The Lambrechts–Stanley Model of Configuration Spaces

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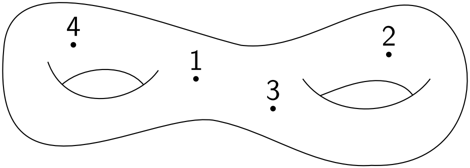
Configuration spaces

M: smooth closed n-manifold (+ future adjectives)

$$\mathrm{Conf}_k(M) = \{(x_1, \ldots, x_k) \in M^{\times k} \mid x_i \neq x_j \ \forall i \neq j\}$$



Goal

Obtain a CDGA model of $\mathrm{Conf}_k(M)$ from a CDGA model of M

Plan

1. The model
2. Action of the Fulton–MacPherson operad
3. Sketch of proof through Kontsevich formality
4. Computing factorization homology

Models

We are interested in rational/real models

$$A \simeq \Omega^*(M) \text{ “forms on } M\text{”} \text{ (de Rham, piecewise polynomial...)}$$

where A is an “explicit” CDGA

M simply connected $\implies$ A contains all the rational/real homotopy type of M

$\mathrm{Conf}_k(M)$ smooth (but noncompact); we're looking for a CDGA $\simeq \Omega^*(\mathrm{Conf}_k(M))$ built from A

Poincaré duality models

Poincaré duality CDGA (A, d, ε) (example: $A = H^*(M)$)

- (A, d): finite type connected CDGA;
- $\varepsilon : A^n \to \Bbbk$ such that $\varepsilon \circ d = 0$;
- $A^k \otimes A^{n-k} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ non degenerate.

Theorem (Lambrechts–Stanley 2008)

Any simply connected manifold has such a model

$$\begin{array}{ccccc} \Omega^*(M) & \xleftarrow{\sim} & \cdot & \xrightarrow{\sim} & \exists A \\ & {\scriptstyle \int_M} \searrow & & \swarrow {\scriptstyle \exists \varepsilon} & \\ & & \Bbbk & & \end{array}$$

Remark

Reasonable assumption: $\exists$ non simply-connected $L \simeq L'$ but $\mathrm{Conf}_k(L) \not\simeq \mathrm{Conf}_k(L')$ for $k \geq 2$ [Longoni–Salvatore].

Diagonal class

In cohomology, diagonal class

$$[M] \in H_n(M) \mapsto \delta_*[M] \in H_n(M \times M) \qquad \delta(x) = (x, x)$$
$$\leftrightarrow \Delta_M \in H^{2n-n}(M \times M)$$

Representative in a Poincaré duality model (A, d, ε):

$$\Delta_A = \sum (-1)^{|a_i|} a_i \otimes a_i^\vee \in (A \otimes A)^n$$

$\{a_i\}$: graded basis and $\varepsilon(a_i a_j^\vee) = \delta_{ij}$ (independent of chosen basis)

Properties

- $(a \otimes 1)\Delta_A = (1 \otimes a)\Delta_A$ "concentrated around the diagonal"
- $A \otimes A \xrightarrow{\mu_A} A$, $\Delta_A \mapsto e(A) = \chi(A) \cdot \mathrm{vol}_A$

The Lambrechts–Stanley model

$\mathrm{Conf}_k(\mathbb{R}^n)$ is a formal space, with cohomology [Arnold–Cohen]:

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = S(\omega_{ij})_{1\le i\neq j\le k}/I, \quad \deg \omega_{ij} = n-1$$
$$I = \langle \omega_{ji} = \pm\omega_{ij},\ \omega_{ij}^2 = 0,\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0 \rangle.$$

$\mathtt{G}_A(k)$ conjectured model of $\mathrm{Conf}_k(M) = M^{\times k} \setminus \bigcup_{i\neq j} \Delta_{ij}$

- "Generators": $A^{\otimes k} \otimes S(\omega_{ij})_{1\le i\neq j\le k}$
- Relations:
  - Arnold relations for the ω_{ij}
  - $p_i^*(a)\cdot\omega_{ij} = p_j^*(a)\cdot\omega_{ij}$. $(p_i^*(a) = 1\otimes\cdots\otimes 1\otimes a\otimes 1\otimes\cdots\otimes 1)$
- $d\omega_{ij} = (p_i^* \cdot p_j^*)(\Delta_A)$.

First examples

$$\mathtt{G}_A(k) = (A^{\otimes k} \otimes S(\omega_{ij})_{1 \le i < j \le k} / J, d\omega_{ij} = (p_i^* \cdot p_j^*)(\Delta_A))$$

$$\mathtt{G}_A(0) = \mathbb{R}\text{: model of } \mathrm{Conf}_0(M) = \{\varnothing\} \quad \checkmark$$

$$\mathtt{G}_A(1) = A\text{: model of } \mathrm{Conf}_1(M) = M \quad \checkmark$$

$$\begin{aligned}
\mathtt{G}_A(2) &= \left(\frac{A \otimes A \otimes 1 \ \oplus \ A \otimes A \otimes \omega_{12}}{1 \otimes a \otimes \omega_{12} \equiv a \otimes 1 \otimes \omega_{12}}, d\omega_{12} = \Delta_A \otimes 1 \right) \\
&\cong (A \otimes A \otimes 1 \ \oplus \ A \otimes_A A \otimes \omega_{12}, \ d\omega_{12} = \Delta_A \otimes 1) \\
&\cong (A \otimes A \otimes 1 \ \oplus \ A \otimes \omega_{12}, \ d\omega_{12} = \Delta_A \otimes 1) \\
&\xrightarrow{\sim} A^{\otimes 2} / (\Delta_A)
\end{aligned}$$

Brief history of $\mathtt{G}_A$

1969 [Arnold–Cohen] $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \approx$ "$\mathtt{G}_{H^*(\mathbb{R}^n)}(k)$"

1978 [Cohen–Taylor] $E^2 = \mathtt{G}_{H^*(M)}(k) \implies H^*(\mathrm{Conf}_k(M))$

~1994 For smooth projective complex manifolds ($\implies$ Kähler):

- [Kříž] $\mathtt{G}_{H^*(M)}(k)$ model of $\mathrm{Conf}_k(M)$
- [Totaro] The Cohen–Taylor SS collapses

2004 [Lambrechts–Stanley] $A^{\otimes 2}/(\Delta_A)$ model of $\mathrm{Conf}_2(M)$ for a 2-connected manifold

~2004 [Félix–Thomas, Berceanu–Markl–Papadima] $\mathtt{G}^\vee_{H^*(M)}(k) \cong$ page E^2 of Bendersky–Gitler SS for $H^*(M^{\times k}, \bigcup_{i\neq j} \Delta_{ij})$

2008 [Lambrechts–Stanley] $H^*(\mathtt{G}_A(k)) \cong_{\Sigma_k-\mathrm{gVect}} H^*(\mathrm{Conf}_k(M))$

2015 [Cordova Bulens] $A^{\otimes 2}/(\Delta_A)$ model of $\mathrm{Conf}_2(M)$ for $\dim M = 2m$

First part of the theorem

Theorem (I.)

Let M be a smooth, closed, simply connected manifold of dimension ≥ 4. Then $\mathtt{G}_A(k)$ is a model over $\mathbb{R}$ of $\mathrm{Conf}_k(M)$ for all $k \geq 0$.

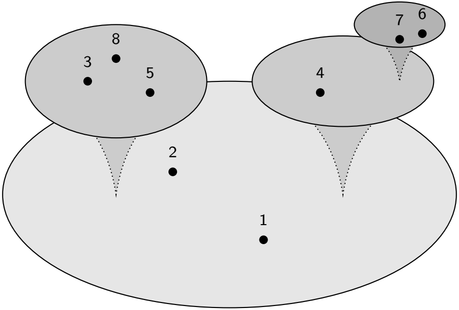
$\dim M \geq 3 \implies \mathrm{Conf}_k(M)$ is simply connected when M is (cf. Fadell–Neuwirth fibrations).

Corollary

All the real homotopy type of $\mathrm{Conf}_k(M)$ is contained in (A, d, ε).

Fulton–MacPherson compactification

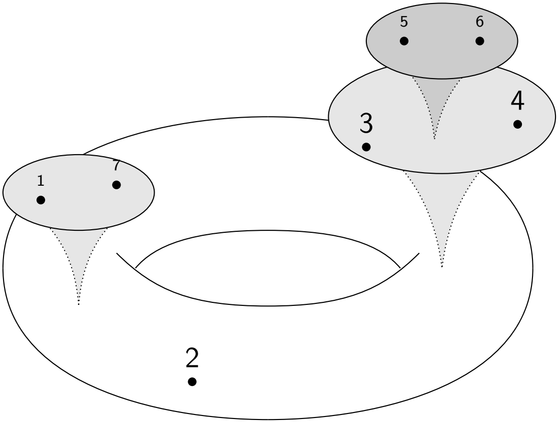
$\mathtt{FM}_n(k)$: Fulton–MacPherson compactification of $\mathrm{Conf}_k(\mathbb{R}^n)$



(+ normalization to deal with $\mathbb{R}^n$ being noncompact)

Fulton–MacPherson compactification (2)

$\mathtt{FM}_M(k)$: similar compactification of $\mathrm{Conf}_k(M)$

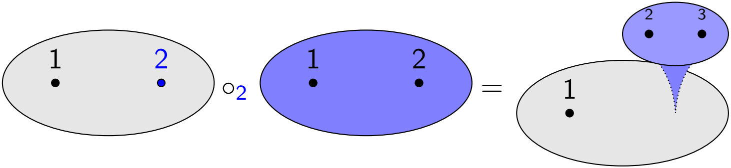


Operads

Idea

Study all of $\{\mathrm{Conf}_k(M)\}_{k \geq 0} \implies$ more structure.

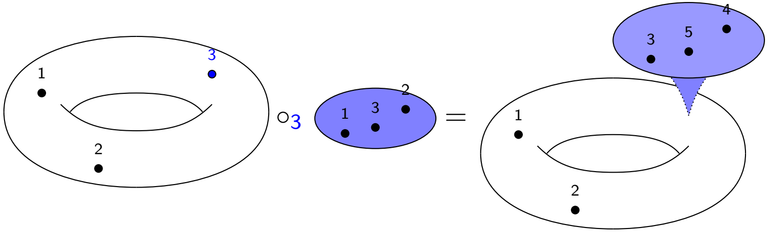
$\mathtt{FM}_n = \{\mathtt{FM}_n(k)\}_{k \geq 0}$ is an operad: we can insert an infinitesimal configuration into another



$$\mathtt{FM}_n(k) \times \mathtt{FM}_n(l) \xrightarrow{\circ_i} \mathtt{FM}_n(k + l - 1), \quad 1 \leq i \leq k$$

Structure de module

M framed $\implies \mathtt{FM}_M = \{\mathtt{FM}_M(k)\}_{k \geq 0}$ is a right $\mathtt{FM}_n$-module: we can insert an infinitesimal configuration into a configuration on M



$$\mathtt{FM}_M(k) \times \mathtt{FM}_n(l) \xrightarrow{\circ_i} \mathtt{FM}_M(k + l - 1), \quad 1 \leq i \leq k$$

Cohomology of $\mathtt{FM}_n$ and coaction on $\mathtt{G}_A$

$H^*(\mathtt{FM}_n)$ inherits a Hopf cooperad structure
One can rewrite:

$$\mathtt{G}_A(k) = (A^{\otimes k} \otimes H^*(\mathtt{FM}_n(k))/\text{relations}, d)$$

Proposition

$\chi(M) = 0 \implies \mathtt{G}_A = \{\mathtt{G}_A(k)\}_{k \geq 0}$ Hopf right $H^*(\mathtt{FM}_n)$-comodule

Motivation

We are looking for something to put here:

$$\mathtt{G}_A(k) \xleftarrow{\sim} ? \xrightarrow{\sim} \Omega^*(\mathtt{FM}_M(k))$$

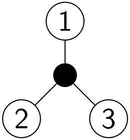
Hunch: if true, then hopefully it fits in something like this!

$$\begin{array}{ccccc} \mathtt{G}_A & \xleftarrow{\quad\sim\quad} & ? & \xrightarrow{\quad\sim\quad} & \Omega^*(\mathtt{FM}_M) \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ H^*(\mathtt{FM}_n) & \xleftarrow{\quad\sim\quad} & ? & \xrightarrow{\quad\sim\quad} & \Omega^*(\mathtt{FM}_n) \end{array}$$

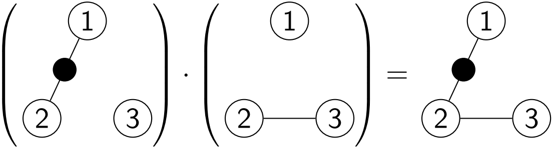
Fortunately, the bottom row is already known: formality of $\mathtt{FM}_n$

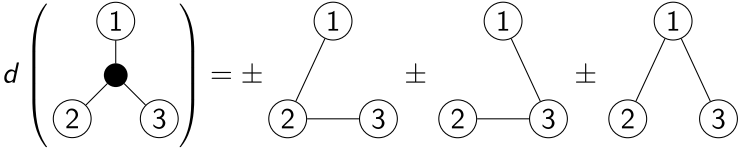
Kontsevich's graph complexes

[Kontsevich] Hopf cooperad $\mathtt{Graphs}_n = \{\mathtt{Graphs}_n(k)\}_{k \geq 0}$



$\in \mathtt{Graphs}_n(3)$



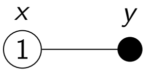


Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

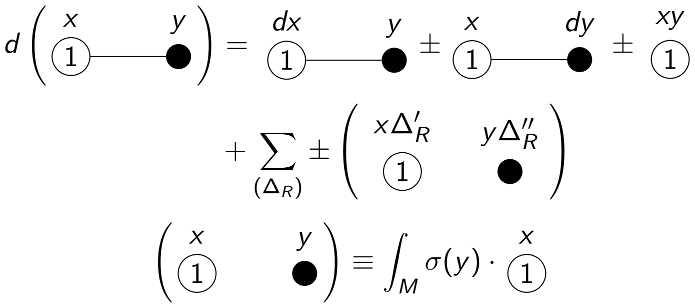
Labeled graph complexes

Recall: $\Omega^*_{\mathrm{PA}}(M) \xleftarrow{\sim} R \xrightarrow{\sim} A$, with $\int_M : \Omega^*_{\mathrm{PA}}(M) \to \Bbbk$ and $\varepsilon : A \to \Bbbk$

$\rightsquigarrow$ labeled graph complex $\mathtt{Graphs}_R$:



$\in \mathtt{Graphs}_R(1)$ (where $x, y \in R$)



Complete version of the theorem

Theorem (I., complete version)

$$\begin{array}{ccccc} \mathtt{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M) \\ \circlearrowleft^{\dagger} & & \circlearrowleft^{\dagger} & & \circlearrowleft^{\ddagger} \\ H^*(\mathtt{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n) \end{array}$$

† When $\chi(M) = 0$ ‡ When M is framed

Factorization homology

$\mathtt{FM}_n$-algebra: space B + maps

$$\mathtt{FM}_n \circ B = \bigsqcup_{k \geq 0} \mathtt{FM}_n(k) \times B^{\times k} \to B$$

$\to$ "homotopy commutative" (up to degree n) algebra
Factorization homology of M with coefficients in B:

$$\int_M B := \mathtt{FM}_M \circ^{\mathbb{L}}_{\mathtt{FM}_n} B = \text{“}\operatorname{Tor}^{\mathtt{FM}_n}(\mathtt{FM}_M, B)\text{”}$$
$$= \operatorname{hocoeq}(\mathtt{FM}_M \circ \mathtt{FM}_n \circ B \rightrightarrows \mathtt{FM}_M \circ B)$$

Factorization homology (2)

In chain complexes over $\mathbb{R}$:

$$\int_M B := C_*(\mathtt{FM}_M) \circ^{\mathbb{L}}_{C_*(\mathtt{FM}_n)} B.$$

Formality $C_*(\mathtt{FM}_n) \simeq H_*(\mathtt{FM}_n) \implies$

$$\mathrm{Ho}(C_*(\mathtt{FM}_n)\text{-Alg}) \simeq \mathrm{Ho}(H_*(\mathtt{FM}_n)\text{-Alg})$$

$$B \leftrightarrow \tilde{B}$$

Full theorem + abstract nonsense $\implies$

$$\int_M B \simeq \mathtt{G}_A^\vee \circ^{\mathbb{L}}_{H_*(\mathtt{FM}_n)} \tilde{B}$$

$\rightsquigarrow$ much more computable (as soon as $\tilde{B}$ is known)

Comparison with a theorem of Knudsen

Theorem (Knudsen, 2016)

$$\mathtt{Lie}\text{-Alg} \underset{\text{forgetful}}{\overset{\exists U_n,\ \perp}{\rightleftarrows}} \mathtt{FM}_n\text{-Alg} \qquad \int_M U_n(\mathfrak{g}) \simeq C_*^{\mathrm{CE}}(A_{\mathrm{PL}}^{-*}(M) \otimes \mathfrak{g})$$

Abstract nonsense $\implies$

$$C_*(\mathtt{FM}_n)\text{-Alg} \longleftrightarrow H_*(\mathtt{FM}_n)\text{-Alg}$$

$$U_n(\mathfrak{g}) \longleftrightarrow S(\Sigma^{1-n}\mathfrak{g})$$

Proposition

$$\mathtt{G}_A^\vee \circ^{\mathbb{L}}_{H_*(\mathtt{FM}_n)} S(\Sigma^{1-n}\mathfrak{g}) \xrightarrow{\sim} \mathtt{G}_A^\vee \circ_{H_*(\mathtt{FM}_n)} S(\Sigma^{1-n}\mathfrak{g}) \cong C_*^{\mathrm{CE}}(A^{-*} \otimes \mathfrak{g})$$

Thanks!

Thank you for your attention!

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