Curved Koszul duality of algebras over unital versions of binary operads

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February 23, 2021

We develop a curved Koszul duality theory for algebras presented by quadratic-linear-constant relations over unital versions of binary quadratic operads. As an application, we study Poisson n-algebras given by polynomial functions on a standard shifted symplectic space. We compute explicit resolutions of these algebras using curved Koszul duality. We use these resolutions to compute derived enveloping algebras and factorization homology on parallelized simply connected closed manifolds with coefficients in these Poisson n-algebras.

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Introduction

Koszul duality was initially developed by Priddy [Pri70] for associative algebras. Given an augmented associative algebra A, there is a Koszul dual algebra $A^!$, and there is an equivalence (subject to some conditions) between parts of the derived categories of A and $A^!$. The Koszul dual $A^!$ is actually the linear dual of a certain coalgebra $A^¡$ up

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to suspension. If the algebra A satisfies what is called the Koszul property, then the cobar construction of $A^{¡}$ is a quasi-free resolution of the algebra A. In this sense, Koszul duality is a tool to produce resolutions of algebras.

An operad is a kind of combinatorial object which governs categories of "algebras" in a wide sense, for example associative algebras, commutative algebras, or Lie algebras. After insights of Kontsevich [Kon93], Koszul duality was generalized with great success to binary quadratic operads by Ginzburg–Kapranov [GK94] (see also Getzler–Jones [GJ94]), and then to quadratic operads by Getzler [Get95] (see also [Fre04; Mar96]). It was for example realized that the operad governing commutative algebras and the operad governing Lie algebras are Koszul dual to each other. This duality explains the links between the approaches of Sullivan [Sul77] and Quillen [Qui69] to rational homotopy theory, which rely respectively on differential graded (dg-) commutative algebras and dg-Lie algebras.

Koszul duality of quadratic operads works roughly as follows. Given an augmented quadratic operad P, there is a Koszul dual cooperad $\mathsf{P}^{¡}$. If P is Koszul, then the operadic cobar construction of $\mathsf{P}^{¡}$ is a quasi-free resolution of the operad P. In this sense, operadic Koszul duality provides a tool to produce resolutions of augmented quadratic operads.

Operadic Koszul duality was then generalized to several different settings (see Section 1.1). Two of them will interest us. The first, due to Hirsh–Millès [HM12], is curved Koszul duality applied to (pr)operads with quadratic-linear-constant relations, by analogy with curved Koszul duality for associative algebras [Pos93; PP05]. The other, due to Millès [Mil12], is Koszul duality for monogenic algebras over quadratic operads, a generalization of quadratic algebras over binary operads.

Our aim will thus be to combine, in some sense, the approaches of Millès [Mil12] and Hirsh–Millès [HM12] in order to develop a curved Koszul duality theory for algebras with quadratic-linear-constant relations over unital versions of binary quadratic operads. The general philosophy is as follows: operads are monoids in the category of symmetric sequence. Hence, the results of Hirsh–Millès [HM12] are in some sense results about the associative operad, which is itself an operad with QLC relations. With this point of view, we reuse the ideas of Millès [Mil12] to define curved Koszul duality for algebras over any operad. Our main theorem is the following:

Theorem A (Theorem 3.7)**.** *Let P be a binary quadratic operad and let $u\mathsf{P}$ be a unital version of P (Def. 1.8). Let A be a $u\mathsf{P}$-algebra with a quadratic-linear-constant presentation (Def. 3.1). Let qA be the P-algebra given by the quadratic reduction of A (Def. 3.3). Finally let $A^{¡} = (qA^{¡}, d_{A^{¡}}, \theta_{A^{¡}})$ be the curved $\mathsf{P}^{¡}$-coalgebra given by the Koszul dual of A (Section 3.2). If the P-algebra qA is Koszul in the sense of [Mil12], then the canonical morphism $\Omega_\kappa A^{¡} \xrightarrow{\sim} A$ is a quasi-isomorphisms of $u\mathsf{P}$-algebras.*

Our motivation is the following. If P is a Koszul operad, then there is a functorial way of obtaining resolutions of P-algebras by considering the bar-cobar construction. However, this resolution is big, and explicit computations are not always easy. On the other hand, the theory of Millès [Mil12] provides resolutions for Koszul monogenic algebras over Koszul quadratic operads which are much smaller, when they exist (cf. Remark 4.6). But the construction is unavailable when the operad is not quadratic and/or when the algebra is not monogenic. Our theorem allows us to construct resolutions of certain non-monogenic algebras over certain non-quadratic operads.

Note that by applying Theorem 3.7 theory to different kinds of operads, we recover some already existing notions of "curved algebras" and "Koszul duality of curved algebras". For example, when applied to associative algebras, we recover the notion of a curved coalgebra of Lyubashenko [Lyu17]. When applied to Lie algebras, we recover (the dual of) curved Lie algebras [CLM16; Mau17].

As an example of application, we study unital Poisson n-algebra. For $n \in \mathbb{Z}$ and $D \geq 0$, consider the Poisson n-algebra $A_{n;D} = \mathbb{R}[x_1, \dots, x_D, \xi_1, \dots, \xi_D]$, where $\deg x_i = 0$, $\deg \xi_j = 1 - n$, and the shifted Lie bracket is given by $\{x_i, \xi_j\} = \delta_{ij}$. We may view $A_{n;D}$ as the algebra $\mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^D[1-n])$ of polynomial functions on the shifted cotangent space of $\mathbb{R}^D$, with x_i being a coordinate function on $\mathbb{R}^D$, and ξ_j being the vector field $\partial/\partial x_j$. This algebra admits a presentation with quadratic-linear-constant relations over the operad $u\mathsf{Pois}_n$ governing unital Poisson n-algebra. It is Koszul, thus the cobar construction on its Koszul dual $\Omega_\kappa A^¡$ provides an explicit cofibrant replacement of A. We explicitly describe this cofibrant replacement.

We use $\Omega_\kappa A^¡$ to compute the derived enveloping algebra of $A_{n;D}$, which we prove is quasi-isomorphic to the underived enveloping algebra of A. We also compute the factorization homology $\int_M A_{n;D}$ of a simply connected parallelized closed manifold M with coefficients in $A_{n;D}$. We prove that the homology of $\int_M A_{n;D}$ is one-dimensional for such manifolds (Proposition 4.17). This fits in with the physical intuition that the expected value of a quantum observable, which should be a single number, lives in $\int_M A$, see e.g. [CG17] for a broad reference.

Note that a computation for a similar object was performed by Markarian [Mar17]. Moreover, shortly after the first version of this paper, Döppenschmitt [Döp18] released a preprint containing the computation of the factorization homology of a twisted version of $A_{n;D}$, using physical methods. See Remark 4.19.

Outline In Section 1, we lay out our conventions and notations, as well as background for the rest of the paper. This section does not contain any original result. We give a quick tour of Koszul duality (Section 1.1), recall the definition of "unital version" of a quadratic operad (Section 1.2), and give some background on factorization homology (Section 1.3). In Section 2, we define curved coalgebras and semi-augmented algebras. We also define the bar and cobar constructions. We prove that they are adjoint to each other. In Section 3, we prove our main theorem. We define algebras with QLC relations and the Koszul dual curved coalgebra of such an algebra. We prove that if the quadratic reduction of the algebra is Koszul, then the cobar construction on the Koszul dual of the algebra is a cofibrant replacement of the algebra. In Section 4, we apply the theory to the symplectic Poisson n-algebras. We explicitly describe the cofibrant replacement obtained by Koszul duality. We use it compute their derived enveloping algebras and factorization homology.

Acknowledgments The author is thankful to Julien Ducoulombier, Benoit Fresse, Thomas Willwacher, and Lukas Woike for helpful discussions and comments, and to the anonymous referee for a thorough report and many helpful suggestions. The completion of this work was partially supported by ERC StG 678156–GRAPHCPX

1 Conventions, background, and recollections

We work with $\mathbb{Z}$-graded chain complexes over some base field $\Bbbk$ of characteristic zero, which we call "dg-modules". Given a dg-module V, its suspension ΣV is given by $(\Sigma V)_n = V_{n-1}$ with a signed differential.

We work extensively with (co)operads and (co)algebras over (co)operads. We refer to e.g. [LV12] or [Fre17, Part I(a)] for a detailed treatment. Briefly, a (symmetric, one-colored) operad P is a collection $\{\mathsf{P}(n)\}_{n\geq 0}$ of dg-modules, with each $\mathsf{P}(n)$ equipped with an action of the symmetric group Σ_n, a unit $\eta \in \mathsf{P}(1)$, and composition maps $\circ_i : \mathsf{P}(k) \otimes \mathsf{P}(l) \to \mathsf{P}(k+l-1)$ for $1 \leq i \leq k$ satisfying the usual equivariance, unit, and associativity axioms. For such an operad P and a dg-module V, we define $\mathsf{P}(V) = \bigoplus_{n\geq 0} \mathsf{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$. A P-algebra is a dg-module A equipped with a structure map $\gamma_A : \mathsf{P}(A) \to A$ satisfying the usual axioms. For a dg-module V, the algebra $\mathsf{P}(V)$ is the free P-algebra on V. Cooperads and coalgebras are defined dually. We will generally consider conilpotent cooperads, i.e. cooperads for which iterated cocompositions vanish on any element after enough iterations.

Example 1.1. Some examples of operads will appear several times: (i) the operad Ass governing associative algebras; (ii) the operad Com, governing commutative algebras; (iii) the operad Lie, governing Lie algebra; (iv) the operad Pois_n governing Poisson n-algebras, i.e. algebras with a commutative product and a Lie bracket of degree $n-1$ which is a biderivation. As a symmetric sequence, Pois_n is isomorphic to $\mathsf{Com} \circ \mathscr{S}^{1-n}\mathsf{Lie}$. The operad structure is induced by those of Com and Lie, as well as a distributive law stating that the bracket is a biderivation.

If $E = \{E(n)\}_{n\geq 0}$ is a symmetric sequence, then we write $\mathrm{Free}(E)$ for the free operad generated by E. It can be described in terms of rooted trees with internal vertices decorated by elements of E. Operadic composition is given by grafting of trees. If $S \subset \mathsf{P}$ is a subsequence of an operad P, then we write $\mathsf{P}/(S)$ for the quotient by the operadic ideal generated by S. Given an operad P, the suspended operad $\mathscr{S}\mathsf{P}$ is defined such that a $\mathscr{S}\mathsf{P}$-algebra structure on ΣA is the same thing as a P-algebra structure on A. It is also sometimes denoted $\Lambda^{-1}\mathsf{P}$. The suspended cooperad $\mathscr{S}^c\mathsf{C}$ is defined similarly. On (co)free (co)algebras, we have $\mathscr{S}\mathsf{P}(\Sigma V) = \Sigma\mathsf{P}(V)$ and $\mathscr{S}^c\mathsf{C}(\Sigma V) = \Sigma\mathsf{C}(V)$.

1.1 Koszul duality for...

We now briefly recall several incarnations of Koszul duality in order to set up the notations and definitions.

1.1.1 ...associative algebras

Let A be a quadratic associative algebra, i.e. an algebra equipped with a presentation with quadratic relations. There is an associated Koszul dual coalgebra A^{i} to A [Pri70]. These can be used to define the Koszul complex $A \otimes_\kappa A^{\text{i}}$. By definition, A is Koszul if this Koszul complex is acyclic. The coalgebra A^{i} is a sub-coalgebra of the bar construction BA, and A is Koszul if and only if the inclusion is a quasi-isomorphism. There is a also canonical morphism from the cobar construction ΩA^{i} to A, and A is Koszul if and only if this canonical morphism is a quasi-isomorphism. Since the cobar construction ΩA^{i} is quasi-free as an associative algebra, this allows to produce an explicit, small, quasi-free resolution of any Koszul algebra.

Example 1.2. Let $A = \mathbb{R}[x_1, \dots, x_k]$ be a free commutative algebra on k variables of degree zero. The Koszul dual coalgebra $A^{\text{i}} = \Lambda^c(dx_1, \dots, dx_k)$ is the exterior coalgebra on k variables of degree 1. The Koszul complex $(A \otimes A^{\text{i}}, d_\kappa)$ has a differential similar to the de Rham differential. It is acyclic, therefore A is Koszul.

1.1.2 ...quadratic operads

We refer to [LV12] for a detailed treatment. Let $\mathsf{P} = \mathrm{Free}(E)/(R)$ be an operad generated $E = \{E(n)\}_{n \geq 0}$ with relations $R \subset \mathrm{Free}(E)$. This presentation is quadratic if the relations R is a subsequence of the weight 2 component $\mathrm{Free}(E)^{(2)}$. We call P quadratic if it admits such a presentation. One can define the Koszul dual cooperad P^{i}, the cofree cooperad on cogenerators ΣE subject to the corelations $\Sigma^2 R$ (i.e. P^{i} is the smallest sub-cooperad of the cofree cooperad on ΣE that contains $\Sigma^2 R$). One can also define the Koszul dual operad $\mathsf{P}^!$ as the shifted linear dual of P^{i}.

Example 1.3. The operad Ass is auto-dual: $\mathsf{Ass}^! = \mathsf{Ass}$. The operads Com and Lie are Koszul dual to each other: $\mathsf{Com}^! = \mathsf{Lie}$ and $\mathsf{Lie}^! = \mathsf{Com}$. The operad $\mathsf{Pois}_n = \mathsf{Com} \circ \mathscr{S}^{1-n}\mathsf{Lie}$ is auto-dual up to suspension: $\mathsf{Pois}_n^! = \mathscr{S}^{n-1}\mathsf{Pois}_n$.

The cooperad P^{i} is a sub-cooperad of $B\mathsf{P}$, the operadic bar construction of P. The Rosetta stone [LV12, Theorem 6.5.7] implies that morphisms $\mathsf{P}^{\text{i}} \to B\mathsf{P}$ are in bijection with twisting morphisms, i.e. equivariant maps $\kappa : \mathsf{P}^{\text{i}} \to \mathsf{P}$ of degree -1 that satisfy the Maurer–Cartan equation $\partial\kappa + \kappa \star \kappa$. The operation $\star$ is the preLie convolution product on $\mathrm{Hom}(\mathsf{P}^{\text{i}}, \mathsf{P})$, defined on $f, g : \mathsf{P}^{\text{i}} \to \mathsf{P}$ by:

$$f \star g : \mathsf{P}^{\text{i}} \xrightarrow{\Delta_{(1)}} \mathsf{P}^{\text{i}} \circ_{(1)} \mathsf{P}^{\text{i}} \xrightarrow{f \circ_{(1)} g} \mathsf{P} \circ_{(1)} \mathsf{P} \xrightarrow{\gamma_{(1)}} \mathsf{P}. \tag{1.1}$$

Here, the twisting morphism κ induced by $\mathsf{P}^{\text{i}} \subset B\mathsf{P}$ is defined by:

$$\kappa : \mathsf{P}^{\text{i}} \twoheadrightarrow \Sigma E \xrightarrow{\Sigma^{-1}} E \hookrightarrow \mathsf{P}, \tag{1.2}$$

The same Rosetta stone moreover implies that such twisting morphisms are in bijection with morphisms from the operadic cobar construction $\Omega\mathsf{P}^{\text{i}}$ to P. The operad is said to be Koszul if this morphism is a quasi-isomorphism.

The twisting morphism $\kappa : \mathsf{P}^{\mathsf{i}} \to \mathsf{P}$ induces an adjunction $\Omega_\kappa \dashv B_\kappa$ between the categories of P^{i}-coalgebras and P-algebras. If P is Koszul, then $\Omega_\kappa B_\kappa$ is a functorial cofibrant replacement functor.

Example 1.4. The operads Ass, Com, Lie, and Pois_n are all Koszul. For the operad Ass, $\Omega_\kappa \dashv B_\kappa$ gives the usual bar-cobar resolution. For Com, the resolution obtained is (up to degree shifts) the algebra of Chevalley–Eilenberg cochains on the Harrison complex.

1.1.3 ...monogenic algebras over operads

Millès [Mil12] extended Koszul duality to monogenic algebras over quadratic operads, a notion which generalizes quadratic associative algebras. Given a quadratic operad $\mathsf{P} = \mathrm{Free}(E)/(R)$, a monogenic P-algebra A is an algebra equipped with a presentation $A = \mathsf{P}(V)/(S)$, where $S \subset E(V)$ is a set of relations. To such an algebra, one associates the Koszul dual P^{i}-coalgebra $A^{\mathsf{i}} = \Sigma\mathsf{P}^{\mathsf{i}}(V, \Sigma S)$, i.e. the suspension of the cofree P^{i}-coalgebra on V subject to the corelations ΣS. There is a canonical algebra-twisting morphism $\varkappa$ of degree -1,

$$\varkappa : A^{\mathsf{i}} \twoheadrightarrow \Sigma V \xrightarrow{\Sigma^{-1}} V \hookrightarrow A, \tag{1.3}$$

Let $\kappa : \mathsf{P}^{\mathsf{i}} \to \mathsf{P}$ be the operad-twisting morphism of Equation (1.2). The κ-star product $\star_\kappa(\varkappa)$ is given by the composition:

$$\star_\kappa(\varkappa) : \Sigma A^{\mathsf{i}} \xrightarrow{\Delta_{A^{\mathsf{i}}}} \Sigma\mathsf{P}^{\mathsf{i}}(A^{\mathsf{i}}) \xrightarrow{\kappa\circ\varkappa} \mathsf{P}(A) \xrightarrow{\gamma_A} A. \tag{1.4}$$

The element $\varkappa$ satisfies the Maurer–Cartan equation $\star_\kappa(\varkappa) = 0$ (the differential vanishes). It thus defines a morphism $f_\varkappa : \Omega_\kappa A^{\mathsf{i}} \to A$.

The algebra A is said to be Koszul if this morphism $f_\varkappa$ is a quasi-isomorphism. Millès [Mil12] proves that this is equivalent to a certain Koszul complex being acyclic, and also equivalent to the adjoint canonical morphism $A^{\mathsf{i}} \to BA$ being a quasi-isomorphism. In this case, the algebra $\Omega_\kappa A^{\mathsf{i}}$ is an explicit, small resolution of A. For example, if $\mathsf{P} = \mathsf{Ass}$, this recovers the usual Koszul duality/resolution of associative algebras.

1.1.4 ...operads with QLC relations

Curved Koszul duality is a generalization of Koszul duality for *unital* associative algebras [Pos93; PP05]. This was generalized by Hirsh–Millès [HM12] for (pr)operads with quadratic-linear-constant (QLC) relations (see also Lyubashenko [Lyu11] for the case of the unital associative operad, and [GTV12] for operads with quadratic-linear relations.) Let I be the unit operad, i.e. $\mathsf{I}(1) = \Bbbk$ and $\mathsf{I}(n) = 0$ for $n \neq 1$. A QLC presentation of an operad is a presentation $\mathsf{P} = \mathrm{Free}(E)/(R)$, where E is some module of generators and $R \subset \mathsf{I} \oplus E \oplus \mathrm{Free}(E)^{(2)}$ is some module of relations with constant (i.e. multiple of id_{P}), linear, and quadratic terms.

Example 1.5. The operad $u\mathsf{Ass}$ governing unital associative algebras has a QLC presentation. It is generated by the unit $\mathbb{1} \in u\mathsf{Ass}(0)$ and the product $\mu \in u\mathsf{Ass}(2)$. The relations are $\mu \circ_1 \mu = \mu \circ_2 \mu$ (quadratic) and $\mu \circ_1 \mathbb{1} = \mathrm{id} = \mu \circ_2 \mathbb{1}$ (quadratic-constant).

The quadratic reduction $q\mathsf{P}$ is the quadratic operad $\mathrm{Free}(E)/(qR)$, where qR is the projection of R onto $\mathrm{Free}(E)^{(2)}$. Hirsh and Millès impose some conditions on this presentation: the space of generators is minimal, i.e. $R \cap \langle \mathsf{I} \oplus E\rangle = 0$, and the space of relations is maximal, i.e. $R = (R) \cap \langle \mathsf{I} \oplus E \oplus \mathrm{Free}(E)^{(2)}\rangle$. Therefore R is the graph of some map $\varphi = (\varphi_0 + \varphi_1) : qR \to \mathsf{I} \oplus E$, i.e. $R = \{X + \varphi(X) \mid X \in qR\}$.

From this data, they define the Koszul dual cooperad P^{i}, which is a *curved* cooperad. This curved cooperad is a triplet $(q\mathsf{P}^{\mathsf{i}}, d_{\mathsf{P}^{\mathsf{i}}}, \theta_{\mathsf{P}^{\mathsf{i}}})$, where:

- $q\mathsf{P}^{\mathsf{i}}$ is the Koszul dual cooperad of the quadratic cooperad $q\mathsf{P}$;
- the predifferential $d_{\mathsf{P}^{\mathsf{i}}}$ is the unique degree -1 coderivation of $q\mathsf{P}^{\mathsf{i}}$ whose corestriction (composition with the projection) onto ΣE is given by $q\mathsf{P}^{\mathsf{i}} \twoheadrightarrow \Sigma^2 qR \xrightarrow{\varphi_1} \Sigma E$;
- the curvature $\theta_{\mathsf{P}^{\mathsf{i}}}$ is the map of degree -2 obtained by $q\mathsf{P}^{\mathsf{i}} \twoheadrightarrow \Sigma^2 qR \xrightarrow{\varphi_0} I$.

The axioms satisfied by this data imply that the cobar construction $\Omega(q\mathsf{P}^{\mathsf{i}}) = \big(\mathrm{Free}(\Sigma^{-1} q\mathsf{P}^{\mathsf{i}}), d_2\big)$ is equipped with an extra differential $d_0 + d_1$ defined from $\theta_{\mathsf{P}^{\mathsf{i}}}$ and $d_{\mathsf{P}^{\mathsf{i}}}$. The canonical morphism $\Omega(q\mathsf{P}^{\mathsf{i}}) \to q\mathsf{P}$ extends to a canonical morphism $\Omega\mathsf{P}^{\mathsf{i}} := \big(\Omega(q\mathsf{P}^{\mathsf{i}}), d_0 + d_1\big) \to \mathsf{P}$. If the quadratic operad $q\mathsf{P}$ is Koszul, then $\Omega\mathsf{P}^{\mathsf{i}} \to \mathsf{P}$ is a quasi-isomorphism [HM12, Theorem 4.3.1]. The operad P is therefore said to be Koszul if the quadratic operad $q\mathsf{P}$ is Koszul in the usual sense.

Remark 1.6. Le Grignou [LeG17] defined a model category structure on the category of curved cooperads, which is Quillen equivalent to the model category of operads using the bar/cobar adjunction.

1.2 Unital versions of operads

In what follows, we will only work with algebras over unital versions of binary quadratic operads. Let $\mathsf{P} = \mathrm{Free}(E)/(R)$ be a an operad presented by binary generators $E = E(2)$ and quadratic relations $R \subset \big(E(2)^{\otimes 2}\big)_{\Sigma_2}$. Moreover, we assume that the differential of P is zero.

Remark 1.7. While most of this paper could be carried out without the assumption that P is binary, we need this assumption to be able to prove Lemma 2.2 and Proposition 3.4. Proposition 3.4 could be proved for non-binary operads by modifying the weight grading, but not Lemma 2.2.

Definition 1.8 (Adapted from [HM12, Definition 6.5.4])**.** A unital version of P is an operad $u\mathsf{P}$ equipped with a presentation of the form $u\mathsf{P} = \mathrm{Free}(E \oplus \mathbb{1})/(R + R')$, where $\mathbb{1}$ is a generator of arity 0 and degree 0, and such that

(i) the inclusion $E \subset E \oplus \mathbb{1}$ induces an injective morphism of operads $\mathsf{P} \to u\mathsf{P}$;

(ii) we have an isomorphism $\mathbb{1} \oplus \mathsf{P} \cong qu\mathsf{P}$ induced by the inclusions $\mathsf{P} \subset u\mathsf{P}$ and $\mathbb{1} \subset u\mathsf{P}$;

(iii) the QLC relations in R' have no linear terms, only quadratic-constant.

Example 1.9. Examples include the operads (i) $u\mathsf{Ass}$, encoding unital associative algebras; (ii) $u\mathsf{Com}$, encoding unital commutative algebras; (iii) $c\mathsf{Lie}$, encoding Lie algebras equipped with a central element; (iv) $u\mathsf{Pois}_n$, encoding Poisson n-algebras equipped with an element which is a unit for the product and a central element for the shifted Lie bracket.

1.3 Factorization homology

Let us now briefly introduce factorization homology (as in the previous sections, we do not claim any originality). Factorization homology [AF15], also known as topological chiral homology [BD04], is an invariant of manifolds with coefficients in a homotopy commutative algebra, just like standard homology is an invariant of topological spaces with coefficients in a commutative ring. One possible definition of factorization homology is the following [Fra13], which we only give for parallelized manifold for simplicity.

Fix some integer $n \geq 0$. Let $\mathsf{Disk}_n^{\mathrm{fr}}$ be the endomorphism operad of $\mathbb{R}^n$ in the category of parallelized manifolds and embeddings preserving parallelization. In each arity, we have $\mathsf{Disk}_n^{\mathrm{fr}}(k) \coloneqq \mathrm{Emb}^{\mathrm{fr}}\big((\mathbb{R}^n)^{\sqcup k}, \mathbb{R}^n\big)$, and operadic composition is given by composition of embeddings. This operad is weakly equivalent to the usual little n-disks operad, i.e. it is an E_n-operad. In particular, its homology $H_*(\mathsf{Disk}_n^{\mathrm{fr}})$ is the unital associative operad $u\mathsf{Ass}$ for $n = 1$, and the unital n-Poisson operad $u\mathsf{Pois}_n$ for $n \geq 2$ (see Example 1.9).

Moreover, given a parallelized n-manifold M, there is a right $\mathsf{Disk}_n^{\mathrm{fr}}$-module given by $\mathsf{Disk}_M^{\mathrm{fr}}(k) \coloneqq \mathrm{Emb}^{\mathrm{fr}}\big((\mathbb{R}^n)^{\sqcup k}, M\big)$. For a topological $\mathsf{Disk}_n^{\mathrm{fr}}$-algebra A (i.e. an E_n-algebra), the factorization homology of M with coefficients in A, denoted by $\int_M A$, is given by the derived composition product:

$$\int_M A \coloneqq \mathsf{Disk}_M^{\mathrm{fr}} \circ^{\mathbb{L}}_{\mathsf{Disk}_n^{\mathrm{fr}}} A = \mathrm{hocoeq}\big(\mathsf{Disk}_M^{\mathrm{fr}} \circ \mathsf{Disk}_n^{\mathrm{fr}} \circ A \rightrightarrows \mathsf{Disk}_M^{\mathrm{fr}} \circ A\big). \tag{1.5}$$

This definition also makes sense in the category of chain complexes, replacing $\mathsf{Disk}_n^{\mathrm{fr}}$ and $\mathsf{Disk}_M^{\mathrm{fr}}$ by their respective chain complexes. The operad Disk_n is formal over the rationals [Kon99; Tam03; LV14; Pet14; FW20]. Hence, up to homotopy, we may replace $C_*(\mathsf{Disk}_n^{\mathrm{fr}}; \mathbb{Q})$ by $H_*(\mathsf{Disk}_n^{\mathrm{fr}}; \mathbb{Q}) = u\mathsf{Pois}_n$ for $n \geq 2$.

In [Idr19], given a simply connected closed smooth manifold M with $\dim M \geq 4$, we provided an explicit model for $C_*(\mathsf{Disk}_M^{\mathrm{fr}}; \mathbb{R})$. Our model is a right module over the operad $H_*(\mathsf{Disk}_n^{\mathrm{fr}}) = u\mathsf{Pois}_n$, and the action is compatible with the action of $\mathsf{Disk}_n^{\mathrm{fr}}$ on $\mathsf{Disk}_M^{\mathrm{fr}}$ in an appropriate sense. This allows us to compute factorization homology of such manifolds by replacing $C_*(\mathsf{Disk}_M^{\mathrm{fr}})$ with our model.

This explicit model, denoted $\mathsf{G}_P^\vee$, depends on a Poincaré duality model P of M, i.e. an (upper-graded) commutative differential graded algebra equipped with a linear form $\varepsilon : P^n \to \mathbb{Q}$ satisfying $\varepsilon \circ d = 0$ and inducing a non-degenerate pairing $P^k \otimes P^{n-k} \to \mathbb{Q}$, $x \otimes y \mapsto \varepsilon(xy)$ for all $k \in \mathbb{Z}$. It is moreover a rational model for M in the sense of Sullivan rational homotopy theory. These Poincaré duality models exist for all simply connected closed manifolds [LS08].

We will not give the original definition of $\mathsf{G}_P^\vee$. Instead, we give the alternative description of [Idr19, Section 5]. Let $\mathsf{Lie}_n = \mathscr{S}^{1-n}\mathsf{Lie}$ be the operad governing shifted

Lie algebras. For convenience, let us also define $\mathsf{L}_n(k) \coloneqq \Sigma^{n-1}\mathsf{Lie}_n(k)$, which satisfies $\mathsf{L}_n(V) \cong \mathsf{Lie}(\Sigma^{n-1}V)$ for all spaces V. This symmetric sequence is a Lie algebra in the category of right Lie_n-modules: the right Lie_n-module of Lie_n is unaffected by the shift, and the Lie algebra structure is the shift of the canonical left action of Lie_n on itself.

Given a Lie algebra $\mathfrak{g}$, let its Chevalley–Eilenberg chain complex (with trivial coefficients) be $C_*^{CE}(\mathfrak{g}) \coloneqq (\bar{S}^c(\Sigma\mathfrak{g}), d_{CE})$, with differential defined by $d_{CE}(x_1 \dots x_k) = \sum_{i<j} \pm x_1 \dots [x_i, x_j] \dots \widehat{x_j} \dots x_k$. As a right Lie_n-module, our model is given by [Idr19, Lemma 5.2]:

$$\mathsf{G}_P^\vee \cong_{\mathsf{Lie}_n-\mathrm{RMod}} C_*^{CE}(P^{-*} \otimes \mathsf{L}_n), \tag{1.6}$$

where P^{-*} is P with grading reversed. (The right $u\mathsf{Com}$-module structure, which is not explicitly described in [Idr19], will be described in Section 4.4.1.)

Using theorems about preservations of weak equivalences by the relative operadic composition product, we find that given a $u\mathsf{Pois}_n$-algebra A, the factorization homology of M with coefficients in A over $\mathbb{R}$ is given, up to quasi-isomorphism and under the hypotheses stated above, by:

$$\int_M A \simeq \mathsf{G}_P^\vee \circ^{\mathbb{L}}_{u\mathsf{Pois}_n} A. \tag{1.7}$$

As an example, if $A = S(\Sigma^{1-n}\mathfrak{g})$ is the universal enveloping n-algebra of $\mathfrak{g}$, then we recovered in [Idr19] a theorem of Knudsen [Knu18] which states that $\int_M S(\Sigma^{1-n}\mathfrak{g}) \simeq C_*^{CE}(P^{n-*} \otimes \mathfrak{g})$.

2 Curved bar-cobar adjunction

From now on, we fix a binary, homogeneous quadratic operad $\mathsf{P} = \mathrm{Free}(E)/(R)$ and a unital version $u\mathsf{P} = \mathrm{Free}(\text{\textbf{!}} \oplus E)/(R + R')$ as in Section 1.2, with the same notations. Note that the operad P is automatically augmented.

2.1 Curved coalgebras and semi-augmented algebras

Let C be a coaugmented conilpotent cooperad, with zero differential. Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism [LV12, Section 6.4.3], i.e. a map satisfying the Maurer–Cartan equation $\varphi \star \varphi = 0$, where $\star$ is the convolution product of Equation (1.1) (and the differential of φ vanishes because d_{C} and d_{P} do). Suppose moreover that $\operatorname{im}\varphi \subset E = \mathsf{P}(2)$.

Given a C-coalgebra C and a map $\theta : C \to \Bbbk\text{\textbf{!}}$, we denote by

$$\Theta : C \xrightarrow{\theta} \Bbbk\text{\textbf{!}} \hookrightarrow u\mathsf{P}(C) \tag{2.1}$$

the composition, where $u\mathsf{P}(C)$ is the free $u\mathsf{P}$-algebra on C. Let us also define a morphism:

$$\begin{aligned} \varphi \circ' \Theta : \mathsf{C}(C) &\to u\mathsf{P}(u\mathsf{P}(C)) \\ x(c_1, \dots, c_n) &\mapsto \sum_{i=1}^{n} (-1)^{i-1} \varphi(x)\big(c_1, \dots, c_{i-1}, \Theta(c_i), c_{i+1}, \dots, c_n\big). \end{aligned} \tag{2.2}$$

Definition 2.1. The φ-star product of θ is the composition:

$$\star_{\varphi}(\theta) : C \xrightarrow{\Delta_C} \mathsf{C}(C) \xrightarrow{\varphi \circ' \Theta} u\mathsf{P}(u\mathsf{P}(C)) \xrightarrow{\gamma_{u\mathsf{P}(C)}} u\mathsf{P}(C). \tag{2.3}$$

Lemma 2.2. *Given a twisting morphism $\varphi : \mathsf{C} \to \mathsf{P}$, a C-coalgebra C, and a map $\theta : C \to \Bbbk^{\dagger}$, the image of $\star_{\varphi}(\theta)$ is included in $C \subset u\mathsf{P}(C)$.*

Proof. This follows from our hypotheses that $\operatorname{im} \varphi \subset \mathsf{P}(2)$, and from conditions (ii) and (iii) of Definition 1.8, which imply that the inhomogeneous relations of $u\mathsf{P}$ have no linear terms. ☐

Lemma 2.3. *Let C be a C-coalgebra, $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism, and $\theta : C \to \Bbbk^{\dagger}$ a linear map. The map $\star_{\varphi}(\theta) : C \to C$ is a coderivation.*

Proof. We need to check that $\Delta_C \circ (\star_{\varphi}(\theta)) = (\mathrm{id} \circ' \star_{\varphi}(\theta)) \circ \Delta_C$. Let $x \in C$ be some element. Using Sweedler's notation, $\Delta_C(x) = \mathrm{id}_{\mathsf{C}}(x) + \sum_{(x)} \alpha_{(0)} \otimes x_{(1)} \otimes x_{(2)} + X$, where $\alpha_{(0)} \in \mathsf{C}(2)$, $x_{(1)}, x_{(2)} \in C$, and $X \in \bigoplus_{r \geq 3} \mathsf{C}(r) \otimes C^{\otimes r}$. Since $\varphi(\mathsf{C}) \subset \mathsf{P}(2)$, we then get that

$$\star_{\varphi}(\theta)(x) = \sum_{(x)} \varphi(\alpha_{(0)})(\Theta(x_{(1)}), x_{(2)}) - \sum_{(x)} \varphi(\alpha_{(0)})(x_{(1)} \otimes \Theta(x_{(2)})). \tag{2.4}$$

Thanks to Definition 1.8, the expressions appearing in both sums are respectively scalar multiples of $x_{(1)}$ and $x_{(2)}$, and the scalar in question only depends on $\alpha_{(0)}$ and, respectively, $x_{(2)}$ or $x_{(1)}$. Thanks to the coassociativity of Δ_C, applying the map Δ_C to $\star_{\varphi}(\Theta)(x)$ thus amounts to applying Δ_C twice to x and then summing the application of $\varphi(-)(\theta(-), -)$ to the first two tensors, and $\varphi(-)(-, \Theta(-))$ to the last two tensors. Using again the coassociativity of Δ_C and the Σ_2-equivariance of φ, this is precisely what we get when we compute $(\mathrm{id} \circ' \star_{\varphi}(\theta)) \circ \Delta_C$, as the other terms (where Θ is applied to one of the middle elements of $\Delta_C^2(x)$) all appear twice with opposing signs. ☐

Example 2.4. Let $u\mathsf{P} = u\mathsf{Ass}$ be the operad governing unital associative operads and $\kappa : \mathsf{Ass}^{\mathsf{i}} \to \mathsf{Ass}$ the Koszul twisting morphism. Then, given a coalgebra C and a map $\theta : C \to \Bbbk$, the κ-star product of θ is given by:

$$\star_{\kappa}(\theta) = (\theta \otimes \mathrm{id} - \mathrm{id} \otimes \theta)\Delta_C : C \to C. \tag{2.5}$$

(Recall that $(f \otimes g)(a \otimes b) = (-1)^{|f| \cdot |a|} f(a) \otimes g(b)$.)

We introduce the following definition, inspired by the definition of a curved coproperad in [HM12, Section 3.2.1]. (Informally, we can think of the definition of [HM12] as the case where the twisting morphism φ is the Koszul morphism from the colored operad of operads to its Koszul dual.)

Definition 2.5. A φ-curved C-coalgebra is a triple (C, d_C, θ_C) where:

- C is a C-coalgebra (with no differential);
- $d_C : C \to C$ is a degree -1 coderivation (the "predifferential");

- $\theta_C : C \to \mathbb{k}\mathbb{1}$ is a degree -2 linear map (the "curvature");

satisfying:

$$d_C^2 = \star_\varphi(\theta_C), \qquad\qquad \theta_C d_C = 0. \tag{2.6}$$

Remark 2.6. This notion is a generalization of the notion of coalgebra over a curved cooperad from [HM12, Section 5.2.1]. A coalgebra over a curved cooperad $(\mathsf{C}, d_\mathsf{C}, \theta_\mathsf{C})$ is a pair (C, d_C) where C is a C-coalgebra, d_C is a coderivation of C, and $d_C^2 = (\theta_\mathsf{C} \circ \mathrm{id}_C)\Delta_C$. In our setting, the curvature is part of the data of the coalgebra, rather than the cooperad itself, and we have an extra condition $\theta_C d_C = 0$. Moreover, our notion of curved coalgebra depends on the data of a twisting morphism $\varphi : \mathsf{C} \to \mathsf{P}$, whereas in the other setting this is extra data required to define a bar/cobar adjunction. Le Grignou [LeG19] endowed the category of coalgebras over a curved cooperad with a model category structure, such that the bar/cobar adjunction defines a Quillen equivalence with the category of unital algebras.

Example 2.7. Consider the case $u\mathsf{P} = u\mathsf{Ass}$, then $\mathsf{C} = \mathsf{Ass}^{¡} = \mathscr{S}^c\mathsf{Ass}^\vee$, and $\varphi = \kappa$ is the twisting morphism of Koszul duality. A κ-curved $\mathsf{Ass}^{¡}$-coalgebra is a (shifted) coassociative coalgebra C, equipped with a predifferential d_C and a curvature $\theta_C : C \to \mathbb{k}$, satisfying $\theta_C d_C = 0$ and: :

$$d_C^2 = \star_{\bar\kappa}(\theta) : C \xrightarrow{\Delta} C \otimes C \xrightarrow{\theta\otimes\mathrm{id} - \mathrm{id}\otimes\theta} C. \tag{2.7}$$

This recovers the notion of curved homotopy coalgebra of Lyubashenko [Lyu17].

Example 2.8. For Lie algebras and the Koszul twisting morphism $\varphi = \kappa : \mathsf{Lie}^{¡} \to \mathsf{Lie}$, this definition recovers (the dual of) the notion of curved Lie algebras [CLM16; Mau17], i.e. a Lie algebras $\mathfrak{g}$ equipped with a derivation d of degree -1 and an element ω of degree -2 such that $d^2 = [\omega, -]$.

We also define the notion of a semi-augmented algebra over $u\mathsf{P}$ (the terminology is adapted from [HM12]). This is necessary because, in general, the bar construction of an algebra is not a curved coalgebra.

Definition 2.9. A semi-augmented $u\mathsf{P}$-algebra is a dg-$u\mathsf{P}$-algebra A equipped with a linear map $\varepsilon_A : A \to \mathbb{k}$ (not necessarily compatible with the dg-algebra structure), such that $\varepsilon_A(\mathbb{1}) = 1$. Given such a semi-augmented $u\mathsf{P}$-algebra, we let $\bar{A}$ be the kernel of ε_A.

Given a semi-augmented $u\mathsf{P}$-algebra (A, ε_A), the exact sequence $0 \to \bar{A} \to A \xrightarrow{\varepsilon} \mathbb{k}\mathbb{1} \to 0$ defines an isomorphism of graded modules $A \cong \bar{A} \oplus \mathbb{k}\mathbb{1}$. This isomorphism is not compatible with the differential or the algebra structure in general. This allows to define a "composition" $\bar\gamma_A : u\mathsf{P}(\bar{A}) \to \bar{A}$ (which is generally not associative or a chain map) and a "differential" $\bar{d}_{\bar{A}} : \bar{A} \to \bar{A}$ (which does not square to zero in general), by using the inclusion and the projection $\bar{A} \to \bar{A} \oplus \mathbb{k}\mathbb{1} \cong A \to \bar{A}$.

2.2 Cobar construction

Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism, i.e. an element satisfying the Maurer–Cartan equation $\varphi \star \varphi = 0$. Let $C = (C, d_C, \theta_C)$ be a φ-curved C-coalgebra as in Definition 2.5. We adapt the definition of [HM12, Section 5.2.5].

Definition 2.10. The cobar construction of C with respect to φ is:

$$\Omega_\varphi(C) := (u\mathsf{P}(\Sigma^{-1}C), d_\Omega = -d_0 + d_1 - d_2), \tag{2.8}$$

where each d_i is a derivation of degree -1 defined on generators by:

$$d_0|_{\Sigma^{-1}C} : \Sigma^{-1}C \xrightarrow{\Sigma\theta_C} \mathbb{k}⫯ \hookrightarrow u\mathsf{P}(\Sigma^{-1}C) \tag{2.9}$$

$$d_1|_{\Sigma^{-1}C} : \Sigma^{-1}C \xrightarrow{d_C} \Sigma^{-1}C \hookrightarrow u\mathsf{P}(\Sigma^{-1}C) \tag{2.10}$$

$$d_2|_{\Sigma^{-1}C} : \Sigma^{-1}C \xrightarrow{\Delta} \mathsf{C}(\Sigma^{-1}C) \xrightarrow{\varphi(\mathrm{id})} u\mathsf{P}(\Sigma^{-1}C) \tag{2.11}$$

It is equipped with the semi-augmentation $\varepsilon_\Omega : \Omega_\varphi(C) \to \mathbb{k}$ given by the projection $u\mathsf{P} \to ⫯$.

Proposition 2.11. *Given a φ-curved C-coalgebra $C = (C, d_C, \theta_C)$, the cobar construction $\Omega_\varphi(C)$ is a well-defined semi-augmented $u\mathsf{P}$-algebra.*

Proof. All we need to check is that the derivation d_Ω squares to zero. There is a weight decomposition (denoted ω) of $u\mathsf{P}(\Sigma^{-1}C)$ obtained by assigning $\Sigma^{-1}C$ the weight 1. For example, d_0 is of weight -2, d_1 is of weight -1, and d_2 is of weight 0. We may then decompose d_Ω^2 in terms of this weight:

$$d_\Omega^2 = \underbrace{d_0^2}_{\omega=-4} - \underbrace{d_0 d_1 - d_1 d_0}_{\omega=-3} + \underbrace{d_1^2 + d_0 d_2 + d_2 d_0}_{\omega=-2} - \underbrace{(d_1 d_2 + d_2 d_1)}_{\omega=-1} + \underbrace{d_2^2}_{\omega=0}. \tag{2.12}$$

and it suffices to check that each summand vanishes. Each summand is itself a derivation (because $d_\Omega^2 = \frac{1}{2}[d_\Omega, d_\Omega]$ is a derivation thus so are its weight components), so it even suffices to check that they vanish on generators.

- $d_0^2 = 0$: the image of d_0 is included in $\mathbb{k}⫯$, and every derivation vanishes on $⫯$;
- $d_1 d_0 = d_0 d_1 = 0$ respectively because $d_1(⫯) = 0$ and $\theta_C d_C = 0$;
- $d_1^2 + d_0 d_2 = d_2 d_0 = 0$: we have that $d_2 d_0 = 0$, again because $d_2(⫯) = 0$, and $d_1^2 + d_0 d_2 = 0$ comes from $d_C^2 = \star_\varphi(\theta_C)$ and the Koszul rule of signs;
- $d_1 d_2 + d_2 d_1 = 0$ comes from the fact that d_C is a derivation, that relation being post-composed by φ to obtain $d_1 d_2 + d_2 d_1 = 0$;

- $d_2^2 = 0$ comes from the Maurer–Cartan equation $\varphi \star \varphi = 0$ and the commutativity of the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"] & \mathsf{C}(C) \arrow[r, "\varphi \circ \mathrm{id}"] \arrow[d, "\mathrm{id} \circ' \Delta_C"] & u\mathsf{P}(C) \arrow[d, "\mathrm{id} \circ' \Delta_C"] & & \\
\mathsf{C}(C) \arrow[r, "\Delta_{(1)} \circ \mathrm{id}"] & \mathsf{C}(C; \mathsf{C}(C)) \arrow[r, "\varphi(\mathrm{id};\mathrm{id})"] & u\mathsf{P}(C; \mathsf{C}(C)) \arrow[r, "\mathrm{id}(\mathrm{id};\varphi(\mathrm{id}))"] & (u\mathsf{P} \circ_{(1)} u\mathsf{P})(C) \arrow[r, "\gamma_{(1)}"] & u\mathsf{P}(C).
\end{tikzcd}$$

□

2.3 Bar construction

We now define the adjoint of the cobar construction, namely the bar construction (cf. [HM12, Section 3.3.2] for the (pr)operadic case). Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism such that $\operatorname{im} \varphi \subset \mathsf{P}(2)$ and let A be a semi-augmented dg-$u\mathsf{P}$-algebra (see Definition 2.9). Let $\varepsilon_A : A \to \Bbbk$ be the semi-augmentation and $\bar{A} = \ker \varepsilon_A$. Recall that we have induced linear maps $\bar{\gamma}_A : u\mathsf{P}(\bar{A}) \to \bar{A}$ and $\bar{d}_{\bar{A}} : \bar{A} \to \bar{A}$.

We now define the φ-curved C-algebra (cf. [HM12, Section 5.2.3])

$$B_\varphi A \coloneqq (\mathsf{C}(\Sigma \bar{A}), d_B, \theta_B). \tag{2.13}$$

The underlying C-coalgebra of $B_\varphi(A)$ is merely the cofree coalgebra $\mathsf{C}(\Sigma \bar{A})$. The predifferential d_B is the sum $d_1 + d_2$, where d_1 and d_2 are the unique coderivations whose corestrictions are respectively:

$$d_2|^{\Sigma \bar{A}} : \mathsf{C}(\Sigma \bar{A}) \xrightarrow{\varphi \circ \mathrm{id}} \mathsf{P}(\Sigma \bar{A}) \xrightarrow{\bar{\gamma}_A} \Sigma \bar{A}; \tag{2.14}$$

$$d_1|^{\Sigma \bar{A}} : \mathsf{C}(\Sigma \bar{A}) \twoheadrightarrow \Sigma \bar{A} \xrightarrow{\bar{d}_{\bar{A}}} \Sigma \bar{A}. \tag{2.15}$$

Let $\varepsilon_{\mathsf{C}} : \mathsf{C} \to I$ be the counit of the cooperad C. The curvature $\theta_B : \mathsf{C}(\Sigma A) \to \Bbbk$ is the map of degree -2 given by:

$$\mathsf{C}(\Sigma \bar{A}) \xrightarrow{(\varepsilon_{\mathsf{C}} \oplus \varphi)(\mathrm{id}_A)} \Sigma \bar{A} \oplus u\mathsf{P}(\Sigma \bar{A}) \xrightarrow{d_A + \gamma_A} \Sigma A \xrightarrow{\varepsilon_A} \Bbbk. \tag{2.16}$$

Concretely, let us say that $c(\Sigma a_1, \dots, \Sigma a_n) \in B_\varphi A$ has weight n. Then:

- $\theta_B(\mathrm{id}_{\mathsf{C}}(\Sigma a)) = \varepsilon(d_A a)$ on elements of weight 1;
- $\theta_B(c(\Sigma a, \Sigma a')) = \varepsilon_A(\gamma_A(\varphi(c), a_1, a_2))$ on elements of weight 2;
- θ_B vanishes on all elements of weight ≥ 3 thanks to $\operatorname{im} \varphi \subset \mathsf{P}(2)$.

Compare the following with [HM12, Lemma 3.3.3].

Proposition 2.12. *The data $B_\varphi A = (\mathsf{C}(\Sigma A), d_B, \theta_B)$ defines a φ-curved coalgebra from the semi-augmented $u\mathsf{P}$-algebra A.*

Proof. Let us first check that $\theta_B d_B = 0$. The curvature θ_B is only nonzero on elements of weight ≤ 2. The summand d_1 of the differential preserves the weight. The summand d_2 decreases the weight by exactly 1, thanks to our hypothesis that $\operatorname{im} \varphi \subset \mathsf{P}(2)$. We thus only need to check the equality on elements $x \in B_\varphi A$ of weight ≤ 3.

- If $x = c(\Sigma a)$ has weight 1, then either $c = \mathrm{id}_{\mathsf{C}}$, in which case $\theta_B(d_B(\mathrm{id}_{\mathsf{C}}(\Sigma a))) = \varepsilon(\overline{d_A^2}a) = 0$, or c is not a multiple of id_{C}, in which case $\theta_B(d_B x) = 0$ by definition.

- If $x = c(\Sigma a, \Sigma a')$ has weight 2, then

$$d_B x = (-1)^{|c|} c(\Sigma \bar{d}_A a, \Sigma a') + (-1)^{|c|+|a|+1} c(\Sigma a, \Sigma \bar{d}_A a') \\ + \mathrm{id}_{\mathsf{C}}(\Sigma \bar{\gamma}_A(\varphi(c), a, a')). \quad (2.17)$$

  and thus $\theta_B d_B x = 0$ simply follows from the compatibility of the P-structure with the differential.

- If $x = c(\Sigma a, \Sigma a', \Sigma a'')$, then $\theta_B d_1 x = 0$ for weight reasons. Moreover, using the associativity of the P-structure on A, we can compute easily that $\theta_B d_B x = \varepsilon_A(\gamma_A(\varphi \star \varphi, a, a', a''))$, which vanishes from the Maurer–Cartan equation $\varphi \star \varphi = 0$.

Let us now check that $d_B^2 = \star_\varphi(\theta_B)$. It is enough to check this when projected on cogenerators, as $d_B^2 = \frac{1}{2}[d_B, d_B]$ and $\star_\varphi(\theta_B)$ are coderivations (Lemma 2.3). Thanks to the explicit description of Equation (2.4), we find that the projection of $\star_\varphi(\theta_B)(x)$ on cogenerators is nonzero only on elements of weight 2 and 3. Moreover, $d_1^2(x) = 0$, the projection of $(d_1 d_2 + d_2 d_1)(x)$ can only be nonzero on elements of weight 2, and the projection of $d_2^2(x)$ is only nonzero on elements of weight 3. If $x = c(\Sigma a, \Sigma a')$, then:

$$\star_\varphi(\theta_B)(x)|^{\Sigma \bar{A}} = \gamma_{u\mathsf{P}}(\varphi(c), \varepsilon_A(da)^{\dagger}, a') - \gamma_{u\mathsf{P}}(\varphi(c), a, \varepsilon_A(da')). \quad (2.18)$$

This is easily seen to be equal to the projection of $(d_1 d_2 + d_2 d_1)(x)$ from the computations above. A similar computation shows that if x has weight 3, then $\star_\varphi(\theta_B)(x)$ has the same projection on cogenerators as $d_2^2(x)$. $\square$

2.4 Adjunction

Definition 2.13. Let $\varphi : \mathsf{C} \to \mathsf{P}$ be a twisting morphism with $\mathrm{im}\,\varphi \subset \mathsf{P}(2)$, $C = (C, d_C, \theta_C)$ be a φ-curved C-coalgebra, and A be a semi-augmented $u\mathsf{P}$-algebra. The set of φ-twisting morphisms from C to A is:

$$\mathrm{Tw}_\varphi(C, A) \coloneqq \{\beta : C \to \bar{A} \mid \partial(\beta) + \hat{\star}_\varphi(\beta) = \Theta^A\}, \quad (2.19)$$

where $\partial(\beta) = d_A \beta - \beta d_C$, and $\hat{\star}_\varphi(\beta)$ and Θ^A are given by:

$$\hat{\star}_\varphi(\beta) : C \xrightarrow{\Delta_C} \mathsf{C} \circ C \xrightarrow{\varphi \circ \beta} u\mathsf{P} \circ A \xrightarrow{\gamma_A} A, \quad (2.20)$$

$$\Theta^A : C \xrightarrow{\theta_C} \Bbbk \dagger \to A, \quad (2.21)$$

and $\Bbbk \dagger \to A$ is defined using the action of $\dagger \in u\mathsf{P}(0)$ on A.

We then have the following “Rosetta stone” (cf. [LV12]):

Proposition 2.14. *Let C be a φ-curved C-coalgebra and A be a semi-augmented $u\mathsf{P}$-algebra. Then there are natural bijections (in particular, Ω_φ and B_φ are adjoint):*

$$\begin{aligned}\mathrm{Hom}_{\text{sem.aug.}u\mathsf{P}\text{-alg}}(\Omega_\varphi C, A) &\cong \mathrm{Tw}_\varphi(C, A)\\ &\cong \mathrm{Hom}_{\varphi\text{-curved }\mathsf{C}\text{-coalg}}(C, B_\varphi A).\end{aligned} \tag{2.22}$$

Proof. Let us first prove the existence of the first bijection. Given $\beta \in \mathrm{Tw}_\varphi(C, A)$, we let $f_\beta : u\mathsf{P}(\Sigma^{-1}C) \to A$ be the $u\mathsf{P}$-algebra morphism given on generators by β. We must check that $f_\beta d_\Omega = d_A f_\beta$. As we are working with derivations and morphisms, it is enough to check this on the generators $\Sigma^{-1}C$. Recall that $d_\Omega = -d_0 + d_1 - d_2$, where the summands are respectively defined using the curvature, the predifferential, and the coalgebra structure of C. The restrictions of the maps involved are:

- $f_\beta d_0|_{\Sigma^{-1}C} = -\Theta^A$ (since d_0 is defined using $\Sigma\theta_C$, a sign occurs);
- $f_\beta d_1|_{\Sigma^{-1}C} = \beta d_C$;
- $f_\beta d_2|_{\Sigma^{-1}C} = (\Sigma^{-1}C \xrightarrow{\Delta_C} \mathsf{C} \circ C \xrightarrow{\varphi\circ\beta} u\mathsf{P}(A) \xrightarrow{\gamma_A} A) = \hat{\star}_\varphi\,\beta$;
- $d_A f_\beta|_{\Sigma^{-1}C} = d_A\beta$.

We can thus compute that:

$$\begin{aligned}(f_\beta d_\Omega - d_A f_\beta)|_{\Sigma^{-1}C} &= (-fd_0 + fd_1 - fd_2 - d_A f)|_{\Sigma^{-1}C}\\ &= \Theta^A + \beta d_C - \hat{\star}_\varphi\beta - d_A\beta\\ &= \Theta^A - \partial(\beta) - \hat{\star}_\varphi\beta.\end{aligned} \tag{2.23}$$

The Maurer–Cartan equation $\partial(\beta) + \hat{\star}_\varphi(\beta) = \Theta^A$ then implies $f_\beta d_\Omega = d_A f_\beta$.

Conversely, given $f : \Omega_\varphi C \to A$, then we can define $\beta \coloneqq f|_{\Sigma^{-1}C}$. The same proof as above but in the reverse direction shows that the compatibility of f with the differentials imply the Maurer–Cartan equation. Moreover, the two constructions are inverse to each other'.

The definition of the second bijection is similar (see also the proof of [HM12, Theorem 3.4.1] for the case of (co)operads). Given a twisting morphism $\beta \in \mathrm{Tw}_\varphi(C, A)$, the morphism $g_\beta : C \to B_\varphi A$ is defined as the unique morphism of C-coalgebras $C \to \mathsf{C}(\Sigma\bar{A})$ with corestriction β. The fact that g_β commutes with the predifferentials and the curvatures of C and $B_\varphi A$ follows from a similar argument from the Maurer–Cartan equation.

All that remains is checking that the two bijections are natural in terms of C and A. This is a simple exercise in commutative diagrams. □

3 Koszul duality of unitary algebras

3.1 Algebras with quadratic-linear-constant relations

We now define the type of algebras for which we will developed a Koszul duality theory, namely algebras with quadratic-linear-constant (QLC) relations. For this we adapt the notion of a monogenic algebra of [Mil12, Section 4.1] (see Section 1.1.3). We still assume that we are given a unital version $u\mathsf{P} = \mathrm{Free}(\upharpoonleft \oplus E)/(R + R')$ of a binary quadratic operad $\mathsf{P} = \mathrm{Free}(E)/(R)$ as in Section 1.2.

Definition 3.1. An $u\mathsf{P}$-algebra with QLC relations is a $u\mathsf{P}$-algebra A with $d_A = 0$, equipped with a presentation by generators and relations:

$$A = u\mathsf{P}(V)/I, \tag{3.1}$$

satisfying the two conditions:

1. the ideal I is generated by $S := I \cap (\upharpoonleft \oplus V \oplus E(V))$ (where $E(V) = E \otimes_{\Sigma_2} V^{\otimes 2}$),
2. the relations in S all contain quadratic terms, $S \cap (\upharpoonleft \oplus V) = 0$.

Remark 3.2. Such an algebra is automatically semi-augmented (Definition 2.9) as we are working over a field $\Bbbk$.

Definition 3.3. Given a $u\mathsf{P}$-algebra A with QLC relations as above, let qS be the projection of S onto $E(V)$. Then the quadratic reduction qA of A is the monogenic P-algebra obtained by:

$$qA := \mathsf{P}(V)/(qS). \tag{3.2}$$

Definition 3.1 implies that S is the graph of some map

$$\alpha = (\alpha_0 \oplus \alpha_1) : qS \to \Bbbk\upharpoonleft \oplus V, \tag{3.3}$$

i.e. $S = \{x + \alpha(x) \mid x \in qS\}$.

Until the end of this section, A will be a $u\mathsf{P}$-algebra with QLC relations, using the same notations.

3.2 Koszul dual coalgebra

Let $\mathsf{P}^{¡}$ be the Koszul dual cooperad of P, with an operad-twisting morphism $\kappa : \mathsf{P}^{¡} \to \mathsf{P}$ (see Section 1.1.2). The theory of [Mil12] (see Section 1.1.3) defines a Koszul dual $\mathscr{S}^c\mathsf{P}^{¡}$-coalgebra $qA^{¡}$ from the quadratic reduction qA:

$$qA^{¡} := \Sigma\mathsf{P}^{¡}(V, \Sigma qS). \tag{3.4}$$

Using the map α from Equation (3.3), we may define $d_{A^{¡}} : qA^{¡} \to \Sigma\mathsf{P}^{¡}(V)$ to be the unique coderivation whose corestriction is given by:

$$d_{A^{¡}}|^{\Sigma V} : qA^{¡} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-1}\alpha_1} \Sigma V. \tag{3.5}$$

Moreover, define $\theta_{A^{¡}}$ by:

$$\theta_{A^{¡}} : qA^{¡} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-2}\alpha_0} \Bbbk\upharpoonleft. \tag{3.6}$$

We now define the Koszul dual coalgebra of A by adapting [HM12, Section 4.2]. The proof of the following proposition is heavily inspired from that of [HM12, Lemma 4.1.1].

Proposition 3.4. *The following data defines a κ-curved $\mathsf{P}^{¡}$-coalgebra, called the Koszul dual curved coalgebra of A:*

$$A^{¡} \coloneqq (qA^{¡}, d_{A^{¡}}, \theta_{A^{¡}}). \tag{3.7}$$

Proof. We must show that $d_{A^{¡}}(qA^{¡}) \subset qA^{¡}$, that $\star_{\bar{\kappa}}(\theta_{A^{¡}}) = d_{A^{¡}}^2$, and that $\theta_{A^{¡}} d_{A^{¡}} = 0$. Just like in the proof of Proposition 2.12, we can decompose $qA^{¡} = \Sigma \mathsf{P}^{¡}(V, \Sigma qS)$ by weight, i.e. the number n of generators $v_i \in V$ in an expression of the form $\Sigma x(v_1, \dots, v_n)$. The differential $d_{A^{¡}}$ decreases the weight by exactly 1, while $\theta_{A^{¡}}$ is only nonzero on elements of weight 2. Using considerations similar to the proof of Proposition 2.12, we only need to check the three equalities above on elements of $qA^{¡}$ of weight 3.

Let thus $Y \in (qA^{¡})^{(3)}$ be an element of weight 3. The coproduct $\Delta(Y) \in \mathsf{P}^{¡}(qA^{¡})$ must belong, by definition of $qA^{¡}$ as a cofree coalgebra with corelations, to the subspace

$$\Delta(Y) \in E \otimes (\Sigma V \otimes \Sigma^2 qS) \cap E \otimes (\Sigma^2 qS \otimes \Sigma V). \tag{3.8}$$

In other words, we must have two decompositions

$$\Delta(Y) = \sum_i \rho_i(\Sigma v_i, \Sigma^2 X_i) = \sum_j \rho'_j(\Sigma^2 X'_j, \Sigma v'_j), \tag{3.9}$$

where $\rho_i, \rho'_j \in E$, $v_i, v'_j \in V$, and $X_i, X'_j \in qS$. Then the rule of signs shows that:

$$\begin{aligned} d_{A^{¡}}(Y) &= -\sum_i \rho_i(\Sigma v_i, (\Sigma^{-1}\alpha_1)(\Sigma^2 X_i)) + \sum_j \rho'_j((\Sigma^{-1}\alpha_1)(\Sigma^2 X'_j), \Sigma v'_j) \\ &= -\sum_i \rho_i(\Sigma v_i, (\Sigma\alpha_1(X_i))) + \sum_j \rho'_j(\Sigma\alpha_1(X'_j), \Sigma v'_j) \in E(\Sigma V)). \end{aligned} \tag{3.10}$$

And similarly:

$$\star_{\bar{\kappa}}(\theta_{A^{¡}})(Y) = -\sum_i \rho_i(\Sigma v_i, \Sigma\alpha_0(X_i)) + \sum_j \rho'_j(\Sigma\alpha_0(X'_j), \Sigma v'_j) \in V. \tag{3.11}$$

Recall that S is the graph of α from Equation (3.3): for $X \in qS$, we have that $f(X) \coloneqq (\mathrm{id} + \alpha_0 + \alpha_1)(X) \in S$. Therefore, if we apply the map

$$g \coloneqq \gamma \circ (\mathrm{id}_E \otimes (\mathrm{id}_{\Sigma V} \otimes \Sigma^{-1} f)) \tag{3.12}$$

to an element in $E \otimes (\Sigma V \otimes \Sigma^2 qS)$, then the result must belong to $\Sigma^2(S)$. The same holds if we apply

$$g' \coloneqq \gamma \circ (\mathrm{id}_E \otimes (\Sigma^{-1} f \otimes \mathrm{id}_{\Sigma V})) \tag{3.13}$$

to an element of $E \otimes (\Sigma^2 qS \otimes \Sigma V)$.

We know that $\Delta(Y)$ belongs to these two spaces, thus $g(\Delta(Y))$ and $g'(\Delta(Y))$ must belong to $\Sigma^2(S)$. Hence so does $(g - g')(\Delta(Y))$. Signs cancel out in $(g - g')(\Delta(Y))$, and therefore:

$$\begin{aligned} (g - g')(\Delta(Y)) &= d_{A^{¡}}(Y) + \star_{\bar{\kappa}}(\theta_{A^{¡}})(Y) \\ &\in \Sigma^2(S) \cap (\text{⫯} \oplus V \oplus E(V)) = \Sigma^2 S. \end{aligned} \tag{3.14}$$

We know that S is the graph of α, hence the expression of Equation (3.14) is of the form $x+\alpha_1(x)+\alpha_0(x)$ for some $x \in qS$. The element $d_{A^{\mathrm{i}}}(Y)$ is of weight 2, while $\star_{\bar{\kappa}}(\theta_{A^{\mathrm{i}}})(Y)$ is of weight 1. Thus by identifying each weight component:

- $d_{A^{\mathrm{i}}}(Y)$ belongs to $\Sigma^2 qS = (qA^{\mathrm{i}})^{(2)}$;
- the element $d^2_{A^{\mathrm{i}}}(Y)$ is equal to $(\Sigma^{-1}\alpha_1)(d_{A^{\mathrm{i}}}(Y))$, which we know from Equation (3.14) is equal to $\star_{\bar{\kappa}}(\theta_{A^{\mathrm{i}}})(Y)$;
- similarly, $\theta_{A^{\mathrm{i}}} d_{A^{\mathrm{i}}}(Y)$ is equal to $\Sigma^{-2}\alpha_0(d_{A^{\mathrm{i}}}(Y))$, which vanishes because there is no element of weight 0 in Equation (3.14). □

3.3 Main theorem

Let us now define $\varkappa : qA^{\mathrm{i}} \to A$ of degree -1 by:

$$\varkappa : qA^{\mathrm{i}} \twoheadrightarrow \Sigma V \xrightarrow{\Sigma^{-1}} V \hookrightarrow A. \tag{3.15}$$

Proposition 3.5. *The morphism $\varkappa$ satisfies the curved Maurer–Cartan equation, i.e. it is an element of* $\mathrm{Tw}_{\varphi}(qA^{\mathrm{i}}, A)$ *from Proposition 2.14:*

$$\partial(\varkappa) + \hat{\star}_{\bar{\kappa}}(\varkappa) = \Theta^A. \tag{3.16}$$

Proof. We can rewrite the equation as:

$$-\varkappa d_{A^{\mathrm{i}}} + \gamma_A(\bar{\kappa} \circ \varkappa)\Delta_{qA^{\mathrm{i}}} = \Theta^A. \tag{3.17}$$

Moreover, by checking the definitions, we see that:

- $\varkappa d_{A^{\mathrm{i}}}$ is obtained as $qA^{\mathrm{i}} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-1}\alpha_1} \Sigma V \xrightarrow{\Sigma^{-1}} V \hookrightarrow A$;
- Θ^A is obtained as $qA^{\mathrm{i}} \twoheadrightarrow \Sigma^2 qS \xrightarrow{\Sigma^{-2}\alpha_0} \Bbbk\mathbf{1} \hookrightarrow A$;
- $\hat{\star}_{\bar{\kappa}}(\varkappa)$ vanishes everywhere except on $(qA^{\mathrm{i}})^{(2)} = \Sigma^2 qS$, where it is equal to $\gamma_A \circ \Sigma^{-2}\Delta_{qA^{\mathrm{i}}}$.

Checking signs carefully, we see that the image of $\hat{\star}_{\bar{\kappa}}(\varkappa) - \varkappa d_{A^{\mathrm{i}}} - \Theta^A$ is included in the image of the graph of α under γ_A. But this graph is S, and $\gamma_A(S) = 0$ because these are part of the relations of A. □

Definition 3.6. The $u\mathsf{P}$-algebra A is said to be Koszul if the P-algebra qA is Koszul in the sense of [Mil12].

Using the Rosetta stone (Proposition 2.14), $\varkappa$ defines a morphism $f_\varkappa : \Omega_\kappa A^{\mathrm{i}} \to A$. Recall that qA is Koszul if and only if the induced morphism $\Omega_\kappa(qA^{\mathrm{i}}) \to qA$ is a quasi-isomorphisms [Mil12, Theorem 4.9]. Our definition is justified by the following theorem:

Theorem 3.7. *If A is Koszul, then $f_\varkappa : \Omega_\kappa A^{¡} \to A$ is a cofibrant resolution of A in the semi-model category of $u\mathsf{P}$-algebras defined in [Fre09, Theorem 12.3.A].*

Proof. Let us filter A and $\Omega_\kappa A^{¡}$ by the weight in terms of V. It is clear that $f_\varkappa$ is compatible with this filtration. The summands d_0 and d_1 of d_Ω strictly lower this filtration, while d_2 preserves it. Thus, on the first pages of the associated spectral sequences, we obtain the morphism:

$$\Omega_\kappa(qA^{¡}) \oplus \Bbbk\text{⫯} \to qA \oplus \Bbbk\text{⫯}. \tag{3.18}$$

Our hypothesis on qA and [Mil12, Theorem 4.9] implies that this is a quasi-isomorphism, i.e. we have an isomorphism on the second pages of the spectral sequences. The filtration is exhaustive and bounded below, therefore $f_\varkappa$ itself is a quasi-isomorphism (see e.g. [McC01, Theorem 3.5]).

It remains to check that $\Omega_\kappa A^{¡}$ is cofibrant in the semi-model category from [Fre09, Theorem 12.3.A], which applies as we are working over a field of characteristic zero and so $u\mathsf{P}$ is always Σ_*-cofibrant. The cobar construction is quasi-free, i.e. free as an algebra if we forget the differential. It is moreover equipped with a filtration where $\Sigma x(v_1, \dots, v_n) \in A^{¡}$ (for $x \in \mathsf{P}^{¡}$ and $v_i \in V$) is in filtration level n. Let us check that this filtration satisfies the hypotheses of [Fre09, Proposition 12.3.8]. The summands d_0 and d_1 of the differential decrease n. Since P is binary, $\operatorname{im}\kappa \subset \mathsf{P}^{¡}(2)$, so d_2 decomposes an element of weight n as a product of elements of weights $n_1 + n_2 = n$ and so $n_1, n_2 < n$ (as there is no arity zero generator in $\mathsf{P}^{¡}$). It follows that $\Omega_\kappa A^{¡}$ is cofibrant. ☐

4 Application: symplectic Poisson n-algebras

4.1 Definition

In this section, with deal with Poisson n-algebras for some integer n. We will abbreviate as Lie_n the operad of Lie algebras shifted by $n-1$, i.e. $\mathsf{Lie}_n \coloneqq \mathscr{S}^{1-n}\mathsf{Lie}$. Recall from Example 1.9 the operad $u\mathsf{Pois}_n \cong \mathsf{Com} \circ \mathsf{Lie}_n$, generated by two binary operations μ (product) and λ (bracket) and a unary operation ⫯.

Definition 4.1. The Dth symplectic Poisson n-algebra is defined by:

$$A_{n;D} \coloneqq \big(\Bbbk[x_1, \dots, x_D, \xi_1, \dots, \xi_D], \{,\}\big). \tag{4.1}$$

where the generators x_i have degree 0 and the ξ_i have degree $1-n$. The algebra $A_{n;D}$ is free as a unital commutative algebra, and the Lie bracket is defined on generators by:

$$\{x_i, x_j\} = 0 \qquad \{\xi_i, \xi_j\} = 0 \qquad \{x_i, \xi_j\} = \delta_{ij}. \tag{4.2}$$

The algebra $A_{n;D} = u\mathsf{Pois}_n(V_{n;D})/(S_{n;D})$ is equipped with a QLC presentation. The space of generators is $V_{n;D} \coloneqq \mathbb{R}\langle x_i, \xi_j\rangle$, a graded vector space of dimension $2D$. We check that the ideal of relations $I_{n;D}$ is generated by the set $S_{n;D}$ given by the three sets of relations fixing the Lie brackets of the generators, namely

$$S_{n;D} = \mathbb{R}\langle \{x_i, x_j\}, \{\xi_i, \xi_j\}, \{x_i, \xi_j\} - \delta_{ij}\text{⫯}\rangle. \tag{4.3}$$

Remark 4.2. We may view $A_{n;D}$ as the Poisson n-algebra of polynomial functions on the standard shifted symplectic space $T^*\mathbb{R}^D[1-n]$. The element x_i is a polynomial function on the coordinate space $\mathbb{R}^D$, and the element ξ_j can be viewed as the vector field $\partial/\partial x_j$, which is a function on $T^*\mathbb{R}^D[1-n]$.

We will drop the indices n and D from the notation in what follows.

4.2 Koszul property and explicit resolution

In this section, we prove:

Proposition 4.3. *The $u\mathsf{Pois}_n$-algebra A is Koszul.*

Lemma 4.4. *The quadratic reduction qA of A is a free symmetric algebra with trivial Lie bracket.*

Proof. Let $V = \mathbb{R}\langle x_1, \dots, x_D, \xi_1, \dots, \xi_D\rangle$ be the generators of A. We check that $qS = \lambda(V) = \lambda \otimes_{\Sigma_2} V^{\otimes 2}$, i.e. in the quadratic reduction, all Lie brackets vanish. Therefore, $qA = \mathsf{Pois}_n(V)/(qS) = \mathsf{Pois}_n(V)/(\lambda(V)) = \mathsf{Com}(V)$ is a free symmetric algebra, and the Lie bracket vanishes. ☐

Let Com^c be the cooperad governing cocommutative coalgebras. Up to a suspension, it is the Koszul dual of the Lie operad. Since we are working over a field of characteristic zero, we can identify $\mathsf{Com}^c(X)$ with:

$$\bar{S}^c(X) \coloneqq \bigoplus_{i \geq 1} (X^{\otimes i})_{\Sigma_i} \tag{4.4}$$

where the coproduct is given by shuffles. For a shorter notation we will also write $L(X)$ for the free Lie algebra on X, $S(X)$ for the free unital symmetric algebra, and $\bar{S}(X)$ for the free symmetric algebra without unit.

Lemma 4.5. *The Koszul dual coalgebra of qA is given by:*

$$qA^{¡} = \Sigma^{1-n} \bar{S}^c(\Sigma^n V). \tag{4.5}$$

Proof. Recall that if $\mathsf{P} = \mathsf{Q}_1 \circ \mathsf{Q}_2$ is obtained by a distributive law between two finitely-generated binary quadratic operads $\mathsf{Q}_i = \mathrm{Free}(E_i)/(R_i)$ $(i = 1, 2)$, then $\mathsf{P}^! = \mathsf{Q}_2^! \circ \mathsf{Q}_1^!$ with the transpose distributive law [LV12, Proposition 8.6.7]. If $A = \mathsf{Q}_1(V) = \mathsf{P}(V)/(E_2(V))$, it follows that $A^! = \mathsf{P}^!(V^*)/(E_2(V)^\perp) = (\mathsf{Q}_2^! \circ \mathsf{Q}_1^!)(V^*)/(E_1^\vee(V^\vee)) \cong \mathsf{Q}_2^!(V^\vee)$ is simply given by the free $\mathsf{Q}_2^!$-algebra with a trivial action of the generators of Q_1. By dualizing this statement, we thus find that $A^{¡} = \Sigma \mathsf{Q}_2^{¡}(V)$.

Applied to our case, we obtain that the Koszul dual coalgebra of $qA = \mathsf{Com}(V)$ is $qA^{¡} = \Sigma(\mathsf{Lie}_n)^{¡}(V)$. Thanks to the Koszul duality between Com and Lie, this is identified with $\Sigma^{1-n}\mathsf{Com}^c(\Sigma^n V) = \Sigma^{1-n}\bar{S}^c(\Sigma^n V)$. ☐

Proof of Proposition 4.3. Let $\kappa : \mathsf{Pois}_n^{\text{¡}} \to \mathsf{Pois}_n$ be the twisting morphism of Koszul duality. Then the cobar construction of $qA^{\text{¡}}$ is given by:

$$\Omega_\kappa qA^{\text{¡}} = \big(\underbrace{\bar{S}(\Sigma^{1-n}L(\Sigma^{n-1}}_{=\mathsf{Pois}_n}\Sigma^{-1}\underbrace{\Sigma^{1-n}\bar{S}^c(\Sigma^n V)}_{=qA^{\text{¡}}})), d_2\big). \tag{4.6}$$

Here d_2 is the derivation of Pois_n-algebras whose restriction on $\Sigma^{-n}\bar{S}^c(\Sigma^n V)$ is given by (forgetting about suspension for ease of notation):

$$d_2|_{\Sigma^{-1}qA^{\text{¡}}}(u) = \sum_{(u)} \frac{1}{2}[u_{(1)}, u_{(2)}], \tag{4.7}$$

where the bracket is the bracket of the free Lie algebra appearing in $\Omega_\kappa qA^{\text{¡}}$. This 1/2-factor is due to the identification of Com^c, which is initially defined using invariants under the symmetric group action, with $\bar{S}^c(X)$, which is defined using coinvariants.

The twisting morphism $\varkappa \in \mathrm{Tw}_\kappa(qA^{\text{¡}}, qA)$ is given by $\varkappa(\Sigma x_i) = x_i$ and $\varkappa(\Sigma\xi_i) = \xi_i$ on terms of weight 1, and it vanishes on terms of weight ≥ 2. This twisting induces a morphism $\Omega_\kappa(qA^{\text{¡}}) \to qA$. We easily see that this morphism is the image under S of the bar-cobar resolution of the abelian Lie_n algebra V:

$$\big(\Sigma^{1-n}L(\Sigma^{-1}\bar{S}^c(\Sigma^n V)), d_2\big) \xrightarrow{\sim} V, \tag{4.8}$$

which is indeed a quasi-isomorphism thanks to the Koszul property of Lie_n. The functor S preserves quasi-isomorphisms as we are working over a field of characteristic zero. Therefore we obtain that $\Omega_\kappa(qA^{\text{¡}}) \to qA$ is a quasi-isomorphism, thus qA is Koszul, and therefore by definition A is Koszul. ☐

We then obtain a small resolution of the $u\mathsf{Pois}_n$-algebra A using the cobar construction of its Koszul dual coalgebra. Let us now describe it. The map $\alpha : qS \to \mathbb{k}⫯ \oplus V$ from Equation (3.3) is given by $\alpha_1 = 0$, $\alpha_0(\{x_i, \xi_i\}) = -⫯$ for all i, and $\alpha_0 = 0$ on all other basis elements. Let us write as a shorthand:

$$v_1 \vee \cdots \vee v_k \coloneqq \frac{1}{k!}\sum_{\sigma \in \Sigma_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \in \Sigma^{1-n}\bar{S}^c(\Sigma^n V). \tag{4.9}$$

Then the Koszul dual $A^{\text{¡}} = (qA^{\text{¡}}, d_{A^{\text{¡}}}, \theta_{A^{\text{¡}}})$ is such that $d_{A^{\text{¡}}} = 0$, and

$$\begin{aligned} \theta : \Sigma^{1-n}\bar{S}^c(\Sigma^n V) &\to \mathbb{k}⫯ \\ x_i \vee \xi_i &\mapsto -⫯, \qquad \text{for all } i \\ \text{other basis elements} &\mapsto 0. \end{aligned}$$

We then obtain

$$\Omega_{\bar{\kappa}} A^{\text{¡}} = (S(\Sigma^{1-n}L(\Sigma^{-1}\bar{S}^c(\Sigma^n V))), d_0 + d_2) \xrightarrow{\sim} A, \tag{4.10}$$

where (by abuse of notation) we still denote by d_2 the Chevalley–Eilenberg differential from before, satisfying $d_2(⫯) = 0$. The derivation d_0 is the one whose restriction to $\Sigma^{-1}A^{\text{¡}}$ is given by $\Sigma\theta$:

$$d_0|_{\Sigma^{-1}qA^{\text{¡}}} : \Sigma^{-n}\bar{S}^c(\Sigma^n V) \xrightarrow{\Sigma\theta} \mathbb{k}⫯ \hookrightarrow u\mathsf{Pois}_n(\Sigma^{-1}qA^{\text{¡}}). \tag{4.11}$$

Remark 4.6. Let us compare $\Omega_\kappa A^{¡}$ to the resolution that would be obtained if one applied curved Koszul duality at the level of operads (see Section 1.1.4) with the bar/cobar resolution from the theory of [HM12]. Without the suspensions, our resolution is just $SL\bar{S}^c(V)$, i.e. it is the free symmetric algebra on the free Lie algebra on the cofree symmetric coalgebra on V.

The resolution that would be obtained from [HM12] would be much bigger (although it can be made explicit). Indeed, the quadratic reduction of $u\mathsf{Pois}_n$ is not just Pois_n, it is in fact the direct sum $\mathsf{Pois}_n \oplus \mathbb{1}$. It follows from [HM12, Proposition 6.1.4]) that $(qu\mathsf{Pois}_n)^{¡}(r)$ is spanned by elements of the type $\bar{\alpha}_S$, where $\alpha \in \mathsf{Pois}_n(k)$, $S \subset \{1, \dots, k\}$ and $r = k - \#S$. Roughly speaking, S represents inputs of α that have been "plugged" by the counit $\mathbb{1}$. The bar construction of A is then the cofree $(qu\mathsf{Pois}_n)^{¡}$-coalgebra on $A = S(V)$ (plus some differential and we forget about suspensions). Then the bar-cobar resolution of A is the free $u\mathsf{Pois}_n$-algebra on this bar construction. It contains as a subspace $SL\bar{S}^c L^c(A) = SL\bar{S}^c L^c S(V)$, which is already quite bigger than $\Omega_\kappa A^{¡}$, and then we also need to add all operations where inputs have been plugged in by the unit.

The difference can roughly speaking be explained as follows. The bar-cobar resolution of [HM12] knows nothing about the specifics of the algebra A, thus it must resolve everything in A: the Lie bracket, the symmetric product, and the relations involving the unit. This has the advantage of being a general procedure that is independent of A (and functorial). But with our specific A, we may find a smaller resolution: we know that the product of A has no relations, and the unit is not involved in nontrivial relations (a consequence of the QLC condition), hence they do not need to be resolved.

4.3 Derived enveloping algebras

4.3.1 General constructions

Given an operad P and a P-algebra A, the enveloping algebra $\mathcal{U}_{\mathsf{P}}(A)$ is a unital associative algebra such that the left modules of $\mathcal{U}_{\mathsf{P}}(A)$ are precisely the operadic left modules of A (see e.g. [Fre09, Section 4.3]). Let $\mathsf{P}[1]$ be the operadic right P-module given by $\mathsf{P}[1] = \{\mathsf{P}(n+1)\}_{n \geq 0}$. Then the enveloping algebra $\mathcal{U}_{\mathsf{P}}(A)$ can be obtained as the relative composition product:

$$\mathcal{U}_{\mathsf{P}}(A) \cong \mathsf{P}[1] \circ_{\mathsf{P}} A = \operatorname{coeq}\big(\mathsf{P}[1] \circ \mathsf{P} \circ A \rightrightarrows \mathsf{P}[1] \circ A\big). \tag{4.12}$$

Example 4.7. The enveloping algebra $\mathcal{U}_{\mathsf{Lie}}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the usual universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. We view it as a free associative algebra on symbols X_f, for $f \in \mathfrak{g}$, subject to the relations $X_{[f,g]} = X_f X_g - (-1)^{|g| \cdot |f|} X_g X_f$. The universal enveloping algebra $\mathcal{U}_{\mathsf{cLie}}(\mathfrak{g})$ of a Lie algebra equipped with a central $\mathbb{1} \in \mathfrak{g}$ is the quotient $\mathcal{U}(\mathfrak{g})/(X_{\mathbb{1}})$.

Example 4.8. The enveloping algebra $\mathcal{U}_{\mathsf{Com}}(B)$ of a commutative algebra B is $B_+ = \Bbbk 1 \oplus B$, where 1 is an extra unit. The enveloping algebra $\mathcal{U}_{u\mathsf{Com}}(B)$ of a unital commutative algebra B is B itself (strictly speaking, the quotient of $\Bbbk 1 \oplus B$ by the relation $1 - 1_B$).

Suppose now that P is any operad (potentially unital, e.g. we could take $\mathsf{P} = u\mathsf{Pois}_n$) and let A be a P-algebra. Let $\mathsf{P}_\infty \xrightarrow{\sim} \mathsf{P}$ be a cofibrant resolution of P. The given morphism $\mathsf{P}_\infty \xrightarrow{\sim} \mathsf{P}$ induces a Quillen equivalence between the semi-model categories of P- and P_∞-algebras. The right adjoint allows us to view A as a P_∞-algebra.

Proposition 4.9. *Let $R_\infty \xrightarrow{\sim} A$ be a cofibrant resolution of A as a P_∞-algebra, and let $R \coloneqq \mathsf{P} \circ_{\mathsf{P}_\infty} R_\infty$. Then there is an equivalence*

$$\mathcal{U}_{\mathsf{P}_\infty}(A) \simeq \mathcal{U}_{\mathsf{P}}(R). \tag{4.13}$$

Proof. The proposition follows from the following diagram:

$$\begin{tikzcd} \mathcal{U}_{\mathsf{P}_\infty}(A) \cong \mathsf{P}_\infty[1] \circ_{\mathsf{P}_\infty} A & \mathsf{P}_\infty[1] \circ_{\mathsf{P}_\infty} R_\infty \arrow[l, "\sim"'] \arrow[d, "\sim"] \\ \mathsf{P}[1] \circ_{\mathsf{P}} R \cong \mathcal{U}(R) & \mathsf{P}[1] \circ_{\mathsf{P}_\infty} R_\infty \arrow[l, "\cong"'] \end{tikzcd} . \tag{4.14}$$

The upper horizontal equivalence follows from [Fre09, Theorem 17.4.B(b)], and the right vertical one follows from [Fre09, Theorem 17.4.A(a)]. Finally, the bottom horizontal isomorphism follows from the cancellation rule $X \circ_{\mathsf{P}} (\mathsf{P} \circ_{\mathsf{Q}} Y) \cong X \circ_{\mathsf{Q}} Y$ [Fre09, Theorem 7.2.2]. ☐

4.3.2 Poisson case

We now consider the symplectic $u\mathsf{Pois}_n$-algebra $A = (\mathbb{R}[x_i, \xi_j], \{\})$ from Definition 4.1. We have the following proposition.

Proposition 4.10. *The derived enveloping algebra $\mathcal{U}_{(u\mathsf{Pois}_n)_\infty}(A)$ is quasi-isomorphic to $\mathcal{U}(\Omega_\kappa A^{\text{¡}})$, where $\Omega_\kappa A^{\text{¡}}$ is the cobar construction described in Section 4.2.*

Proof. This follows from Proposition 4.9. In order to apply that proposition, we need to resolve A as a $(u\mathsf{Pois}_n)_\infty$-algebra and push forward that resolution to a $u\mathsf{Pois}_n$-algebra. In Definition 2.10, instead of taking the free $u\mathsf{P}$-algebra, we can take the free $(u\mathsf{P})_\infty$-algebra to define the cobar construction and obtain a resolution $R_\infty \to A$ which is cofibrant as a $(u\mathsf{Pois}_n)_\infty$-algebra (using the same result as the end of the proof of Theorem 3.7). Since the differentials in the cobar construction only involve generating operations from $u\mathsf{Pois}_n$, we find that $R = u\mathsf{Pois}_n \circ_{(u\mathsf{Pois}_n)_\infty} R_\infty$ is isomorphic to $\Omega_\kappa A^{\text{¡}}$. ☐

Both A and $\Omega_\kappa A^{\text{¡}}$ are obtained by considering the relative composition product

$$S(\Sigma^{1-n}\mathfrak{g}) \coloneqq u\mathsf{Pois}_n \circ_{c\mathsf{Lie}_n} \Sigma^{1-n}\mathfrak{g}, \tag{4.15}$$

where $\mathfrak{g}$ is some $c\mathsf{Lie}$-algebra, and we consider the embedding $c\mathsf{Lie}_n \hookrightarrow u\mathsf{Pois}_n$. In other words, A and $\Omega_\kappa A^{\text{¡}}$ are free as symmetric algebras on a given Lie algebra, with a central element identified with the unit of the symmetric algebra. The differential and the bracket are both extended from the differential and bracket of $\mathfrak{g}$ as (bi)derivations. Recall from Examples 4.7 and 4.8 the descriptions of the enveloping algebras of Lie algebras and commutative algebras.

Proposition 4.11 (Explicit description found in [Fre06, Section 1.1.4])**.** *Let $\mathfrak{g}$ be a $c\mathsf{Lie}$-algebra and $B = S(\Sigma^{1-n}\mathfrak{g})$ the induced $u\mathsf{Pois}_n$-algebra. Then there is an isomorphism of graded modules:*

$$\mathcal{U}_{u\mathsf{Pois}_n}(B) \cong B \otimes \mathcal{U}_{c\mathsf{Lie}_n}(\Sigma^{1-n}\mathfrak{g}). \tag{4.16}$$

The algebra $\mathcal{U}_{c\mathsf{Lie}_n}(\Sigma^{1-n}\mathfrak{g})$ is generated by symbols X_f for $f \in \mathfrak{g}$, with $\deg X_f = \deg f$. We have the following relations in $\mathcal{U}_{u\mathsf{Pois}_n}(B)$ (where $f, g \in \mathfrak{g}$, whose suspensions belong to $B \subset \mathcal{U}_{u\mathsf{Pois}_n}(B)$):

$$\begin{aligned} X_{\bullet} &= 0, \\ X_{fg} &= (\Sigma^{1-n}f) \cdot X_g + (-1)^{(|f|+1-n)\cdot(|g|+1-n)}(\Sigma^{1-n}g) \cdot X_f, \\ X_f \cdot (\Sigma^{1-n}g) &= (\Sigma^{1-n}\{f,g\}) + (-1)^{|f|\cdot(|g|+1-n)}(\Sigma^{1-n}g) \cdot X_f, \\ X_{\{f,g\}} &= X_f \cdot X_g - (-1)^{|f|\cdot|g|}X_g \cdot X_f. \end{aligned} \tag{4.17}$$

In particular, elements of B and $\mathcal{U}_{c\mathsf{Lie}_n}(\Sigma^{1-n}\mathfrak{g})$ do not necessarily commute. The differential is the sum of the differential of B and the differential given by $dX_f := X_{df}$, where $df \in B = S(\Sigma^{1-n}\mathfrak{g})$ and we use the relations to get back to $B \otimes \mathcal{U}(\Sigma^{1-n}\mathfrak{g})$.

As explained in [Fre06, Section 1.1.4], if M is an $u\mathsf{Pois}_n$-module over $B = S(\Sigma^{1-n}\mathfrak{g})$, then $\mathcal{U}_{u\mathsf{Pois}_n}(B)$ acts on M in the following way: an element $b \in B$ act by multiplication, while the element X_f acts by $[\Sigma^{1-n}f, -]$. The relations above simply encode the Jacobi and Leibniz identities.

Proof. The extension of the result from [Fre06, Section 1.1.4] to the unital case is immediate. ☐

Proposition 4.12. *Let $A = A_{n;D}$ be the symplectic Poisson n-algebra. The derived enveloping algebra, $\mathcal{U}_{(u\mathsf{Pois}_n)_\infty}(A)$, is quasi-isomorphic to the underived one, $\mathcal{U}_{u\mathsf{Pois}_n}(A)$.*

Proof. We use the cobar resolution $\Omega_\kappa A^{\text{!`}}$ and the result of Proposition 4.9 to obtain that this derived enveloping algebra is quasi-isomorphic to $\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\text{!`}})$. From the description of Proposition 4.11, as a dg-module, this is isomorphic to

$$\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\text{!`}}) \cong \big(\Omega_\kappa A^{\text{!`}} \otimes \mathcal{U}_{c\mathsf{Lie}_n}(c\mathsf{Lie}_n(\Sigma^{-1}\bar{S}^c(\Sigma^n V))), d_\Omega + d'\big), \tag{4.18}$$

where $V = \mathbb{R}\langle x_i, \xi_j\rangle$ is the graded module of generators. The product and the generators $X_f \in \mathcal{U}_{c\mathsf{Lie}_n}(c\mathsf{Lie}_n(\Sigma^{-1}\bar{S}^c(\Sigma^n V)))$ of are defined in Equation (4.17). The differential d' is defined on a generator X_f by $d'(X_f) = X_{df}$ (where we use the relations to get back to $\Omega_\kappa A^{\text{!`}} \otimes \mathcal{U}_{c\mathsf{Lie}_n}(\dots)$). We have a chain map, where $d''$ is defined similarly to $d'$ (but we apply the quotient map $\Omega_\kappa A^{\text{!`}} \to A$ to the first factor):

$$\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\text{!`}}) \to \big(A \otimes \mathcal{U}_{c\mathsf{Lie}_n}(c\mathsf{Lie}_n(\Sigma^{-1}\bar{S}^c(\Sigma^n V))), d''\big) \tag{4.19}$$

Let us describe this differential d'' explicitly. Let X_f be a generator of the universal enveloping algebra, for some $f \in \bar{S}^c(\Sigma^n V)$. Then $d''X_f = X_{df} = X_{d_0 f} + X_{d_2 f}$, where d_0 and d_2 were explicitly described in Section 4.2. In both complexes ($\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{\text{!`}})$ and the one at the target of (4.19)), we can filter by the degree of the $\mathcal{U}_{c\mathsf{Lie}_n}$ factor. On the first page of the associated spectral sequence, only the d_Ω differential of the first complex remains, while the differential of the second one vanishes. Since $\Omega_\kappa A^{¡} \to A$ is a quasi-isomorphism, we find that the map (4.19) is a quasi-isomorphism.

Since $d_0 f$ is a multiple of the unit and $X_{\mathbb{1}} = \{\mathbb{1}, -\} = 0$, we obtain that $X_{d_0 f} = 0$. On the other hand,

$$X_{d_2 f} = \frac{1}{2} \sum_{(f)} (X_{f_{(1)}} X_{f_{(2)}} - (-1)^{|f_{(1)}| \cdot |f_{(2)}|} X_{f_{(2)}} X_{f_{(1)}}), \tag{4.20}$$

where we use the shuffle coproduct of $\bar{S}^c(X)$. Thus we see that the differential stays inside the universal enveloping algebra, and is precisely the one of the bar/cobar resolution of the abelian $c\mathsf{Lie}_n$ algebra $V_+ = V \oplus \mathbb{R}\mathbb{1}$. Thanks to Lemma 4.13 below, we know that $\mathcal{U}_{c\mathsf{Lie}_n}$ preserves quasi-isomorphisms (the unit is freely adjoined and hence is not a boundary), and the universal enveloping algebra of an abelian Lie algebra is just a symmetric algebra, hence:

$$\mathcal{U}_{u\mathsf{Pois}_n}(\Omega_\kappa A^{¡}) \xrightarrow{\sim} A \otimes \mathcal{U}_{c\mathsf{Lie}_n}(V_+) \cong A \otimes S(\Sigma^{n-1} V). \tag{4.21}$$

This last algebra is simply $\mathcal{U}_{u\mathsf{Pois}_n}(A)$, as claimed. □

We now state the missing lemma in the previous proof. For simplicity we state it for unshifted Lie algebra; the $c\mathsf{Lie}_n$ case is identical. The functor $\mathcal{U}_{\mathsf{Lie}} = \mathcal{U}$ preserves quasi-isomorphisms: we can filter it by tensor powers and apply Künneth's theorem as we are working over a field. We thus get the following result on $\mathcal{U}_{c\mathsf{Lie}}(-) = \mathcal{U}_{\mathsf{Lie}}(-)/(X_{\mathbb{1}})$:

Lemma 4.13. *Let $f : \mathfrak{g} \to \mathfrak{h}$ be a quasi-isomorphism of $c\mathsf{Lie}$-algebras. The induced morphism $\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g}) \to \mathcal{U}_{c\mathsf{Lie}}(\mathfrak{h})$ is a quasi-isomorphism.*

Proof. The associative algebra $\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g})$ is given by the presentation:

$$\mathcal{U}_{c\mathsf{Lie}}(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - (-1)^{|x| \cdot |y|} y \otimes x = [x, y], \mathbb{1} = 0), \tag{4.22}$$

where $T(-)$ is the tensor algebra. This algebra is isomorphic to $\mathcal{U}_{\mathsf{Lie}}(\mathfrak{g}/\mathbb{1})$. Given our quasi-isomorphism f, we can apply the five lemma to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{1} \arrow[r, hook] \arrow[d, "\cong"] & \mathfrak{g} \arrow[r, two heads] \arrow[d, "f", "\sim"'] & \mathfrak{g}/\mathbb{1} \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
0 \arrow[r] & \mathbb{1} \arrow[r, hook] & \mathfrak{h} \arrow[r, two heads] & \mathfrak{h}/\mathbb{1} \arrow[r] & 0
\end{tikzcd} \tag{4.23}$$

to obtain that $\bar{f} : \mathfrak{g}/\mathbb{1} \to \mathfrak{h}/\mathbb{1}$ is a quasi-isomorphism. Since $\mathcal{U}_{\mathsf{Lie}}$ preserves quasi-isomorphisms by the argument above, the result is established. □

4.4 Factorization homology

As another application, let us compute over $\mathbb{R}$ the factorization homology of a fixed parallelized, simply connected, closed manifold M of dimension $n \geq 4$, with coefficients in A.

4.4.1 Right $u\mathsf{Com}$-module structure on $\mathsf{G}_P^\vee$

Let P be Poincaré duality model of M (see Section 1.3) with augmentation $\varepsilon : P^n \to \mathbb{R}$. Recall from Section 1.3 (or [Idr19, Section 5]) that we may use our explicit real model $\mathsf{G}_P^\vee$ in order to compute factorization homology $\int_M A$. The object $\mathsf{G}_P^\vee$ is a right $u\mathsf{Pois}_n$-module. As $u\mathsf{Pois}_n = u\mathsf{Com} \circ \mathsf{Lie}_n$, describing the $u\mathsf{Pois}_n$-module structure amounts to describing the Lie_n-module structure and the $u\mathsf{Com}$-module structure. We are going to describe these two structure separately.

Let us first describe the right Lie_n-module structure of $\mathsf{G}_P^\vee$, which is found in [Idr19]. As a right Lie_n-module, we have an isomorphism:

$$\mathsf{G}_P^\vee \cong_{\mathsf{Lie}_n} C_*^{CE}(P^{-*} \otimes \Sigma^{n-1}\mathsf{Lie}_n), \tag{4.24}$$

where C_*^{CE} is the Chevalley–Eilenberg chain complex, and $\Sigma^{n-1}\mathsf{Lie}_n = \{\Sigma^{n-1}\mathsf{Lie}_n(k)\}_{k\geq 0}$ is a Lie algebra in the category of right Lie_n-modules.

Let us now describe the right $u\mathsf{Com}$-module structure of $\mathsf{G}_P^\vee$. This module structure is not described in [Idr19, Section 5], but it easily follows from the arguments there. Roughly speaking, one needs to use the distributive law $\mathsf{Lie}_n \circ u\mathsf{Com} \to u\mathsf{Com} \circ \mathsf{Lie}_n$, which states that the bracket is a biderivation with respect to the product, and that the unit is a central element for the bracket. Then we either need to use $\varepsilon : P^n \to \mathbb{R}$, to describe the action of the unit of $u\mathsf{Com}$, or the coproduct $\Delta : P^{n-*} \to (P^{n-*})^{\otimes 2}$ which is the Poincaré dual of the product $P \otimes P \to P$, to describe the action of the product of $u\mathsf{Com}$. This coproduct is the unique linear map such that $(\varepsilon_A \otimes \varepsilon_A)(\Delta(a) \cdot (x \otimes y)) = \varepsilon_A(axy)$ for all $a, x, y \in P$.

In more detail, given $k \geq 0$, we have an isomorphism of graded modules:

$$\mathsf{G}_P^\vee(k) \cong \bigoplus_{r\geq 0} \Bigg(\bigoplus_{\pi \in \mathrm{Part}_r(k)} (A^{n-*})^{\otimes r} \otimes \mathsf{Lie}_n(\#\pi_1) \otimes \cdots \otimes \mathsf{Lie}_n(\#\pi_r) \Bigg)^{\Sigma_r}, \tag{4.25}$$

where the inner sum runs over all partitions $\pi = \{\pi_1, \dots, \pi_r\}$ of $\{1, \dots, k\}$. To describe the right $u\mathsf{Com}$-module structure, we need to say what happens when we insert the two generators, the unit $\mathbb{1}$ and the product μ, at each index $1 \leq i \leq k$, for each summand of the decomposition.

Suppose we are given an element $X = (x_j)_{j=1}^r \otimes \lambda_1 \otimes \cdots \otimes \lambda_r$, where $x_j \in A$ and $\lambda_j \in \mathsf{Lie}_n(\#\pi_j)$. Suppose that $i \in \pi_j$ in the partition. Then:

- $X \circ_i \mathbb{1}$ is obtained by inserting the unit in λ_j:
  - if λ_j has at least one bracket then the result is zero;
  - otherwise, if $\lambda_j = \mathrm{id}$, then the corresponding factor disappears ($c\mathsf{Lie}_n(0) = \mathbb{R}$) and we apply ε to x_j;
- $X \circ_i \mu$ is obtained by inserting the product μ in λ_j. Using the distributive law for Com and Lie_n, we obtain a sum of products of two elements from Lie_n, splitting π_j in two subsets. We then apply the coproduct $\Delta : P^{n-*} \to (P^{n-*})^{\otimes 2}$, which is Poincaré dual to the product of P, to x_j to obtain a tensor in $A \otimes A$, which we assign to the two subsets of π_j in the corresponding summand.

Example 4.14. Given $x \in P$, we can view $x \otimes \mathrm{id}$ as an element of $\mathsf{G}_P^\vee(1) = P^{n-*} \otimes \mathsf{Lie}_n(1)$. Notice that $\mathsf{G}_P^\vee(0) = \mathbb{R}$, and $\mathsf{G}_P^\vee(2) = P^{n-*} \otimes \mathsf{Lie}_n(2) \oplus \big((P^{n-*})^{\otimes 2} \otimes \mathsf{Lie}_n(1)^{\otimes 2}\big)^{\Sigma_2}$. We then have the following relations:

$$(x \otimes \mathrm{id}) \circ_1 \lambda = x \otimes \lambda \in P^{n-*} \otimes \mathsf{Lie}_n(2). \tag{4.26}$$

$$(x \otimes \mathrm{id}) \circ_1 \mu = \Delta(x) \otimes \mathrm{id} \otimes \mathrm{id} \in \big((P^{n-*})^{\otimes 2} \otimes \mathsf{Lie}_n(1)^{\otimes 2}\big)^{\Sigma_2}. \tag{4.27}$$

$$(x \otimes \mathrm{id}) \circ_1 \mathbb{1} = \varepsilon(x) \in \mathbb{R}. \tag{4.28}$$

4.4.2 Computation

Lemma 4.15. *The underived relative composition product $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$ is given by the* unital *Chevalley–Eilenberg homology of the* $c\mathsf{Lie}$*-algebra $P^{-*} \otimes \Sigma^{n-1} V$.*

This unital Chevalley–Eilenberg complex is given by

$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A = \big(S^c(P^{-*} \otimes \Sigma^n V), d_{CE}\big) \tag{4.29}$$

Here the shifted Lie bracket of V (and hence the differential d_{CE}) can produce a unit. In this case, we apply $\varepsilon : P \to \mathbb{R}$ to the corresponding factor, and the result is identified with the counit of $S^c(-)$, i.e. the empty tensor.

Proof. This is almost identical to the case of the universal enveloping algebra of a Lie algebra (with no central element) from [Idr19] (see Section 1.3). The Lie bracket cannot produce a product of two elements of V, only a unit. Therefore we just need to verify what happens to the unit in the isomorphism of [Idr19, Lemma 5.2], which is part of Section 4.4.1. $\square$

Proposition 4.16. *The factorization homology $\int_M A \simeq \mathsf{G}_P^\vee \circ^{\mathbb{L}}_{u\mathsf{Pois}_n} A$ of the symplectic Poisson n-algebra A is quasi-isomorphic to $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$.*

Proof. As we are working with a derived composition product, we take a resolution of A as a $u\mathsf{Pois}_n$-algebra. For this, we use the cobar construction of the Koszul dual algebra, $\Omega_\kappa A^{¡}$, described in Section 4.2. We then have:

$$\int_M A \simeq \mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{¡}. \tag{4.30}$$

The cobar construction $\Omega_\kappa A^{¡}$ is a quasi-free $u\mathsf{Pois}_n$-algebra on the Koszul dual $qA^{¡}$, with some differential. Therefore, by the cancellation rule for relative products over operad ($X \circ_{\mathsf{P}} (\mathsf{P} \circ Y) = X \circ Y$), we obtain that, as a graded module,

$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{¡} \cong \big(\mathsf{G}_P^\vee \circ qA^{¡}, d_\Omega\big), \tag{4.31}$$

with a differential induced by the differential of the cobar construction. The Koszul dual of qA is $qA^{¡} = \Sigma^{1-n} \bar{S}^c(\Sigma^n V)$, where $V = \mathbb{R}\langle x_i, \xi_j \rangle$ is the graded vector space of generators (see Lemma 4.5). Using the explicit form of the right module $\mathsf{G}_P^\vee$ found in Section 4.4.1, we then find that:

$$\int_M A \simeq \big(S^c(P^{-*} \otimes L\bar{S}^c(\Sigma^n V)), d_{CE} + d_0 + d_2\big). \tag{4.32}$$

Let us now write down explicit formulas for the three summands of the differential. As there are two instances of the cofree cocommutative coalgebra appearing, we have to be careful with notations. We will write $\wedge$ for the tensor of the outer coalgebra, and $\vee$ for the tensor of the inner coalgebra. Strictly speaking, we need to consider only elements that are invariant under the symmetric group actions. We will consider all elements, and check that formulas are actually well-defined when passing to invariants. The three parts of the differentials are:

- Given $x_1, \dots, x_k \in P$ and $Y_1, \dots, Y_k \in L\bar{S}^c(\Sigma^n V)$, we have

  $$d_{CE}(x_1 Y_1 \wedge \dots \wedge x_k Y_k) = \sum_{i<j} \pm x_1 Y_1 \wedge \dots \wedge x_i x_j [Y_i, Y_j] \wedge \dots \wedge \widehat{x_j Y_j} \wedge \dots \wedge x_k Y_k. \tag{4.33}$$

- The differential d_2 is defined on the inner $\bar{S}^c(\Sigma^n V)$, extended to a derivation of $L\bar{S}^c(V)$, which is itself extended to the full complex as a coderivation:

  $$d_2(v_1 \vee \dots \vee v_k) = \frac{1}{2} \sum_{i+j=k} \sum_{(\mu,\nu) \in \mathrm{Sh}_{i,j}} \pm [v_{\mu(1)} \vee \dots \vee v_{\mu(i)}, v_{\nu(1)} \vee \dots \vee v_{\nu(j)}], \tag{4.34}$$

  where the inner sum is over all (i,j)-shuffles. (This is the differential of the bar/cobar resolution of the abelian Lie algebra $\Sigma^{n-1} V$).

- The differential d_0 is similarly defined on $\bar{S}^c(\Sigma^n V)$ and extended to the full complex by:

  $$d_0(X) = \begin{cases} -\mathbb{1}, & \text{if } X = \Sigma^n x_i \vee \Sigma^n \xi_i \text{ for some } i; \\ 0 & \text{otherwise.} \end{cases} \tag{4.35}$$

  Note that the unit is appearing here. If the unit is inside a Lie bracket, the result is zero ($\mathbb{1}$ is central). Otherwise, we have to apply $\varepsilon : P \to \mathbb{R}$ to the corresponding element of P in the outer $S^c(-)$, and this factor disappears (it is replaced with a real coefficient).

We can project this complex to

$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A = \big(S^c(P^{-*} \otimes \Sigma^n V), d_{CE}\big), \tag{4.36}$$

i.e. the Chevalley–Eilenberg complex (with constant coefficients) of the $c\mathsf{Lie}$ algebra $P^{-*} \otimes \Sigma^{n-1} V$. The projection from $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{\mathsf{i}}$ is compatible with the differential.

Let i be the number of Lie brackets in an element of the complex, and j be the number of inner tensors ($\vee$). We then observe that d_2 preserves the difference $i - j$, while d_{CE} and d_0 increase them by 1. We can thus filter our first complex by this number to obtain what we will call the "first spectral sequence". The second complex, $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A = \big(S^c(P^{-*} \otimes \Sigma^{n-1}V), d_{CE}\big)$, is also filtered, with the unit in filtration 0 and the rest in filtration 1. This yields a "second spectral sequence". The projection is compatible with this filtration, hence we obtain a morphism from the first spectral sequence to the second one.

On the E^0 page of the first spectral sequence, only d_2 remains. Recall that d_2 is exactly the differential of the bar/cobar resolution $\Sigma^{-1}L\bar{S}^c(\Sigma^n V) \xrightarrow{\sim} \Sigma^{n-1}V$ of the abelian Lie algebra $\Sigma^{n-1}V$. Thus on the E^1 page of the spectral sequence, we obtain an isomorphism of graded modules from the first spectral sequence to the second. The differential d_{CE} of the first spectral sequence vanishes, and the differential d_0 precisely correspond to part of the "unital" Chevalley–Eilenberg differential of the second spectral sequence. Hence we find that the projection $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} \Omega_\kappa A^{\mathrm{i}} \to \mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$ is a quasi-isomorphism. $\square$

Proposition 4.17. *Let A be a symplectic n-Poisson algebra (Definition 4.1) and let M be a simply connected smooth framed manifold of dimension at least 4. Then the homology of $\int_M A$ is one-dimensional.*

Proof. Thanks to Proposition 4.16, we only need to compute the homology of $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$. Let us use the explicit description from Lemma 4.15 as the unital Chevalley–Eilenberg complex of the $c\mathsf{Lie}$-algebra

$$\mathfrak{g}_{P,V} \coloneqq P^{-*} \otimes \Sigma^{n-1}V. \tag{4.37}$$

There is a pairing $\langle -,- \rangle : \mathfrak{g}_{P,V}^{\otimes 2} \to \mathbb{R}$ given by $xv \otimes x'v' \mapsto \varepsilon_P(xx') \cdot \{v, v'\}$. We have the following isomorphism of chain complexes:

$$\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A \cong \left(\bigoplus_{i \geq 0} \big((\Sigma \mathfrak{g}_{P,V})^{\otimes i}\big)_{\Sigma_i}, d_{CE} \right) \tag{4.38}$$

where

$$d_{CE}(\alpha_1 \wedge \dots \wedge \alpha_k) = \sum_{i<j} \pm \langle \alpha_i, \alpha_j \rangle \cdot \alpha_1 \wedge \dots \widehat{\alpha}_i \dots \widehat{\alpha}_j \dots \wedge \alpha_k. \tag{4.39}$$

Recall that since P is a Poincaré commutative dg-algebra, the pairing induced by ε_A is by definition non-degenerate. Moreover, the pairing on V given by the bracket is clearly non-degenerate. It follows that the pairing $\langle -,- \rangle$ defined on $\mathfrak{g}_{P,V}$ is non-degenerate. If $\{a_i\}_{i \in I}$ is a graded basis of P and $\{a_i^\vee\}_{i \in I}$ is its dual basis, then $\{a_i \otimes x_j, a_i \otimes \xi_j\}_{i \in I, 1 \leq j \leq D}$ is a graded basis of $\mathfrak{g}_{P,V}$ and its dual basis is $\{a_i^\vee \otimes \xi_j, a_i^\vee \otimes x_j\}_{i \in I, 1 \leq j \leq D}$.

In order to have lighter notations, write $\{y_k\}_{1 \leq k \leq r}$ for the graded basis of $\mathfrak{g}_{P,V}$ found above, and let $\{y_k^\vee\}$ be its dual basis under the pairing. We can then identify $\mathsf{G}_P^\vee \circ_{u\mathsf{Pois}_n} A$ with the "algebraic de Rham complex":

$$\Omega^*_{adR}(\mathbb{R}^r) = \Big(S(y_1, \dots, y_r) \otimes \Lambda(dy_1, \dots, dy_r), d_{dR} = \sum_k \frac{\partial}{\partial y_k} \cdot dy_k\Big). \tag{4.40}$$

Note that if all the variables y_k had degree zero then this would be isomorphic to the algebra $A_{PL}(\Delta^r) \otimes_{\mathbb{Q}} \mathbb{R}$ of piecewise real polynomial forms on Δ^r. There is an isomorphism (up to a degree shift and reversal) given by:

$$\begin{aligned}\left(\bigoplus_{i\geq 0}\big((\Sigma\mathfrak{g}_{P,V})^{\otimes i}\big)_{\Sigma_i}, d_{CE}\right) &\xrightarrow{\cong} \Omega^*_{adR}(\mathbb{R}^r) \\ y_{k_1}\wedge\cdots\wedge y_{k_\alpha}\wedge y^*_{l_1}\wedge\cdots\wedge y^*_{l_\beta} &\mapsto y_{k_1}\dots y_{k_\alpha}\cdot\prod_{\substack{1\leq l\leq r\\ l\notin\{l_1,\dots,l_\beta\}}} dy_l\end{aligned} \tag{4.41}$$

For example if $r=3$, then the isomorphism sends $y_1\wedge y_2^*$ to $y_1dy_1dy_3$.

The algebraic de Rham complex is a particular example of a Koszul complex and is therefore acyclic There is an explicit homotopy given by $h(dy_i)=y_i$, $h(y_i)=0$ and extended suitably as a derivation. In particular, a representative of the only homology class is the unit of the de Rham complex, which under our identification is $\bigwedge_{j=1}^r y_j^*$. □

Remark 4.18. From a physical point of view, this result is satisfactory: when one wants to compute expected values of observables, one wants a number. The next best thing to a number is a closed element in a complex whose homology is one-dimensional. We thank T. Willwacher for this perspective.

Remark 4.19. This result appears similar to the computation of Markarian [Mar17] for the Weyl n-algebra $\mathcal{W}_n^h(D)$, which is an algebra over the operad $C_*(\mathsf{FM}_n;\mathbb{R}[[h,h^{-1}]])$, where FM_n is the Fulton–MacPherson operad. We do not know the precise relationship between $A_{n;D}$ and $\mathcal{W}_n^h(D)$, though. Curved Koszul duality was conjectured to apply for this computation by Markarian [MT15]. Moreover, Döppenschmitt [Döp18] recently released a preprint containing an analogous computation for a twisted version of A, using a "physical" approach based on AKSZ theory. Our approach is however different from these two approaches. It is also in some sense more general, as we should be able to compute the factorization homology of M with coefficients in any Koszul $u\mathsf{Pois}_n$-algebra, e.g. an algebra of the type $S(\Sigma^{1-n}\mathfrak{g})$ where $\mathfrak{g}$ is a Koszul $c\mathsf{Lie}$-algebra. Finally, let us note that previous results of Getzler [Get99], regarding the computation of the Hodge polynomials (with compact support) of configuration spaces of quasi-projective varieties over a base, involve similar techniques.

Remark 4.20. Using the results from [CILW18], we hope to be able to compute factorization homology of compact manifolds with boundary with coefficients in A.

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