boundary word $\zeta = [\gamma_a, \gamma_b] \in G$ induces a ring homomorphism $\mathcal{A} \to \mathcal{U}$ and hence makes $\mathcal{U}$ a left $\mathcal{A}$-module. In genus one it is given by [BS24, Definition 4.12, Lemma 4.13, and Section 6.1]

$$x \longmapsto 0, \qquad y \longmapsto 2x_a x_b, \qquad y^{[m]} \longmapsto 0 \quad (m \geq 2). \tag{1.8}$$

This is the genus-one specialization of the general assignment $y^{[m]} \mapsto \Omega_{2m}$: here $\Omega_2 = 2x_a x_b$, while $\Omega_{2m} = 0$ for $m \geq 2$ because $(x_a x_b)^2 = 0$. Bianchi–Stavrou identify the cellular complex with the reduced two-sided bar complex [BS24, Proposition 6.1]

$$\mathcal{C}_{\mathbb{Z}} \cong \overline{B}_\bullet(\mathbb{Z}, \mathcal{A}, \mathcal{U}), \qquad \overline{B}_b(\mathbb{Z}, \mathcal{A}, \mathcal{U}) := \mathbb{Z} \otimes \bar{\mathcal{A}}^{\otimes b} \otimes \mathcal{U}, \tag{1.9}$$

where $\bar{\mathcal{A}}$ is the augmentation ideal; a term of bar length b and weight $-n$ has regraded chain degree $b - n$. If $P_i = 2m_i + \epsilon_i$ and $v_j = 2m'_j + \epsilon'_j$, with $\epsilon_i, \epsilon'_j \in \{0, 1\}$, then $t = (b, P, v)$ corresponds to

$$e_{(b,P,v)} \longleftrightarrow 1 \otimes x^{\epsilon_1} y^{[m_1]} \otimes \cdots \otimes x^{\epsilon_b} y^{[m_b]} \otimes \prod_{j \in \{a,b\}} x_j^{\epsilon'_j} y_j^{[m'_j]}.$$

The initial bar face vanishes. The other faces merge adjacent bar entries or apply (1.8) to the last entry before multiplying the surface factor. Geometrically these are, respectively, collisions of adjacent bars and the rightmost bar reaching the bouquet [BS24, Proposition 4.14].

Bianchi–Stavrou's cellular approximation gives a $\Gamma_{1,1}$-action by chain maps on $\mathcal{C}_{\mathbb{Z}}$. Under (1.9), it fixes every bar entry and acts on $\mathcal{U}$ through the induced action on G [BS24, Propositions 4.18 and 4.19].

To describe the action on $\mathcal{U}$, write $[w] \in H_{\mathbb{Z}}$ for the abelianization of $w \in G$, and let

$$(-)_x : H_{\mathbb{Z}} \longrightarrow \mathbb{Z}\{x_a, x_b\}, \qquad (-)_y : H_{\mathbb{Z}} \longrightarrow \mathbb{Z}\{y_a, y_b\}$$

send $[\gamma_i]$ to x_i and y_i, respectively, for $i \in \{a, b\}$, and extend $(-)_x$ to exterior powers. The *content map* is

$$c : \mathbb{Z}[G] \longrightarrow \Lambda H_{\mathbb{Z}}, \qquad c(\gamma_i) = 1 + [\gamma_i], \qquad c(\gamma_i^{-1}) = 1 - [\gamma_i].$$

Let $c_2(w) \in \Lambda^2 H_{\mathbb{Z}}$ be the exterior-degree-two component of $c(w)$. For $\phi \in \Gamma_{1,1}$, Bianchi–Stavrou's action formula on $\mathcal{U}$ is [BS24, Theorem 3.13]

$$x_i \longmapsto [\phi(\gamma_i)]_x, \qquad y_i \longmapsto [\phi(\gamma_i)]_y + [c_2(\phi(\gamma_i))]_x. \tag{1.10}$$

Here and below $i \in \{a, b\}$. The second summand in the image of y_i is the quadratic correction term.

Throughout the article, we let:

$$H := H_{\mathbb{Z}} \otimes \mathbb{F}_p \cong H_1(\Sigma_{1,1}; \mathbb{F}_p), \quad \mathcal{T}_{1,1}(p) := \ker\bigl(\Gamma_{1,1} \longrightarrow \mathrm{SL}(H)\bigr)$$

be the mod-p Torelli subgroup. The quadratic corrections define the homomorphism [BS24, Definition 5.8 and Corollary 5.12]

$$\xi^p_\tau : \mathcal{T}_{1,1}(p) \longrightarrow \mathrm{Hom}_{\mathbb{F}_p}(H, \Lambda^2 H), \qquad \xi^p_\tau(\phi)([\gamma_i]) = c_2(\phi(\gamma_i)) \bmod p.$$

For $\phi \in \mathcal{T}_{1,1}(p)$, formula (1.10) reduces to

$$\phi(x_i) = x_i, \qquad \phi(y_i) = y_i + [\xi^p_\tau(\phi)([\gamma_i])]_x.$$

Bianchi–Stavrou prove that ξ^p_τ is $\Gamma_{1,1}$-equivariant [BS24, Lemma 5.14], so

$$K^\xi := \ker(\xi^p_\tau)$$

is normal in $\Gamma_{1,1}$. Their proof of Proposition 5.22 checks all divided powers $y_i^{[m]}$, including $m \geq p$, and shows that K^ξ acts trivially on $\mathcal{U} \otimes \mathbb{F}_p$. Since the bar entries are fixed, K^ξ acts trivially on the entire mod-p cellular complex.

1.4 Napolitano's cellular decomposition

Napolitano [Nap03] constructs a Borel–Moore cellular decomposition for the homology of $B_k(\Sigma_1)$ which generalizes the Fuks–Vainshtein [Fuk70; Vaĭ78] model for the plane. We will use this model to prove Theorem C. Let us now recall it.

Fix a base point on the circle, e.g., the north pole. The *index* of a configuration $\xi \in B_k(S^1)$ is $\operatorname{ind}(\xi) := (k-v, v)$, where $v=1$ if ξ contains the base point, and $v=0$ otherwise.

Now, consider the projection $\pi : S^1 \times \mathbb{R} \to \mathbb{R}$. Given an element $\xi \in B_k(S^1 \times \mathbb{R})$, we can consider the projection $\pi(\xi)$, which consists of d ordered points; the fiber over any of these points is an unordered configuration ξ_i of k_i points in S^1, with $\sum k_i = k$. We can then define $\operatorname{ind}(\xi) := (\operatorname{ind}(\xi_1), \dots, \operatorname{ind}(\xi_d))$. This defines a cellular decomposition of $(B_k(S^1 \times \mathbb{R}))^+$, with boundaries as indicated in [Nap03, Section 2.3].

Finally, the torus Σ_1 is the union of a circle S^1 and a cylinder $S^1 \times \mathbb{R}$, so a configuration ξ on Σ_1 is the union of a configuration ξ' on the circle and a configuration ξ'' on the cylinder. The *index* of ξ is

$$\operatorname{ind}(\xi) := (\operatorname{ind}(\xi'), \operatorname{ind}(\xi'')) = ((e, v); (m_1, v_1), \dots, (m_d, v_d)), \tag{1.11}$$

where $v_i \in \{0,1\}$, $m_i \in \mathbb{N}$, $m_i + v_i \geq 1$, and $e + v + \sum m_i + \sum v_i = k$. This, again, defines a cellular decomposition of $(B_k(\Sigma_1))^+$, with boundaries as indicated in [Nap03, Section 2.3]. The Borel–Moore degree of the cell associated to the index in Equation (1.11) is $n = e + \sum_i (m_i + 1)$. We will use it in Section 6, where we will also fix orientations and signs for this cellular decomposition.

Remark 1.12. Both Napolitano and Bianchi–Stavrou use an ordered-bar stratification of the same Fuks–Vainshtein type. Napolitano's model [Nap03, Sections 2.3–2.4] treats the closed surface directly and retains the boundary faces arising from pinned points, the bouquet vertex, and the two ends of the cylinder. In the Bianchi–Stavrou model [BS24, Propositions 4.9 and 4.14], the adjacent-bar mergers have the same signed-shuffle incidence coefficients (the parity-binomial numbers denoted $P(a,b)$ in Section 6) while the remaining surface contribution is packaged in the $\mathrm{UMor}_\bullet(2)$-factor and a single term determined by the symplectic class. This reorganization makes the bar description and the mapping-class-group action explicit. We do not identify the two complexes term by term: below we use Napolitano's model for the closed-torus calculation and the Bianchi–Stavrou model for the punctured, equivariant calculation.

1.5 Representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$

We record the representation-theoretic conventions and the two elementary facts used later to separate the source and target of the mod-p Gysin map. We refer to [Hum75] for the representation theory of $\mathrm{SL}_2(\mathbb{F}_p)$. As before, let $H = H_1(\Sigma_{1,1}; \mathbb{F}_p)$ be the standard two-dimensional representation of $\mathrm{SL}_2(\mathbb{F}_p)$. For $0 \leq r \leq p-1$, let the rth divided power of H (of dimension $r+1$) be:

$$V_r := \Gamma^r H.$$

For $0 \leq r \leq p-1$, the canonical map from symmetric tensors to symmetric coinvariants is an $\mathrm{SL}_2(\mathbb{F}_p)$-equivariant isomorphism ($r!$ is invertible in $\mathbb{F}_p$). Thus $V_r \cong \mathrm{Sym}^r H$ is the simple module of highest weight r. In particular, V_0 is trivial and V_{p-1} is the Steinberg module. Write ω for the nondegenerate $\mathrm{SL}_2(\mathbb{F}_p)$-invariant intersection pairing on H. The map $v \mapsto \omega(v, -)$ identifies H equivariantly with $H^\vee$, and hence $V_r^\vee \cong V_r$ in this range.

For $1 \leq r \leq p-2$, the modular Clebsch–Gordan formula gives:

$$[H \otimes \Gamma^r H] = [\Gamma^{r+1} H] + [\Gamma^{r-1} H] \tag{1.13}$$

in the Grothendieck group; see [DH05, Lemma 1.3] (where $\Gamma^r H = L(r)$).

We shall also use two standard facts about modular representations. If $P \trianglelefteq E$ is a normal p-subgroup of a finite group, then P acts trivially on every simple $\mathbb{F}_p[E]$-module; equivalently, every such simple module is inflated from E/P [Ser77, Section 8.3, Corollary to Proposition 26]. Here *inflated* means that an E/P-action is pulled back along the quotient map $E \twoheadrightarrow E/P$, so that P acts trivially. Moreover, if $\phi : V \to W$ is an E-linear map over a field and V has no trivial composition factor, then $\operatorname{im}\phi$ contains no non-zero E-fixed vector: the image is a quotient of V, while a fixed vector spans a trivial submodule.

2 Marked-point transfer and Theorem A

Let us now prove Theorem A using the marked-point transfer.

Definition 2.1. Let X be a topological space. The marked-point unordered configuration space of X is

$$B_k^{\odot}(X) \coloneqq \{(Q,x) \in B_k(X) \times X : x \in Q\}.$$

Lemma 2.2. *If $p \nmid k$, then $H_*(B_k(\Sigma_1);\mathbb{Z}_{(p)})$ is a direct summand of $H_*(B_k^{\odot}(\Sigma_1);\mathbb{Z}_{(p)})$.*

Proof. The forgetful map $\pi : B_k^{\odot}(\Sigma_1) \to B_k(\Sigma_1)$ is a k-sheeted covering. Its transfer satisfies

$$\pi_* \circ \operatorname{tr} = k \cdot \operatorname{id}_{H_*(B_k(\Sigma_1);\mathbb{Z}_{(p)})}.$$

Since k is a unit in $\mathbb{Z}_{(p)}$, the rescaled transfer splits π_*. □

Lemma 2.3. *There is a homeomorphism*

$$B_k^{\odot}(\Sigma_1) \cong \Sigma_1 \times B_{k-1}(\Sigma_{1,1}^{\circ}) = \Sigma_1 \times B_{k-1}(\Sigma_{1,1}^{\circ}).$$

Proof. Regard Σ_1 as an abelian topological group with identity element 0. The maps

$$\begin{aligned} \Sigma_1 \times B_{k-1}(\Sigma_1 \setminus \{0\}) &\longrightarrow B_k^{\odot}(\Sigma_1), & (x,R) &\longmapsto (\{x\} \cup (x+R), x),\\ B_k^{\odot}(\Sigma_1) &\longrightarrow \Sigma_1 \times B_{k-1}(\Sigma_1 \setminus \{0\}), & (Q,x) &\longmapsto (x, (Q-x) \setminus \{0\}) \end{aligned}$$

are mutually inverse homeomorphisms. Composing with a homeomorphism $\Sigma_1 \setminus \{0\} \cong \Sigma_{1,1}^{\circ}$, which exists since both spaces are once-punctured tori, gives the result. □

Proof of Theorem A. Suppose $p < k < 2p$. Theorem 1.4 implies that $H_*(B_{k-1}(\Sigma_{1,1}^{\circ});\mathbb{Z}_{(p)})$ is free because $k-1 < 2p$. By Lemma 2.3 and the Künneth theorem over $\mathbb{Z}_{(p)}$, $H_*(B_k^{\odot}(\Sigma_1);\mathbb{Z}_{(p)})$ is free. Since $p \nmid k$, Lemma 2.2 makes $H_*(B_k(\Sigma_1);\mathbb{Z}_{(p)})$ a direct summand of a free $\mathbb{Z}_{(p)}$-module, hence free. □

3 The Gysin exact sequence

The puncture-filling inclusion gives a Gysin sequence whose connecting map controls whether the punctured torsion class survives on the closed torus. After writing this sequence, we reduce survival to a mod-p equivariant separation problem and identify the action on its target line.

Recall the choices $q \in D \subset \Sigma_1$ and the identification $\Sigma_{1,1} = \Sigma_1 \setminus \mathring{D}$. Inside the unordered configuration space $B_{2p}(\Sigma_1)$, we identify the closed submanifold of codimension 2 consisting of configurations containing the point q with the unordered configuration space of $2p-1$ points in the punctured torus:

$$\{Q \in B_{2p}(\Sigma_1) : q \in Q\} \cong B_{2p-1}(\Sigma_{1,1}^{\circ}). \tag{3.1}$$

Its open complement is $B_{2p}(\Sigma_{1,1}^{\circ})$, and its normal bundle is the oriented plane $T_q\Sigma_1$. The Gysin exact sequence therefore contains

$$\cdots \to H_{2p-3}(B_{2p-1}(\Sigma_{1,1}^{\circ});\mathbb{Z}_{(p)}) \xrightarrow{\partial} H_{2p-2}(B_{2p}(\Sigma_{1,1}^{\circ});\mathbb{Z}_{(p)}) \xrightarrow{j_*} H_{2p-2}(B_{2p}(\Sigma_1);\mathbb{Z}_{(p)}) \to \cdots \tag{3.2}$$

3.1 Reduction modulo p

We convert the integral survival question into the assertion that β_0 is not in the image of the mod-p connecting map. Since $B_{2p-1}(\Sigma^\circ_{1,1})$ lies below the punctured threshold, its homology with $\mathbb{Z}_{(p)}$-coefficients is free; this gives the coefficient-change identification for the domain of ∂ used below. The codomain need not be torsion-free: it contains the order-p class τ_p, whose reduction modulo p is non-zero.

Let τ_p be the unique order-p class in $H_{2p-2}(B_{2p}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)})$ from (1.5). By exactness,

$$j_*(\tau_p) \neq 0 \quad \Longleftrightarrow \quad \tau_p \notin \operatorname{im} \partial. \tag{3.3}$$

The universal coefficient theorem over $\mathbb{Z}_{(p)}$ gives a natural short exact sequence

$$0 \longrightarrow H_i(Y; \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \longrightarrow H_i(Y; \mathbb{F}_p) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}_{(p)}}\big(H_{i-1}(Y; \mathbb{Z}_{(p)}), \mathbb{F}_p\big) \longrightarrow 0. \tag{3.4}$$

Since $2p-1 < 2p$, Theorem 1.4 implies that $H_*(B_{2p-1}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)})$ is free. Applying (3.4) in degree $2p-3$ therefore yields a natural identification

$$M := H_{2p-3}(B_{2p-1}(\Sigma^\circ_{1,1}); \mathbb{F}_p) \cong H_{2p-3}(B_{2p-1}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p. \tag{3.5}$$

Set

$$N := H_{2p-2}(B_{2p}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$$

and let $\bar\partial = \partial \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p : M \to N$. Under the natural coefficient-change injection

$$N \hookrightarrow H_{2p-2}(B_{2p}(\Sigma^\circ_{1,1}); \mathbb{F}_p),$$

the element $\overline{\tau_p} \in N$ is the class β_0 of (1.6). It is non-zero because the p-primary torsion summand generated by τ_p is $\mathbb{Z}_{(p)}/(p)$, and $\mathbb{Z}_{(p)}/(p) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \cong \mathbb{F}_p$.

The reduction maps fit into the commutative square

$$\begin{tikzcd} H_{2p-3}(B_{2p-1}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)}) \arrow[r, "\partial"] \arrow[d, "x \mapsto \overline{x}"'] & H_{2p-2}(B_{2p}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)}) \arrow[d, "y \mapsto \overline{y}"] \\ M \arrow[r, "\bar\partial"'] & N \end{tikzcd} \tag{3.6}$$

We use only the commutativity of this square; we do not need tensoring (3.2) with $\mathbb{F}_p$ to preserve exactness.

Lemma 3.7. *If $\beta_0 \notin \operatorname{im} \bar\partial$, then $0 \neq j_*(\tau_p) \in H_{2p-2}(B_{2p}(\Sigma_1); \mathbb{Z}_{(p)})$.*

Proof. Suppose, to the contrary, that $j_*(\tau_p) = 0$. Exactness of (3.2) then gives an element $x \in H_{2p-3}(B_{2p-1}(\Sigma^\circ_{1,1}); \mathbb{Z}_{(p)})$ such that $\partial x = \tau_p$. Commutativity of (3.6) now gives

$$\bar\partial(\overline{x}) = \overline{\partial x} = \overline{\tau_p} = \beta_0,$$

contradicting $\beta_0 \notin \operatorname{im} \bar\partial$. Hence $j_*(\tau_p) \neq 0$. $\square$

Theorem B therefore reduces to showing that β_0 is not in the image of $\bar\partial$.

3.2 Mapping-class equivariance and the target line

We place the mod-p Gysin map in the category of modules over the finite Johnson quotient $\widetilde{G}$. We first show that the target line spanned by β_0 is a trivial $\widetilde{G}$-representation for every odd prime. The restriction $p \geq 5$ enters only later, when we exclude trivial composition factors from the source.

Continue with the notation of Section 1.3, and write $H = H_1(\Sigma_{1,1}; \mathbb{F}_p) \cong \mathbb{F}_p^2$. Because $\Gamma_{1,1}$ fixes the boundary pointwise, it should not be confused with $\mathrm{SL}_2(\mathbb{Z})$: its integral homological action surjects onto $\mathrm{SL}_2(\mathbb{Z})$, but the Dehn twist about the boundary component lies in the kernel. Set

$$A = \mathrm{im}(\xi_\tau^p), \qquad \widetilde{G} = \Gamma_{1,1}/K^\xi. \tag{3.8}$$

Reduction modulo p of the homological action gives a surjection $\Gamma_{1,1} \twoheadrightarrow \mathrm{SL}(H) = \mathrm{SL}_2(\mathbb{F}_p)$ [FM12, Section 2.2]. Since $K^\xi \subset \mathcal{T}_{1,1}(p)$ and $\mathcal{T}_{1,1}(p)/K^\xi \cong A$, there is an exact sequence

$$1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow \mathrm{SL}_2(\mathbb{F}_p) \longrightarrow 1. \tag{3.9}$$

Thus $\widetilde{G}$ is finite and A is an elementary abelian p-group.

Extending boundary-fixing diffeomorphisms across D by the identity preserves the submanifold (3.1) and its oriented normal bundle. Hence the Gysin sequence (3.2) and its reduction $\bar{\partial}$ are $\Gamma_{1,1}$-equivariant. By Section 1.3, K^ξ acts trivially on the mod-p cellular complex and therefore on M. It also acts trivially on N, by equivariance of the coefficient-change injection $N \hookrightarrow H_{2p-2}(B_{2p}(\Sigma_{1,1}^\circ); \mathbb{F}_p)$. Thus M and N are $\mathbb{F}_p[\widetilde{G}]$-modules and $\bar{\partial} : M \to N$ is $\widetilde{G}$-linear.

Lemma 3.10. *As an $\mathrm{SL}_2(\mathbb{F}_p)$-module, A is either 0 or the standard module H.*

Remark 3.11. Bianchi–Stavrou's Proposition 5.17 [BS24] would give $A \cong \Lambda^3 H = 0$ in our notation, but the cited proof of the generation statement used there does not establish the genus-one case. More precisely, it uses [BS24, Theorem 5.2], attributed to Cooper [Coo15] and Perron [Per08], which states that $\mathcal{T}_{g,1}(p)$ is generated by $\mathcal{T}_{g,1}$ and the p-th powers of all Dehn twists. Perron's note contains no proof but points to a theorem of Bass–Milnor–Serre [BMS67]. Cooper [Coo15, Lemma 5.1] gave an argument for the corresponding statement which invokes a normal-generation result stated as [Coo15, Theorem 5.4] and attributed to Bass–Milnor–Serre [BMS67]. The underlying symplectic theorem is stated only in rank at least two [BMS67, Theorem 12.4]; hence this proof does not cover genus one. In fact, the unrestricted statement is false in genus one for $p \geq 7$.

Indeed, project the symplectic representation of $\Gamma_{1,1}$ to $\mathrm{PSL}_2(\mathbb{Z})$, and let T denote the image of a non-separating Dehn twist. The ordinary Torelli group maps trivially. The subgroup generated by the p-th powers of all Dehn twists is normal, since $f D_c^p f^{-1} = D_{f(c)}^p$ for every mapping class f and Dehn twist D_c. Separating twists act trivially on homology and all non-separating twists are conjugate, so its image in $\mathrm{PSL}_2(\mathbb{Z})$ is precisely the normal closure $\langle\langle T^p \rangle\rangle$. For odd p, the projective image of $\mathcal{T}_{1,1}(p)$ is precisely the projective principal congruence subgroup

$$\overline{\Gamma}(p) = \ker\bigl(\mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{F}_p)\bigr).$$

Indeed, a projective class trivial modulo p has a representative congruent to either I or $-I$, and changing its sign if necessary gives a representative congruent to I. Thus, if [BS24, Theorem 5.2] held in genus one, this normal closure would equal $\overline{\Gamma}(p)$. However, the standard presentation $\mathrm{PSL}_2(\mathbb{Z}) = \langle s, r \mid s^2 = r^3 = 1 \rangle$, with $T = sr$, gives

$$\mathrm{PSL}_2(\mathbb{Z})/\langle\langle T^p \rangle\rangle \cong \mathrm{D}(2,3,p) = \langle s, r \mid s^2 = r^3 = (sr)^p = 1 \rangle,$$

where $\mathrm{D}(2,3,p)$ is the orientation-preserving triangle group (or von Dyck group). This triangle group on the right is hyperbolic, and hence infinite, for $p \geq 7$, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{p} < 1$, whereas $\mathrm{PSL}_2(\mathbb{Z})/\overline{\Gamma}(p) \cong \mathrm{PSL}_2(\mathbb{F}_p)$ is finite. Thus the two subgroups cannot coincide.

Our argument does not require the conclusion of [BS24, Proposition 5.17]: Lemma 3.10 uses only the equivariance of ξ_τ^p to obtain the dichotomy $A = 0$ or $A = H$, and either case suffices below.

Proof of Lemma 3.10. In genus 1, the target of ξ_τ^p is

$$\operatorname{Hom}(H, \Lambda^2 H) \cong H^\vee \otimes \det(H) \cong H.$$

The second isomorphism uses the symplectic self-duality of H and the triviality of the determinant on $\mathrm{SL}_2(\mathbb{F}_p)$. By [BS24, Lemma 5.14], A is an $\mathrm{SL}_2(\mathbb{F}_p)$-submodule of this standard module. The standard module is irreducible, so $A = 0$ or $A = H$. ☐

Lemma 3.12. *If $p \geq 5$, then $\widetilde{G}$ is perfect (i.e., its abelianization vanishes).*

Proof. Since A is abelian, conjugation by $\widetilde{G}$ induces an $\mathrm{SL}_2(\mathbb{F}_p)$-action on A. The map $A \to \widetilde{G}^{\mathrm{ab}}$ kills every element $g{\cdot}a - a$ and therefore factors through the coinvariants $A_{\mathrm{SL}_2(\mathbb{F}_p)} = A/\langle g{\cdot}a - a\rangle_{g \in \mathrm{SL}_2(\mathbb{F}_p),\ a \in A}$. Moreover, the quotient map $\widetilde{G} \to \mathrm{SL}_2(\mathbb{F}_p)$ induces an exact sequence

$$A_{\mathrm{SL}_2(\mathbb{F}_p)} \longrightarrow \widetilde{G}^{\mathrm{ab}} \longrightarrow \mathrm{SL}_2(\mathbb{F}_p)^{\mathrm{ab}} \longrightarrow 0. \tag{3.13}$$

We first reprove the classical fact that $\mathrm{SL}_2(\mathbb{F}_p)$ is perfect for $p \geq 5$. Choose $\lambda \in \mathbb{F}_p^\times$ with $\lambda^2 \neq 1$, and set

$$d = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \ell(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Direct calculation gives

$$[d, u(t)] = u\big((\lambda^2 - 1)t\big), \qquad [d, \ell(t)] = \ell\big((\lambda^{-2} - 1)t\big).$$

Both displayed coefficients are non-zero. Hence every upper and lower unipotent matrix is a commutator. These matrices generate $\mathrm{SL}_2(\mathbb{F}_p)$, so $\mathrm{SL}_2(\mathbb{F}_p)$ is perfect and $\mathrm{SL}_2(\mathbb{F}_p)^{\mathrm{ab}} = 0$.

It remains to treat the left-hand term. By Lemma 3.10, either $A = 0$, in which case $A_{\mathrm{SL}_2(\mathbb{F}_p)} = 0$, or $A = H$. In the latter case, let e, f be the standard basis of H. Then

$$(u(1) - 1)f = e, \qquad (\ell(1) - 1)e = f.$$

Thus the relations defining the coinvariants contain both basis vectors, and consequently $(H)_{\mathrm{SL}_2(\mathbb{F}_p)} = 0$. Both outer terms in (3.13) therefore vanish, so $\widetilde{G}^{\mathrm{ab}} = 0$. Equivalently, $\widetilde{G}$ is perfect. ☐

Lemma 3.14. *For every odd prime p, the line $\mathbb{F}_p\{\beta_0\} \subset N = H_{2p-2}(B_{2p}(\Sigma_{1,1}^\circ); \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ is the trivial one-dimensional representation of $\widetilde{G}$.*

Proof. By Theorem 1.4, the p-primary torsion subgroup of $H_{2p-2}(B_{2p}(\Sigma_{1,1}^\circ); \mathbb{Z}_{(p)})$ is $T_p = \mathbb{Z}_{(p)}/(p)\{\tau_p\}$. This subgroup is characteristic. Hence the $\Gamma_{1,1}$-action preserves T_p and therefore also its non-zero reduction $T_p \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p = \mathbb{F}_p\{\beta_0\}$, where non-vanishing follows from $\mathbb{Z}_{(p)}/(p) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \cong \mathbb{F}_p$. Since the action on N factors through $\widetilde{G}$, this is a $\widetilde{G}$-stable line. Let $\chi : \widetilde{G} \to \mathbb{F}_p^\times$ be its character. If $p \geq 5$, then χ is trivial because $\widetilde{G}$ is perfect by Lemma 3.12.

Suppose that $p = 3$. Since A is an elementary abelian 3-group and $\mathbb{F}_3^\times$ has order 2, the restriction of χ to A is trivial. Thus χ factors through $\mathrm{SL}_2(\mathbb{F}_3)$. The upper and lower unipotent matrices $u(1)$ and $\ell(1)$ generate $\mathrm{SL}_2(\mathbb{F}_3)$ and both have order 3. Their images under a character with values in $\mathbb{F}_3^\times$ are therefore trivial, so χ is trivial in this case as well. Thus every element of $\widetilde{G}$ fixes β_0, as claimed. ☐

4 The source representation

We compute the composition factors of the $\widetilde{G}$-module $M = H_{2p-3}(B_{2p-1}(\Sigma_{1,1}^\circ); \mathbb{F}_p)$, the source of the mod-$p$ Gysin map $\bar{\partial} : M \to N$. We use a filtration to reduce the action to $\mathrm{SL}_2(\mathbb{F}_p)$ on the associated graded, identify the resulting complex with a small Chevalley–Eilenberg complex, and compute the required bidegree.

4.1 The divided-power filtration

We filter the Bianchi–Stavrou complex by divided-power degree in its surface variables. On the associated graded, the quadratic correction term in (1.10) disappears, while the composition factors of the source homology are preserved.

Let $\mathcal{C} = \mathcal{C}_{\mathbb{Z}} \otimes \mathbb{F}_p$, where $\mathcal{C}_{\mathbb{Z}}$ is the integral Bianchi–Stavrou complex (1.7), and write $\mathcal{C}^{(k)}$ for its summand of weight $-k$. Each fixed weight is finite-dimensional, and $\mathcal{C}$ uses the regraded cellular degree, so equivariant Poincaré–Lefschetz duality gives [BS24, Proposition 2.3]

$$H_q\big(\mathcal{C}^{(k)}\big) \cong \bar{H}^{\mathrm{BM}}_{2k+q}(B_k(\Sigma^\circ_{1,1});\mathbb{F}_p), \qquad H_q\big(\mathcal{C}^{(k)}\big)^\vee \cong H_{-q}(B_k(\Sigma^\circ_{1,1});\mathbb{F}_p). \tag{4.1}$$

In particular, $M = H_{2p-3}(B_{2p-1}(\Sigma^\circ_{1,1});\mathbb{F}_p)$ is dual to $H_{-(2p-3)}(\mathcal{C}^{(2p-1)})$, whose Borel–Moore degree is $2p+1$.

After reducing (1.9) modulo p, a basis term has the form

$$1 \otimes a_1 \otimes \cdots \otimes a_b \otimes x_{i_1} \cdots x_{i_t} y_a^{[r]} y_b^{[s]},$$

where each $a_i = x^{\epsilon_i} y^{[m_i]} \neq 1$ is a basis monomial of $\bar{A} \otimes \mathbb{F}_p$. Its *surface divided-power degree* is $r+s$; divided powers in the bar entries a_i are not counted. Let $F_d\mathcal{C}$ be spanned by terms of surface divided-power degree at most d. This is an increasing finite filtration in each fixed weight. With $F_{-1}\mathcal{C} = 0$, we write

$$\mathrm{gr}_F\, \mathcal{C} = \bigoplus_{d \geq 0} F_d\mathcal{C}/F_{d-1}\mathcal{C}$$

for the associated-graded chain complex.

Lemma 4.2. *The $\widetilde{G}$-action preserves the divided-power filtration on $\mathcal{C}$, and its quadratic correction term in (1.10) strictly lowers the surface divided-power degree. The associated-graded action factors through $\mathrm{SL}_2(\mathbb{F}_p)$ and is the standard linear action on both pairs (x_a, x_b) and (y_a, y_b).*

Proof. Write the images in (1.10) as

$$\phi(y_a) = L_a + \lambda_a q, \qquad \phi(y_b) = L_b + \lambda_b q, \qquad q = x_a x_b,$$

where L_a, L_b and $\lambda_a, \lambda_b \in \mathbb{F}_p$ are the reductions of the integral linear and quadratic coefficients in $\phi(y_a), \phi(y_b)$. Since $q^2 = 0$, the divided-power binomial identity gives

$$\phi\big(y_i^{[m]}\big) = L_i^{[m]} + \lambda_i q L_i^{[m-1]}, \qquad i \in \{a, b\},$$

with the second term interpreted as zero when $m = 0$. This identity is obtained in the torsion-free integral algebra before reduction modulo p, so it remains valid even when $m \geq p$. Here

$$(\alpha y_a + \beta y_b)^{[m]} = \sum_{r+s=m} \alpha^r \beta^s y_a^{[r]} y_b^{[s]}.$$

Thus the linear part preserves surface divided-power degree, whereas each term involving the quadratic correction lowers it by one. The action is multiplicative and fixes the bar entries, so it preserves $F_d\mathcal{C}$.

On the associated graded, all the lower-degree correction terms disappear. In particular, the subgroup A in (3.9) acts trivially, and the action factors through $\widetilde{G}/A \cong \mathrm{SL}_2(\mathbb{F}_p)$. The resulting action is the standard linear action on both pairs (x_a, x_b) and (y_a, y_b). $\square$

Lemma 4.3. *The differential on $\mathcal{C}$ is homogeneous of surface divided-power degree zero. Consequently,*

$$\mathrm{gr}_F\, H_*(\mathcal{C}) \cong H_*(\mathrm{gr}_F\, \mathcal{C}). \tag{4.4}$$

Proof. The bar differential is homogeneous in surface divided-power degree. Indeed, its internal faces leave the surface factor unchanged, while its final face multiplies that factor by the image of (1.8), which lies in $\Lambda(x_a, x_b)$. Hence $\mathcal{C}$ is the direct sum of its subcomplexes of exact surface divided-power degree. Taking homology degree by degree gives (4.4). □

Corollary 4.5. *The filtration on $\mathcal{C}$ induces a finite $\widetilde{G}$-stable filtration on*

$$M = H_{2p-3}(B_{2p-1}(\Sigma_{1,1}^{\circ}); \mathbb{F}_p).$$

The action on its associated graded $\mathrm{gr}_F M$ factors through $\mathrm{SL}_2(\mathbb{F}_p)$, and the Jordan–Hölder multiset of M as a $\widetilde{G}$-module is obtained by inflating the Jordan–Hölder multiset of $\mathrm{gr}_F M$ along $\widetilde{G} \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_p)$.

Proof. Passing to the cochain dual via (4.1) gives a $\widetilde{G}$-stable filtration on M. By Lemmas 4.2 and 4.3, its successive quotients are $\mathrm{SL}_2(\mathbb{F}_p)$-modules inflated along $\widetilde{G} \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_p)$. The Jordan–Hölder multiset of a finite filtered module is the multiset union of those of its successive quotients, which proves the claim. □

4.2 The low-weight Chevalley–Eilenberg model and comparison

Lie and Chevalley–Eilenberg models for configuration-space homology arise from Knudsen's description via factorization homology, and their positive-characteristic low-weight forms are developed through spectral-Lie methods by Chen–Zhang and Zhang [Knu17; CZ24; Zha25]. We identify the associated-graded Bianchi–Stavrou complex in weights below $2p$ with the corresponding small Chevalley–Eilenberg complex. The resulting $\mathrm{SL}_2(\mathbb{F}_p)$-equivariant comparison turns the required source representation into a concrete bidegree calculation.

Since $\Sigma_{1,1}^{\circ}$ is connected and noncompact, its compactly supported cohomology has no degree-zero class. Using the natural identification $H_c^*(\Sigma_{1,1}^{\circ}; \mathbb{F}_p) \cong H^*(\Sigma_{1,1}, \partial\Sigma_{1,1}; \mathbb{F}_p)$, choose a basis

$$H_c^*(\Sigma_{1,1}^{\circ}; \mathbb{F}_p) = \mathbb{F}_p\{a, b, c\}, \qquad |a| = |b| = 1, \qquad |c| = 2,$$

with, up to graded commutativity, the single non-zero product

$$a \smile b = c. \tag{4.6}$$

Via the intersection pairing, identify $H_c^1(\Sigma_{1,1}^{\circ}; \mathbb{F}_p)$ with the standard module $H = H_1(\Sigma_{1,1}; \mathbb{F}_p)$, and use the same letters a, b for the resulting symplectic basis. The class $c = a \smile b$ spans $\det(H)$ and is fixed by $\mathrm{SL}_2(\mathbb{F}_p)$.

Let $\mathfrak{l}_{\mathrm{small}}$ be the two-step *shifted* Lie algebra with basis x_2, w_2, where x_2 has degree 2 and weight 1, w_2 has degree 3 and weight 2, and

$$[x_2, x_2] = w_2, \qquad [x_2, w_2] = [w_2, w_2] = 0.$$

For $p \geq 5$, this is the free shifted Lie algebra on x_2. Set

$$\mathfrak{g}_{\mathrm{small}} = H_c^{-*}(\Sigma_{1,1}^{\circ}; \mathbb{F}_p) \otimes \mathfrak{l}_{\mathrm{small}}. \tag{4.7}$$

The superscript $-*$ records the regrading from cohomological to homological degrees: a class in H_c^j is placed in homological degree $-j$. We suppress the tensor symbol in pure tensors, e.g., $ax_2 = a \otimes x_2$. If $y \in H_c^j(\Sigma_{1,1}^{\circ}; \mathbb{F}_p)$ and v is homogeneous in $\mathfrak{l}_{\mathrm{small}}$, then

$$\deg(yv) = |v| - j, \qquad \mathrm{weight}(yv) = \mathrm{weight}(v). \tag{4.8}$$

The six elements in Table 1 form a homogeneous basis of $\mathfrak{g}_{\mathrm{small}}$.

Element	Weight	Degree	$\mathrm{SL}_2(\mathbb{F}_p)$-type	CE factor
$E_0 = cx_2$	1	0	$\mathbf{1}$	Γ (even)
$O_a = ax_2,\ O_b = bx_2$	1	1	H	Λ (odd)
$O_2 = cw_2$	2	1	$\mathbf{1}$	Λ (odd)
$F_a = aw_2,\ F_b = bw_2$	2	2	H	Γ (even)

Table 1: The Chevalley–Eilenberg cogenerators of $\mathfrak{g}_{\mathrm{small}}$.

The Chevalley–Eilenberg chain complex of $\mathfrak{g}_{\mathrm{small}}$ is

$$\mathcal{E} \coloneqq CE(\mathfrak{g}_{\mathrm{small}}) = \Gamma(E_0, F_a, F_b) \otimes \Lambda(O_a, O_b, O_2). \tag{4.9}$$

We use the same symbols for the six basis elements of $\mathfrak{g}_{\mathrm{small}}$ and for the corresponding Chevalley–Eilenberg cogenerators. Each odd cogenerator occurs at most once in a monomial, while an even cogenerator z has divided powers $z^{[r]}$ for $r \geq 0$. Weight and degree are additive; in particular,

$$F_a^{[i]} F_b^{[r-i]}, \qquad 0 \leq i \leq r,$$

form the standard basis of $\Gamma^r H$.

The cup product (4.6) gives the only non-zero bracket between the elements in Table 1. Thus

$$[O_a, O_b] = O_2 \text{ in } \mathfrak{g}_{\mathrm{small}}, \qquad d(O_a O_b) = O_2 \text{ in } \mathcal{E}, \tag{4.10}$$

and the differential vanishes on each cogenerator.

We now compare $\mathcal{E}$ with the associated graded of the Bianchi–Stavrou complex $\mathrm{gr}_F\,\mathcal{C}$ introduced in Section 4.1.

Write $\mathcal{E}^{(k)}$ for the weight-k summand of $\mathcal{E}$. For each weight k, define a chain complex $\mathcal{D}^{(k)}$ by

$$\mathcal{D}_d^{(k)} = \mathrm{Hom}_{\mathbb{F}_p}\big((\mathrm{gr}_F\,\mathcal{C}^{(k)})_{-d}, \mathbb{F}_p\big),$$

with differential dual to that of $\mathrm{gr}_F\,\mathcal{C}^{(k)}$. By Lemma 4.3 and equivariant Poincaré–Lefschetz duality, for every d there is an $\mathrm{SL}_2(\mathbb{F}_p)$-equivariant isomorphism

$$H_d(\mathcal{D}^{(k)}) \cong \mathrm{gr}_F\, H_d(B_k(\Sigma_{1,1}^\circ); \mathbb{F}_p),$$

where F is the filtration on configuration-space homology induced by the surface divided-power filtration $F_\bullet \mathcal{C}^{(k)}$ under this duality.

Lemma 4.11. *For every $k < 2p$, there is an $\mathrm{SL}_2(\mathbb{F}_p)$-equivariant quasi-isomorphism*

$$\mathcal{D}^{(k)} \simeq \mathcal{E}^{(k)}. \tag{4.12}$$

In particular, for every d, there is an isomorphism of $\mathrm{SL}_2(\mathbb{F}_p)$-modules

$$\mathrm{gr}_F\, H_d(B_k(\Sigma_{1,1}^\circ); \mathbb{F}_p) \cong H_d\big(\mathcal{E}^{(k)}\big). \tag{4.13}$$

Proof. For an augmented algebra A and a left A-module U, the reduced two-sided bar complex and the corresponding free A-resolution are

$$\bar{B}_\bullet(\mathbb{F}_p, A, U) = \mathbb{F}_p \otimes \bar{A}^{\otimes \bullet} \otimes U, \qquad B_\bullet(A, A, U) = A \otimes \bar{A}^{\otimes \bullet} \otimes U.$$

The first computes $\mathbb{F}_p \otimes_A^{\mathbf{L}} U$; since each weight is finite-dimensional, its weightwise dual is the cobar cochain complex $\mathrm{Hom}_A(B_\bullet(A, A, U), \mathbb{F}_p)$ computing $\mathrm{Ext}_A(U, \mathbb{F}_p)$.

Applying this construction to (1.9), Bianchi–Stavrou identify the weightwise dual of $\mathcal{C}$ with the cobar complex used in their Theorem B [BS24, Propositions 6.1 and 6.3]. Since the surface divided-power filtration is a direct sum by exact degree, as in the proof of Lemma 4.3, the same description identifies $\mathcal{D}^{(k)}$ with the weight-k summand of the corresponding associated-graded cobar complex. The Eilenberg–Zilber map and the Künneth decomposition in the proof of that theorem give a quasi-isomorphism from this cobar complex to the tensor product of the three cobar complexes computing

$$\mathrm{Ext}_{\Lambda(x)}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Ext}_{\Gamma(y)}(M_1, \mathbb{F}_p) \otimes \mathrm{Hom}_{\mathbb{F}_p}\big(\Gamma(y_a, y_b), \mathbb{F}_p\big), \tag{4.14}$$

where

$$M_1 = \Lambda(x_a, x_b), \qquad y \cdot 1 = 2x_a x_b,$$

and y annihilates $x_a, x_b, x_a x_b$ [BS24, Definition 6.7 and the proof of Theorem B]. After passage to the associated graded, $\mathrm{SL}_2(\mathbb{F}_p)$ acts linearly on both pairs (x_a, x_b) and (y_a, y_b) by Lemma 4.2.

We compare the three cobar factors underlying (4.14) with $\mathcal{E}$. First, the normalized bars

$$[x|\cdots|x]^\vee, \qquad r \text{ copies of } x,$$

have zero differential because $x^2 = 0$. Sending each such bar to $E_0^{[r]}$ identifies the first factor, as a bigraded complex, with $\Gamma(E_0)$.

Second, the weightwise dual of $\Gamma(y_a, y_b)$ gives the third factor in (4.14). In weight $k < 2p$, its homogeneous degree in y_a, y_b is at most $p - 1$. All relevant factorials are therefore invertible in $\mathbb{F}_p$, and the usual symmetrization map, together with the symplectic identification $H^\vee \cong H$, gives an $\mathrm{SL}_2(\mathbb{F}_p)$-equivariant identification of this factor with

$$\Gamma(F_a, F_b).$$

It remains to examine the middle factor. Let $P = \mathbb{F}_p[t]$, with t of weight -2, and regard M_1 as a P-module by

$$t \cdot 1 = 2x_a x_b, \qquad t \cdot x_a = t \cdot x_b = t \cdot x_a x_b = 0.$$

The homomorphism $P \to \Gamma(y)$ sending t to $y = y^{[1]}$ induces a map

$$\bar{B}_\bullet(\mathbb{F}_p, P, M_1) \longrightarrow \bar{B}_\bullet(\mathbb{F}_p, \Gamma(y), M_1). \tag{4.15}$$

This map is an isomorphism in every total weight $-k$ with $k < 2p$. Indeed, every bar tensor in such a weight involves divided powers whose exponents have sum less than p. For $m < p$, the image of t^m is

$$y^m = m!\, y^{[m]},$$

and $m!$ is a unit in $\mathbb{F}_p$. Moreover, whenever $a + b < p$,

$$\big(a!\, y^{[a]}\big)\big(b!\, y^{[b]}\big) = (a+b)!\, y^{[a+b]},$$

so the internal bar faces agree under (4.15). The final bar face agrees because the P-action just displayed is the restriction of the $\Gamma(y)$-action on M_1.

We may therefore replace the middle cobar factor in weight $k < 2p$ by the dual of the polynomial bar complex. The standard two-term Koszul resolution of the augmentation module over P gives the quasi-isomorphic cochain complex

$$M_1^\vee \otimes \Lambda(\alpha_0) \cong \Lambda(O_a, O_b) \otimes \Lambda(\alpha_0).$$

Here α_0 is the Koszul generator of weight 2 and cochain degree 1. The second identification again uses the symplectic self-duality $H^\vee \cong H$. The module relation $t \cdot 1 = 2x_a x_b$ dualizes to

$$d(O_a O_b) = 2\alpha_0.$$

Since p is odd, we may set $O_2 = 2\alpha_0$; the differential then becomes $d(O_aO_b) = O_2$, as in (4.10).

The strict weight bound is visible in the decomposition

$$\Gamma(y) \cong \bigotimes_{i\geq 0} \mathbb{F}_p[y_i]/(y_i^p), \qquad y_i = y^{[p^i]}.$$

The Koszul generator α_0 for $y_0 = y$ is the generator identified with O_2 above. At weight $2p$, the relation $y_0^p = 0$ contributes the degree-two cobar class β_0, while the new algebra generator $y_1 = y^{[p]}$ contributes the degree-one cobar class α_1. These are precisely the first features of the divided-power cobar complex absent from the polynomial Koszul model, which is why the comparison is restricted to $k < 2p$.

Tensoring the three comparisons gives an $\mathrm{SL}_2(\mathbb{F}_p)$-equivariant quasi-isomorphism with

$$\Gamma(E_0, F_a, F_b) \otimes \Lambda(O_a, O_b, O_2)$$

and its differential is precisely the Chevalley–Eilenberg differential of (4.10). Combining this quasi-isomorphism with (4.4) and the duality regrading gives (4.13). □

4.3 The associated-graded source calculation

We now compute the weight-$2p-1$, degree-$2p-3$ part of $H_*(\mathcal{E})$ and identify its $\mathrm{SL}_2(\mathbb{F}_p)$-module structure.

Lemma 4.16. *The exterior part of the complex (4.9) has homology*

$$H\big(\Lambda(O_a, O_b, O_2), d\big) = \mathrm{span}\{1, O_a, O_b, O_aO_2, O_bO_2, O_aO_bO_2\}. \tag{4.17}$$

Proof. The exterior algebra has basis

$$1,\ O_a,\ O_b,\ O_2,\ O_aO_b,\ O_aO_2,\ O_bO_2,\ O_aO_bO_2.$$

By (4.10), its only non-zero differential is $d(O_aO_b) = O_2$. Hence O_aO_b is not a cycle and O_2 is a boundary. The other six basis elements are cycles; in particular,

$$d(O_aO_bO_2) = O_2^2 = 0,$$

because O_2 is exterior. The image of the differential is the line spanned by O_2, so none of these six cycles is a boundary. □

Lemma 4.18. *In weight $2p-1$ and degree $2p-3$, the homology of $\mathcal{E}$ has a basis consisting of the following two families:*

$$E_0^{[2]} O_\varepsilon F_a^{[i]} F_b^{[p-2-i]}, \qquad O_\varepsilon \in \{O_a, O_b\}, \quad 0 \leq i \leq p-2, \tag{4.19}$$

$$E_0 O_a O_b O_2 F_a^{[i]} F_b^{[p-3-i]}, \qquad 0 \leq i \leq p-3. \tag{4.20}$$

Their dimensions are respectively $2(p-1)$ and $p-2$.

Proof. For a homogeneous generator z, set the *defect*:

$$\mathrm{def}(z) = \mathrm{weight}(z) - \deg(z).$$

Table 1 gives

$$\mathrm{def}(E_0) = \mathrm{def}(O_2) = 1, \qquad \mathrm{def}(O_a) = \mathrm{def}(O_b) = \mathrm{def}(F_a) = \mathrm{def}(F_b) = 0.$$

Since the required weight and degree differ by 2, a contributing monomial must have total defect 2. By Lemma 4.16, this means that the exponent of E_0 plus the exponent of O_2 is 2. As O_2 is exterior, the only possibilities for these two exponents are $(2,0)$ and $(1,1)$.

Suppose first that the exterior class contains no O_2. The possible exterior classes are $1, O_a, O_b$, and the defect condition forces the factor $E_0^{[2]}$. For the class 1, the weight remaining for the F-variables is $2p-3$, which is odd and therefore impossible because both F_a and F_b have weight 2. For O_a or O_b, the remaining weight is

$$(2p-1)-2-1=2(p-2),$$

which gives exactly the monomials in (4.19).

Now suppose that the exterior class contains O_2. The possible homology classes are

$$O_aO_2, \qquad O_bO_2, \qquad O_aO_bO_2.$$

The defect condition forces one factor of E_0. For O_aO_2 and O_bO_2, the remaining weight is $2p-5$, again odd. For $O_aO_bO_2$, it is

$$(2p-1)-1-1-1-2=2(p-3),$$

giving exactly (4.20).

The first family has two choices of O_ε and $p-1$ choices of i, whereas the second has $p-2$ choices of i. □

Proposition 4.21. *In weight $2p-1$ and degree $2p-3$,*

$$\operatorname{gr}_F M \cong \left(H \otimes \Gamma^{p-2} H\right) \oplus \Gamma^{p-3} H, \tag{4.22}$$

as an $\mathrm{SL}_2(\mathbb{F}_p)$-module. Hence the $\widetilde{G}$-composition factors of M are the inflations of the composition factors in (4.22).

As a dimension check, in agreement with Bianchi–Stavrou's fake-basechange formula:

$$\dim(\operatorname{gr}_F M)=2\dim\Gamma^{p-2}H+\dim\Gamma^{p-3}H=2(p-1)+(p-2)=3p-4.$$

Proof. Since $2p-1<2p$, Lemma 4.11 identifies $\operatorname{gr}_F M$ with the bidegree computed in Lemma 4.18.

In the first family, $E_0^{[2]}$ is invariant, $\operatorname{span}\{O_a, O_b\} \cong H$, and the monomials

$$F_a^{[i]} F_b^{[p-2-i]}, \qquad 0 \le i \le p-2,$$

form the standard basis of $\Gamma^{p-2}H$. Hence this family is $H \otimes \Gamma^{p-2}H$.

In the second family, E_0 and O_2 are invariant, while O_aO_b spans $\Lambda^2 H = \det(H)$ and is therefore invariant under $\mathrm{SL}_2(\mathbb{F}_p)$. The remaining F-monomials form $\Gamma^{p-3}H$. Thus the second family is $\Gamma^{p-3}H$, proving (4.22).

The final assertion about $\widetilde{G}$-composition factors follows from Corollary 4.5 and the definition of inflation in Section 1.5. □

5 Proof of Theorem B: torsion on the critical line

We now prove Theorem B by showing that the punctured-torus torsion class τ_p survives to a non-zero class in $H_{2p-2}(B_{2p}(\Sigma_1);\mathbb{Z})$ for $p \ge 5$. Proposition 5.2 proves the assertion for $p \ge 5$ using representation theory, and Proposition 5.3 proves the assertion for $p=3$ using Napolitano's calculation.

5.1 No trivial source factors and survival for $p \geq 5$

We combine the source calculation with the modular Clebsch–Gordan formula to exclude trivial composition factors. Equivariance of the Gysin map then prevents its image from containing the fixed target class β_0.

Lemma 5.1. *If $p \geq 5$, then $\left(H \otimes \Gamma^{p-2} H\right) \oplus \Gamma^{p-3} H$ has composition factors V_{p-1}, V_{p-3}, V_{p-3}, and in particular has no trivial composition factor.*

Proof. Applying (1.13) with $r = p-2$ gives

$$[H \otimes \Gamma^{p-2} H] = [V_{p-1}] + [V_{p-3}]$$

in the Grothendieck group. Hence

$$\left[(H \otimes \Gamma^{p-2} H) \oplus \Gamma^{p-3} H\right] = [V_{p-1}] + 2[V_{p-3}].$$

Since $p \geq 5$, both $p-1$ and $p-3$ are positive; thus neither simple module appearing here is V_0, the trivial module. □

Proposition 5.2. *If $p \geq 5$, then $\beta_0 \notin \operatorname{im} \bar{\partial}$. Consequently $j_*(\tau_p)$ is a non-zero order-p class in $H_{2p-2}(B_{2p}(\Sigma_1); \mathbb{Z})$.*

Proof. Proposition 4.21 and Lemma 5.1 show that M has no trivial $\widetilde{G}$-composition factor. Since $\bar{\partial} : M \to N$ is $\widetilde{G}$-linear, the same is true of its image, which is a quotient of M. On the other hand, Lemma 3.14 says that β_0 is a non-zero fixed vector of N. Therefore $\beta_0 \notin \operatorname{im} \bar{\partial}$.

Lemma 3.7 now gives $j_*(\tau_p) \neq 0$. Since τ_p has order p, its non-zero image also has order p. □

5.2 The exceptional prime $p = 3$

The source–target separation used for $p \geq 5$ fails at $p = 3$. We therefore use Napolitano's integral computation (Proposition 1.1) to determine whether the punctured torsion class survives filling.

At $p = 3$, the associated-graded source has composition factors

$$[H \otimes \Gamma^1 H \oplus \Gamma^0 H] = [V_2] + 2[V_0],$$

so the source contains two trivial composition factors. Although β_0 remains a fixed vector by Lemma 3.14, equivariance no longer prevents it from lying in $\operatorname{im} \bar{\partial}$: this image is a quotient of the source and may inherit one of its trivial composition factors. Using a result of Napolitano [Nap03], we now show that the torsion class τ_3 is indeed in the image of the connecting homomorphism ∂.

Proposition 5.3. *The group $H_4(B_6(\Sigma_1); \mathbb{Z})$ has no 3-torsion, hence $j_*(\tau_3) = 0$ and, equivalently, $\tau_3 \in \operatorname{im} \partial$.*

Proof. For $B_6(\Sigma_1)$, Napolitano's calculation (Proposition 1.1) gives $H^5(B_6(\Sigma_1); \mathbb{Z}) \cong \mathbb{Z}^9 \oplus (\mathbb{Z}/2)^8$. Hence $H^5(B_6(\Sigma_1); \mathbb{Z})$ has no 3-torsion. By the universal coefficient theorem, there is a short exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_4(B_6(\Sigma_1); \mathbb{Z}), \mathbb{Z}) \to H^5(B_6(\Sigma_1); \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(H_5(B_6(\Sigma_1); \mathbb{Z}), \mathbb{Z}) \to 0.$$

The right-hand term is torsion-free, and the left-hand term records the torsion in $H_4(B_6(\Sigma_1); \mathbb{Z})$. Therefore $\operatorname{Tors}_3 H_4(B_6(\Sigma_1); \mathbb{Z}) = 0$.

On the other hand, Theorem 1.4 gives $\operatorname{Tors}_3 H_4(B_6(\Sigma^\circ_{1,1}); \mathbb{Z}) = \mathbb{Z}/3\{\tau_3\}$. The image $j_*(\tau_3) \in H_4(B_6(\Sigma_1); \mathbb{Z})$ has order dividing 3. Since $H_4(B_6(\Sigma_1); \mathbb{Z})$ has no 3-torsion, this image must be zero. □

6 Proof of Theorem C: no torsion in high degree

We now prove Theorem C by analyzing the Borel–Moore chain complex of Napolitano's cellular decomposition [Nap03]. After translating ordinary high-degree torsion into bottom-degree Borel–Moore torsion, we identify the relevant punctured cells and compute the closed–punctured connecting map.

Note that Napolitano computes Borel–Moore homology, not ordinary homology. The unordered configuration space $B_k(\Sigma_1)$ is an orientable real $2k$-manifold. Therefore, Borel–Moore Poincaré duality gives

$$\bar{H}_i^{\mathrm{BM}}(B_k(\Sigma_1);\mathbb{Z}_{(p)}) \cong H^{2k-i}(B_k(\Sigma_1);\mathbb{Z}_{(p)}).$$

The universal coefficient theorem over $\mathbb{Z}_{(p)}$ gives

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}_{(p)}}(H_{q-1}(B_k(\Sigma_1);\mathbb{Z}_{(p)}),\mathbb{Z}_{(p)}) \to H^q(B_k(\Sigma_1);\mathbb{Z}_{(p)}) \to \mathrm{Hom}_{\mathbb{Z}_{(p)}}(H_q(B_k(\Sigma_1);\mathbb{Z}_{(p)}),\mathbb{Z}_{(p)}) \to 0.$$

The Hom-term is torsion-free, while the Ext-term records the p-primary torsion in $H_{q-1}(B_k(\Sigma_1);\mathbb{Z})$. Consequently, we have the following.

Lemma 6.1. *For every n and r, the p-primary torsion of $H_r(B_n(\Sigma_1);\mathbb{Z})$ is recorded in the p-primary torsion of $\bar{H}^{\mathrm{BM}}_{2n-r-1}(B_n(\Sigma_1);\mathbb{Z}_{(p)})$. In particular, torsion-freeness of the latter group implies that $H_r(B_n(\Sigma_1);\mathbb{Z})$ has no p-torsion.*

6.1 The cell model and the closed–punctured exact sequence

We fix orientations and write the cellular differential in the form needed for the torsion calculation. Separating cells according to whether the base north pole is occupied then yields a short exact sequence relating the closed and punctured complexes.

Recall from Section 1.4 that Napolitano's cells are indexed by tuples of the form:

$$(e, v; (m_1, v_1), \ldots, (m_d, v_d)), \tag{6.2}$$

with weight and Borel–Moore dimension:

$$k = e + v + \sum_i (m_i + v_i), \qquad n = e + \sum_i (m_i + 1). \tag{6.3}$$

If we view $\Sigma_1 = S^1 \cup (S^1 \times \mathbb{R})$, v counts whether the north pole of S^1 is occupied, e counts the number of points on the base circle (but not at the north pole), and (m_i, v_i) counts the number of points in the ith fiber of the cylinder and whether that fiber contains the north pole. We will say that the ith fiber is *pinned* if $v_i = 1$. A point not at the north pole is called *off-pole*.

Since signs matter for Lemma 6.10, we now fix a cell orientation convention and explicitly describe the boundary. Orient a cell by listing first the e ordered arc coordinates and then, for each fiber from left to right, its position followed by its m_i ordered off-pole coordinates. Let us denote

$$P(a,b) = \begin{cases} 0, & a, b \text{ both odd}, \\ \dbinom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}, & \text{otherwise.} \end{cases}$$

Consider the cell indexed by (6.2). It has five types of codimension-one boundary cells (if the corresponding condition is not satisfied, then the face does not exist):

1. Two adjacent fibers i, $i+1$ can merge unless both are pinned. The indexing tuple of the boundary face replaces $(m_i, v_i), (m_{i+1}, v_{i+1})$ with $(m_i + m_{i+1}, \max(v_i, v_{i+1}))$, with coefficient $(-1)^{e+i-1+\sum_{j\le i} m_j} P(m_i, m_{i+1})$.

2. If $v_i = 0$ and $m_i \geq 1$, an off-pole point can reach the north pole. The boundary face replaces $(m_i, 0)$ by $(m_i - 1, 1)$, with coefficient $(-1)^{e+i-1+\sum_{j<i} m_j}\bigl(1 + (-1)^{m_i}\bigr)$.

3. If $v = 0$ and $e \geq 1$, a point on the base circle can reach the base north pole. The boundary face replaces $(e, 0)$ by $(e-1, 1)$, leaving the fibers unchanged, with coefficient $-\bigl(1 + (-1)^e\bigr)$.

4. If the base north pole and the leftmost north pole are not both occupied, the leftmost fiber can slide onto the base circle. Its m_1 off-pole points join the base circle, its potential pin is transferred to the base north pole, and the coefficient is $-(-1)^e P(e, m_1)$.

5. The analogous rightmost slide has coefficient $(-1)^{e+S(1+m_d)} P(e, m_d)$, where $S = (d-1) + \sum_{j<d} m_j$.

All other codimension-one degenerations either collide points or escape an interior fiber and therefore land at the compactification basepoint. These five formulas are Napolitano's torus differential in the above orientation.

The full ("closed") complex $C^{\mathrm{cl}}(k)$ is the Borel–Moore chain complex of the cellular decomposition of $B_k(\Sigma_1)$. The punctured complex $C^{\mathrm{pu}}(k)$ is the quotient obtained by taking $v = 0$ and discarding summands that hit $v = 1$. For punctured cells, $n - k = d - \sum_i v_i \geq 0$, so the punctured complex for $B_k(\Sigma_{1,1}^\circ)$ has no cells in Borel–Moore dimension below k. For closed cells, $n - k = d - \sum_i v_i - v \geq -1$, so the closed complex has no cells in Borel–Moore dimension below $k - 1$.

The subcomplex of closed cells with $v = 1$ is naturally identified with the punctured complex in weight $k - 1$. Thus there is a short exact sequence of Borel–Moore chain complexes

$$0 \longrightarrow C^{\mathrm{pu}}(k-1) \longrightarrow C^{\mathrm{cl}}(k) \longrightarrow C^{\mathrm{pu}}(k) \longrightarrow 0. \tag{6.4}$$

6.2 Bottom-dimensional punctured cells

We describe the bottom Borel–Moore homology of the punctured complex in terms of all-pinned cells. We show that pure all-pinned cells generate in weights at most $2p$ and form a basis in weight $2p - 1$.

Recall that there are no cells in Borel–Moore dimension $< k$ in the punctured complex. The bottom-dimensional punctured cells are the *all-pinned cells*:

$$X(e; m_1, \ldots, m_d) = (e, 0; ((m_1, 1), \ldots, (m_d, 1))), \qquad e + d + \sum_i m_i = k. \tag{6.5}$$

The differential vanishes on these cells: merging two pinned fibers is forbidden, and every other codimension-one face (sliding a pinned fiber onto the base circle, or moving a base point to the base north pole) occupies the base north pole and is therefore discarded in the punctured complex.

Let

$$W_k \coloneqq \bar{H}_k^{\mathrm{BM}}(C^{\mathrm{pu}}(k)) \ (\cong H^k(B_k(\Sigma_{1,1}^\circ))).$$

In Borel–Moore dimension $k + 1$, the punctured complex consists of cells with exactly one unpinned fiber. Thus $W_k = C_k^{\mathrm{pu}}(k)/d(C_{k+1}^{\mathrm{pu}}(k))$ has three types of relations that we now describe. Suppose that

$$Y(e; m_1, \ldots, m_d; j) \coloneqq (e, 0; (m_1, 1), \ldots, (m_{j-1}, 1), (m_j, 0), (m_{j+1}, 1), \ldots, (m_d, 1)) \tag{6.6}$$

is a *one-unpinned* punctured cell. Then the boundary of $Y(e; m_1, \ldots, m_d; j)$ has three possible types of terms (we will not need the precise signs here):

1. merging two adjacent fibers, if possible, with coefficients $\pm P(m_{j-1}, m_j)$ or $\pm P(m_j, m_{j+1})$;

2. sliding an off-pole point to the north pole, if $m_j \geq 1$, with coefficient $\pm(1 + (-1)^{m_j})$;

3. sliding the unpinned fiber onto the base circle, if the unpinned fiber is the leftmost or rightmost fiber, with coefficient $\pm P(e, m_j)$.

Let us write $\Pi(e,d) \coloneqq X(e;0^d)$ $(e+d=k)$ for the "pure" all-pinned cell with d fibers and e points on the base circle.

Lemma 6.7. *If $k \le 2p$, then the classes $\Pi(e,d)$ with $e+d=k$ generate $W_k \otimes \mathbb{Z}_{(p)}$.*

Proof. Let us set $s(X) = \sum_i m_i$, where X is a Borel–Moore cell. Note that pure cells have $s(\Pi(e,d)) = 0$. This induces a filtration on W_k and we let $\mathrm{gr}_s W_k$ be the associated graded.

If Y is a cell of type (6.6), then dY is a linear combination of cells of type (6.5), each with s-value at most $s(Y)$: merging two fibers does not change s, sliding an off-pole point to the north pole decreases s by 1, and sliding the unpinned fiber onto the base circle decreases s by $m_j > 0$. Therefore, in $\mathrm{gr}_s W_k$, the remaining relations are:

$$\pm P(m_{j-1}, m_j) X(e; m_1, \ldots, m_{j-1}+m_j, \ldots, m_d) \pm P(m_j, m_{j+1}) X(e; m_1, \ldots, m_j+m_{j+1}, \ldots, m_d) = 0. \tag{6.8}$$

Note that if the unpinned fiber is at the beginning or the end, then there is only one term in (6.8).

Let us now prove that $\mathrm{gr}_s W_k \otimes \mathbb{Z}_{(p)} = 0$ for $s \ge 1$, from which it follows that every class in $W_k \otimes \mathbb{Z}_{(p)}$ is a linear combination of pure cells. We proceed by induction on the position of the first nonzero m_i in the tuple $(m_1, \ldots, m_d)$.

First, consider an all-pinned cell $X(e; m_1, \ldots, m_d)$ with $m_1 > 0$. We can find b, c such that $b+c = m_1$ and $P(b,c)$ is a unit in $\mathbb{Z}_{(p)}$: if m_1 is odd, let $(b,c) = (1, m_1 - 1)$ (then $P(b,c) = 1$), and if m_1 is even, let $(b,c) = (2, m_1 - 2)$ (then $P(b,c) = m_1/2$, which is a unit in $\mathbb{Z}_{(p)}$ because $m_1 < 2p$). Then the relation coming from $Y = Y(e; b, c, m_2, \ldots, m_d; 1)$ is just $\pm P(b,c) X(e; m_1, \ldots, m_d) = 0$, which implies that $X(e; m_1, \ldots, m_d) = 0$ in $\mathrm{gr}_s W_k \otimes \mathbb{Z}_{(p)}$.

Now suppose that the result has been proved for all cells with first nonzero m_i at position $< j$, and consider a cell $X(e; 0, \ldots, 0, m_j, \ldots, m_d)$ with $m_j > 0$ and $m_i = 0$ for all $i < j$. Choose $m_j = b + c$ with $P(b,c)$ a unit in $\mathbb{Z}_{(p)}$ as above, and consider the relation coming from $Y = Y(e; 0, \ldots, 0, b, c, m_{j+1}, \ldots, m_d; j)$, whose unpinned fiber, in position j, has left neighbor $(0,1)$ and right neighbor $(c, 1)$. The relation is

$$\pm P(0,b) X(e; 0, \ldots, 0, b, c, m_{j+1}, \ldots, m_d) \pm P(b,c) X(e; 0, \ldots, 0, m_j, m_{j+1}, \ldots, m_d) = 0,$$

where the first cell has $j-2$ zeros and the second one has $j-1$ zeros. The first cell has its first nonzero entry at position $j-1$, so it vanishes in $\mathrm{gr}_s W_k \otimes \mathbb{Z}_{(p)}$ by the induction hypothesis. Since $P(b,c)$ is a unit in $\mathbb{Z}_{(p)}$, we conclude that $X(e; 0, \ldots, 0, m_j, \ldots, m_d) = 0$ in $\mathrm{gr}_s W_k \otimes \mathbb{Z}_{(p)}$. □

Lemma 6.9. *The $2p$ classes $\Pi(e,d)$ with $e+d = 2p-1$ form a $\mathbb{Z}_{(p)}$-basis of $W_{2p-1} \otimes \mathbb{Z}_{(p)}$.*

Proof. By Lemma 6.7, there is a surjection from $\mathbb{Z}_{(p)}\{\Pi(e,d) \mid e+d = 2p-1\}$ to $W_{2p-1} \otimes \mathbb{Z}_{(p)}$. By [DK17, Proposition 3.5], the $\mathbb{Q}$-dimension of $W_{2p-1} \otimes \mathbb{Q}$ is $2p$, so the kernel has rank 0. A rank 0 submodule of a free module over a PID is trivial, so the surjection is an isomorphism. □

6.3 The connecting map on pure cells

We compute the connecting map on pure generators; its cokernel will give the bottom closed-torus group needed for Theorem C. The map induced by (6.4) is

$$\delta : W_k \otimes \mathbb{Z}_{(p)} \longrightarrow W_{k-1} \otimes \mathbb{Z}_{(p)}.$$

It sends a punctured cycle in weight k to a punctured cycle in weight $k-1$ by lifting it to a closed chain, applying the closed differential, and identifying the result with a punctured cycle.

Lemma 6.10. *If $e + d = k$, then*

$$\delta\Pi(e,d) = -\big(1+(-1)^e\big)\,\mathbb{1}_{e\geq 1}\Pi(e-1,d) + (-1)^e\big((-1)^{d-1}-1\big)\,\mathbb{1}_{d\geq 1}\Pi(e,d-1). \tag{6.11}$$

Proof. Recall that the pure cell $\Pi(e,d)$ has d pinned fibers and e off-pole points on the base circle. It is a cycle in the punctured complex: merging adjacent fibers is impossible, and every other codimension-one face occupies the base north pole and is therefore discarded in the punctured complex.

To compute $\delta\Pi(e,d)$, we lift $\Pi(e,d)$ to the closed complex and apply the closed differential. There are only three kinds of possibly nonzero contributions:

- If $e \geq 1$, then the base north pole can be occupied by sliding an off-pole point from the base circle. The resulting cell is $\Pi(e-1,d)$, with coefficient $-(1+(-1)^e)$.
- If $d \geq 1$, then the leftmost pinned fiber can slide onto the base circle. The resulting cell is $\Pi(e,d-1)$, with coefficient $-(-1)^e P(e,0) = -(-1)^e$.
- If $d \geq 1$, then the rightmost pinned fiber can slide onto the base circle. The resulting cell is $\Pi(e,d-1)$, with coefficient $(-1)^{e+(d-1)}P(e,0) = (-1)^{e+d-1}$.

Since $-(-1)^e+(-1)^{e+d-1} = (-1)^e\big((-1)^{d-1}-1\big)$, the sum of these three contributions gives (6.11). □

Proposition 6.12. *For every odd prime p,*

$$\operatorname{coker}\big(\delta : W_{2p}\otimes\mathbb{Z}_{(p)} \longrightarrow W_{2p-1}\otimes\mathbb{Z}_{(p)}\big) \cong \mathbb{Z}_{(p)}^{p-1}.$$

Proof. By Lemma 6.7, the source $W_{2p}\otimes\mathbb{Z}_{(p)}$ is generated by the pure cells $\Pi(e,d)$ with $e+d=2p$, so the image of δ is spanned by their images. There are $p+1$ such cells with both e and d even, and p such cells with both e and d odd. According to Lemma 6.10, the connecting map kills $\Pi(e,d)$ when both e and d are odd, and satisfies $\delta\Pi(e,d) = \pm 2\Pi(e-1,d) \pm 2\Pi(e,d-1)$ when both e and d are even (if $e=0$ or $d=0$, the corresponding term is absent, as recorded by the indicator functions in (6.11)). Note that 2 is a unit in $\mathbb{Z}_{(p)}$.

By Lemma 6.9, the target $W_{2p-1}\otimes\mathbb{Z}_{(p)}$ is freely spanned by pure cells $\Pi(e,d)$ with $e+d=2p-1$. The supports of the $p+1$ non-zero images $\delta\Pi(e,d)$ (with both e and d even) are pairwise disjoint. Indeed, a target of the form $\Pi(2i-1,2j)$ can only be hit by $\delta\Pi(2i,2j)$, and a target of the form $\Pi(2i,2j-1)$ can only be hit by $\delta\Pi(2i,2j)$. Since the coefficients (± 2) are units in $\mathbb{Z}_{(p)}$, the $p+1$ non-zero images are linearly independent, and the cokernel is free of rank $(2p)-(p+1) = p-1$. □

Proof of Theorem C. Part of the long exact sequence in Borel–Moore homology induced by (6.4) (with $\mathbb{Z}_{(p)}$-coefficients) is:

$$\bar{H}^{\mathrm{BM}}_{2p}(C^{\mathrm{pu}}(2p)) \xrightarrow{\delta} \bar{H}^{\mathrm{BM}}_{2p-1}(C^{\mathrm{pu}}(2p-1)) \longrightarrow \bar{H}^{\mathrm{BM}}_{2p-1}(C^{\mathrm{cl}}(2p)) \longrightarrow \bar{H}^{\mathrm{BM}}_{2p-1}(C^{\mathrm{pu}}(2p)).$$

The last group vanishes as $C^{\mathrm{pu}}(2p)$ has no cells in Borel–Moore dimension $< 2p$. Moreover, we have $\bar{H}^{\mathrm{BM}}_{2p-1}(C^{\mathrm{pu}}(2p-1)) = W_{2p-1}$ and $\bar{H}^{\mathrm{BM}}_{2p}(C^{\mathrm{pu}}(2p)) = W_{2p}$, so Proposition 6.12 gives

$$\bar{H}^{\mathrm{BM}}_{2p-1}(C^{\mathrm{cl}}(2p)) \cong \operatorname{coker}(\delta) \cong \mathbb{Z}_{(p)}^{p-1}.$$

In particular, this is a free $\mathbb{Z}_{(p)}$-module. We conclude by Lemma 6.1 that $H_{2p}(B_{2p}(\Sigma_1);\mathbb{Z})$ has no p-torsion.

Similarly, $\bar{H}^{\mathrm{BM}}_{2p-2}(C^{\mathrm{cl}}(2p)) = 0$, because the closed complex has no cells in Borel–Moore dimension below $2p-1$. Lemma 6.1 then shows that $H_{2p+1}(B_{2p}(\Sigma_1);\mathbb{Z})$ has no p-torsion. □

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