Formalité opéradique et homotopie des espaces de configuration

Operadic Formality and Homotopy of Configuration Spaces

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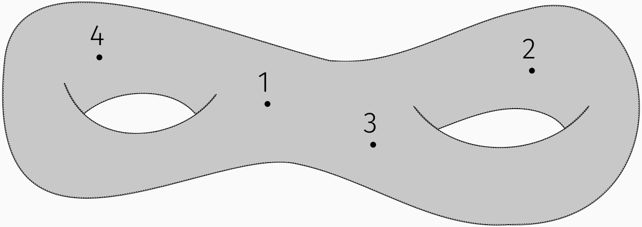
Overall Goal

Study configuration spaces of manifolds:

$$\mathrm{Conf}_k(M) := \{(x_1, \ldots, x_k) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



Idea

Use “formality of the little disks operads” = results for $\mathrm{Conf}_k(\mathbb{R}^n)$.

Plan

Little Disks Operads

Swiss-Cheese Operad and Drinfeld Center

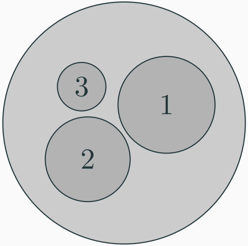
The Lambrechts–Stanley Model of Configuration Spaces

Configuration Spaces of Manifolds with Boundary

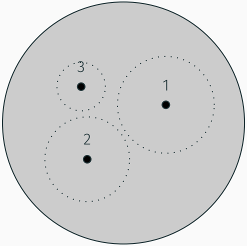
Little Disks Operads

Little Disks Operads

Boardmann–Vogt, May (70's): little disks operads $\mathsf{D}_n = \{\mathsf{D}_n(r)\}_{r \geq 0}$



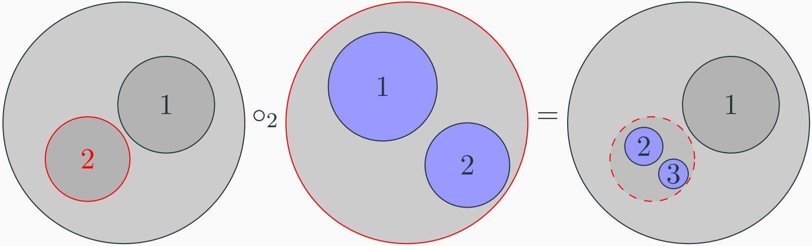
$\in \mathsf{D}_2(3) \mapsto$



$\in \mathrm{Conf}_3(\mathbb{R}^2)$

New structure: insertion

One can insert a configuration into a disk:

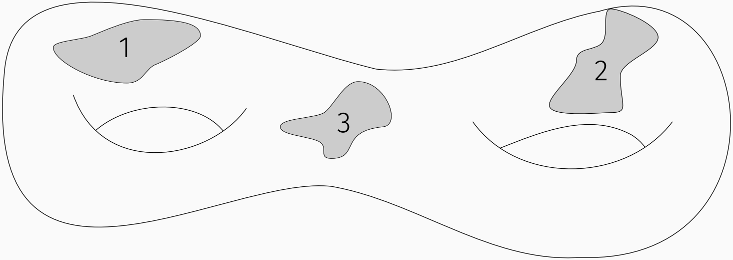


$\Longrightarrow$ operad structure, cannot be seen on $\mathrm{Conf}_\bullet(\mathbb{R}^n)$

Configuration spaces of manifolds

If M is "framed":

$$\mathsf{D}_M(k) := \mathrm{Emb}^{\mathrm{fr}}\left(\mathbb{D}^n \sqcup \cdots \sqcup \mathbb{D}^n, M\right) \xrightarrow{\sim} \mathrm{Conf}_k(M)$$



$\implies \mathsf{D}_M = \{\mathsf{D}_M(k)\}_{k\geq 0}$ is a "right module" over D_n

Idea

Use this extra structure to study $\mathrm{Conf}_k(M)$.

Algebras over D_n in the topological world

An algebra over D_n is a space on which D_n "acts":

$$\mathsf{D}_n(k) \times X^k \to X$$

Theorem (Boardmann–Vogt, May 1972)

- If $X = \Omega^n Y$, then D_n acts on X;
- if D_n acts on X (+ grouplike), then $X \simeq \Omega^n Y$ for some Y.

Algebraic world

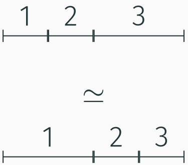
Operad $\mathsf{D}_n \mapsto$ homology $H_*(\mathsf{D}_n)$ (⚠ lose info) -

Theorem (Cohen 1976)

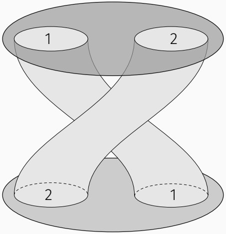
An algebra over $H_*(\mathsf{D}_n)$ is:

- an associative algebra $(A, \cdot)$ for $n = 1$;
- an n-Gerstenhaber algebra $(B, \wedge, [,])$ for $n \geq 2$.

Associativity for $n \geq 1$:



Commutativity for $n \geq 2$:



Swiss-Cheese Operad and Drinfeld Center

Categorical world

Operad $\mathsf{D}_n \mapsto$ fundamental groupoid $\pi\mathsf{D}_n$

Proposition

For $n \in \{1, 2\}$, no loss of information: $\mathsf{D}_n \xrightarrow{\sim} \mathrm{B}(\pi\mathsf{D}_n)$.

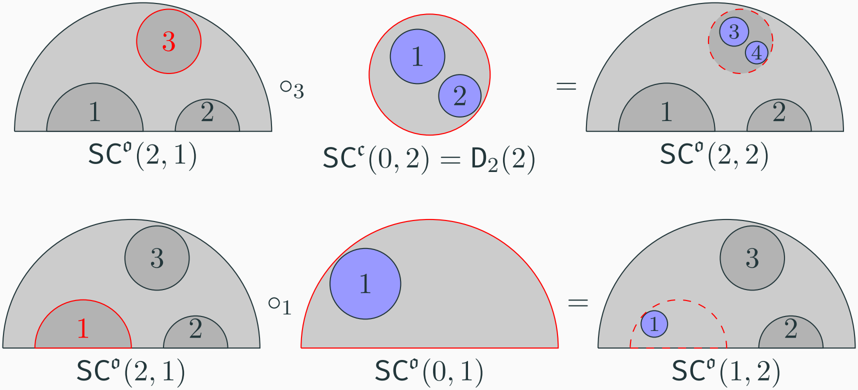
Theorem (Tamarkin, Fresse)

$\pi\mathsf{D}_n \simeq$ operad whose algebras are:

- monoidal categories $(\mathsf{M}, \otimes)$ for $n = 1$;
- braided monoidal categories $(\mathsf{N}, \otimes, \tau)$ for $n = 2$.

Swiss-Cheese operad

Swiss-Cheese operad SC: "D_2-algebras acting on D_1-algebras"



Homology vs fundamental groupoid of SC

Theorem (Voronov 1999, Hoefel 2009)

An algebra over $H_*(\mathsf{SC})$ is a triplet (A, B, f) where:

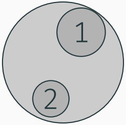
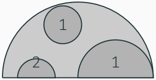
- $(A, \cdot)$ is an associative algebra;
- $(B, \wedge, [,])$ is a Gerstenhaber algebra;
- $f : B \to Z(A)$ is a central morphism of algebras.

Theorem

$\pi\mathsf{SC} \simeq$ an operad whose algebras are triplets $(\mathsf{M}, \mathsf{N}, F)$ where:

- $(\mathsf{M}, \otimes)$ is a monoidal category;
- $(\mathsf{N}, \otimes, \tau)$ is a braided monoidal category;
- $F : \mathsf{N} \to \mathcal{Z}(\mathsf{M})$ is a braided functor to the "Drinfeld center"

Recap

	Topological $\Rightarrow$	Algebraical $H_*(-)$ $\Longleftarrow$	Categorical $\pi(-)$
D_1		associative $(A, \cdot)$	monoidal $(\mathsf{M}, \otimes)$
D_2		Gerstenhaber $(B, \wedge, [,])$	braided $(\mathsf{N}, \otimes, \tau)$
SC		$(B, \wedge, [,]) \xrightarrow{f} Z(A, \cdot)$	$(\mathsf{N}, \otimes, \tau) \xrightarrow{F} \mathcal{Z}(\mathsf{M}, \otimes)$

Remark

I also build a model $\mathsf{PaP}\widehat{\mathsf{CD}}_+^{\phi}$ = "$\mathsf{PaP} \rtimes_\phi \widehat{\mathsf{CD}}_+$" out of a Drinfeld associator ϕ, following Tamarkin's proof of the formality of D_2.

The Lambrechts–Stanley Model of Configuration Spaces

Models

We are interested in rational/real models

$$A \simeq \Omega^*(M) \text{ “forms on } M\text{”}$$ (e.g. de Rham, piecewise polynomial…)

where A is an “explicit” CDGA (= Commutative Differential Graded Algebra)

M nilpotent of finite type $\implies$ A contains all the rational/real homotopy type of M

We’re looking for a CDGA $\simeq \Omega^*(\mathrm{Conf}_k(M))$ built from A

Formality of $\mathrm{Conf}_k(\mathbb{R}^n)$

$\mathrm{Conf}_k(\mathbb{R}^n)$ is a formal space, i.e. [Kontsevich]:

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \simeq \Omega^*(\mathrm{Conf}_k(\mathbb{R}^n))$$

completely determines the rational homotopy type of $\mathrm{Conf}_k(\mathbb{R}^n)$

Theorem (Arnold 1969, Cohen 1976)

- $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = S(\omega_{ij})_{1\leq i\neq j\leq k}/I$
- $\deg \omega_{ij} = n-1$
- $I = \left(\omega_{ji} = \pm\omega_{ij},\ \omega_{ij}^2 = 0,\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0\right)$

Poincaré duality models

Poincaré duality CDGA (A, ε)

(example: M is closed & oriented)

- A: finite type connected CDGA; (e.g. $(H^*(M), d = 0)$)
- $\varepsilon : A^n \to \mathbb{k}$ such that $\varepsilon \circ d = 0$; (e.g. $\int_M(-)$)
- $A^k \otimes A^{n-k} \to \mathbb{k}$, $a \otimes b \mapsto \varepsilon(ab)$ non degenerate. (e.g. $H^k(M) \otimes H^{n-k}(M) \to \mathbb{k}$)

Theorem (Lambrechts–Stanley 2004)

Any simply connected manifold has such a model

$$\begin{array}{ccccc} \Omega^*(M) & \xleftarrow{\sim} & \cdot & \xrightarrow{\sim} & \exists A \\ & {\scriptstyle\int_M}\searrow & \mathbb{k} & \swarrow{\scriptstyle\exists\varepsilon} & \end{array}$$

Remark

By a result of Longoni–Salvatore (2005), $\exists$ non simply-connected $L \simeq L'$ but $\mathrm{Conf}_k(L) \not\simeq \mathrm{Conf}_k(L')$

The Lambrechts–Stanley model

$\mathsf{G}_A(k)$ conjectured model of $\mathrm{Conf}_k(M) = M^{\times k} \setminus \bigcup_{i \neq j} \Delta_{ij}$

$\Delta_{ij} := \{x_i = x_j\}$

- "Generators": $A^{\otimes k} \otimes S(\omega_{ij})_{1 \leq i \neq j \leq k}$
- Relations:
  - Arnold relations $(\omega_{ji} = \pm\omega_{ij}, \omega_{ij}^2 = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0)$
  - $p_i^*(a) \cdot \omega_{ij} = p_j^*(a) \cdot \omega_{ij}$. $(p_i^*(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1)$
- $d\omega_{ij} = (p_i^* \cdot p_j^*)(\Delta_A)$ kills the dual of $[\Delta_{ij}]$.

Theorem (Lambrechts–Stanley 2008)

$$\dim_{\mathbb{Q}} H^i(\mathrm{Conf}_k(M)) = \dim_{\mathbb{Q}} H^i(\mathsf{G}_A(k))$$

First part of the theorem

$\mathsf{G}_A(k)$ was known to be a rational model of $\mathrm{Conf}_k(M)$ in a few cases:

- M smooth projective complex variety [Kriz];
- $k = 2$ and M is 2-connected [Lambrechts–Stanley];
- $k = 2$ and $\dim M$ is even [Cordova Bulens]...

Theorem

Let M be a smooth, closed, simply connected manifold of dimension ≥ 4. Then $\mathsf{G}_A(k)$ is a model over $\mathbb{R}$ of $\mathrm{Conf}_k(M)$ for all $k \geq 0$.

Corollary

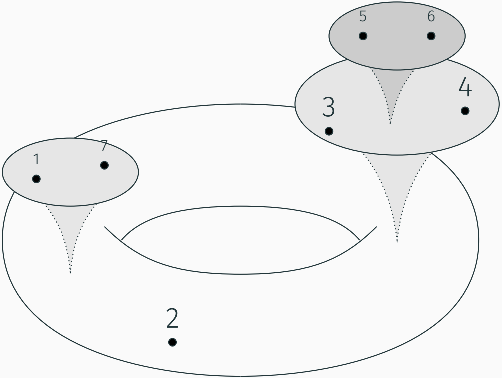
The real homotopy type of $\mathrm{Conf}_k(M)$ only depends on the real homotopy type of M:

$$M \simeq_{\mathbb{R}} N \implies \mathrm{Conf}_k(M) \simeq_{\mathbb{R}} \mathrm{Conf}_k(N).$$

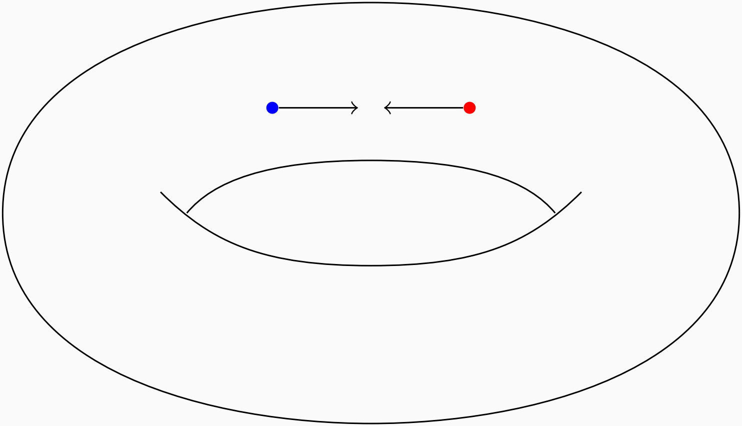
Ideas & Goals

Adapt the construction for D_n & keep track of the D_n-action whenever it exists

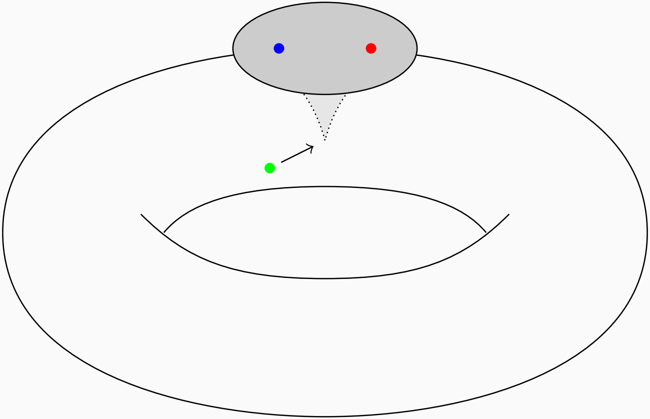
Fulton–MacPherson compactification $\mathrm{Conf}_k(M) \xrightarrow{\sim} \mathsf{FM}_M(k)$



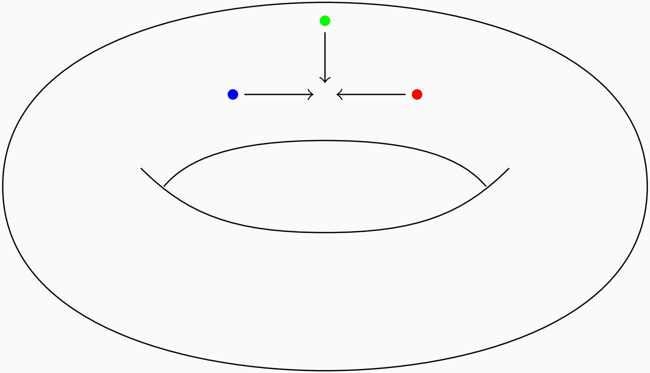
Understanding FM_M (#1)



Understanding FM_M (#2)

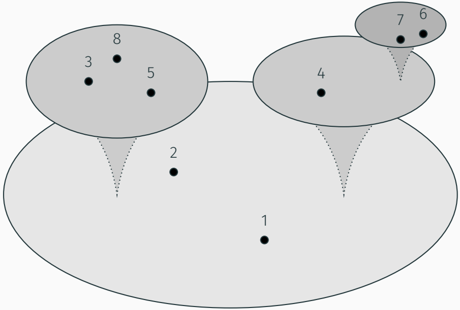


Understanding FM_M (#3)



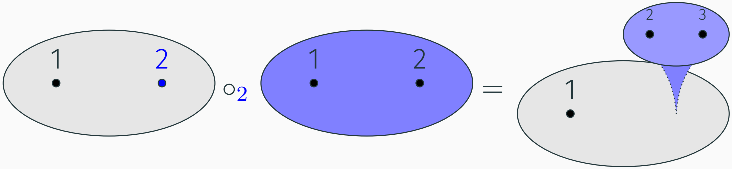
Compactifying $\mathrm{Conf}_k(\mathbb{R}^n)$

Can also compactify $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\sim} \mathrm{Conf}_k(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_+^*) \xhookrightarrow{\sim} \mathsf{FM}_n(k)$



(+ normalization to deal with $\mathbb{R}^n$ being noncompact)

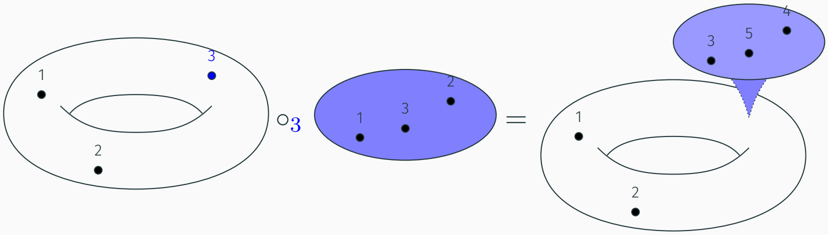
$\mathsf{FM}_n = \{\mathsf{FM}_n(k)\}_{k\geq 0}$ is an operad $\simeq \mathsf{D}_n$



$$\mathsf{FM}_n(k) \times \mathsf{FM}_n(l) \xrightarrow{\circ_i} \mathsf{FM}_n(k+l-1), \quad 1 \leq i \leq k$$

Modules over operads

M framed $\implies$ $\mathsf{FM}_M = \{\mathsf{FM}_M(k)\}_{k\geq 0}$ is a right FM_n-module $\simeq \mathsf{D}_M$



$$\mathsf{FM}_M(k) \times \mathsf{FM}_n(l) \xrightarrow{\circ_i} \mathsf{FM}_M(k+l-1), \quad 1 \leq i \leq k$$

Cohomology of FM_n and coaction on G_A

$H^*(\mathsf{FM}_n)$ inherits a Hopf cooperad structure

One can rewrite:

$$\mathsf{G}_A(k) = (A^{\otimes k} \otimes H^*(\mathsf{FM}_n(k))/\text{relations}, d)$$

Proposition

$\chi(M) = 0 \implies \mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k \geq 0}$ is a Hopf right $H^*(\mathsf{FM}_n)$-comodule

Motivation

We are looking for something to put here:

$$\mathsf{G}_A(k) \xleftarrow{\sim} ? \xrightarrow{\sim} \Omega^*(\mathsf{FM}_M(k))$$

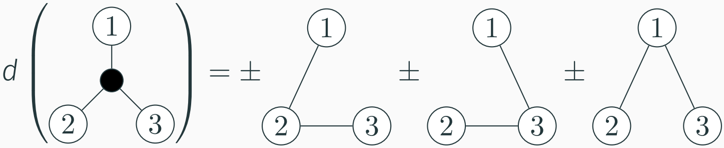
If true, then hopefully it fits in a diagram like this:

$$\begin{array}{ccccc}
\mathsf{G}_A & \xleftarrow{\quad\sim\quad} & ? & \xrightarrow{\quad\sim\quad} & \Omega^*(\mathsf{FM}_M) \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
H^*(\mathsf{FM}_n) & \xleftarrow{\quad\sim\quad} & ? & \xrightarrow{\quad\sim\quad} & \Omega^*(\mathsf{FM}_n)
\end{array}$$

$$\downarrow$$

Already known: formality of the little disks operads

[Kontsevich] Hopf cooperad $\texttt{Graphs}_n = \{\texttt{Graphs}_n(k)\}_{k\geq 0}$



Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

$$H^*(\mathsf{FM}_n;\mathbb{R}) \xleftarrow{\sim} \texttt{Graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$$

$\omega_{ij} \longleftarrow\!\!\shortmid$  $\longmapsto$ explicit representatives

$0 \longleftarrow\!\!\shortmid$  $\longmapsto$ “explicit” integrals

Complete version of the theorem

Idea

Build $\mathtt{Graphs}_R^{z_\varepsilon}$ from $\mathtt{Graphs}_n$ similar to how $\mathtt{G}_A$ is built from $H^*(\mathtt{FM}_n)$

Theorem (Complete version)

M: closed, simply connected, smooth manifold with $\dim \geq 4$

$$\begin{array}{ccccc} \mathtt{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R^{z_\varepsilon} & \overset{\sim}{\dashrightarrow} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M) \\ \circlearrowleft^\dagger & & \circlearrowleft^\dagger & & \circlearrowleft^\ddagger \\ H^*(\mathtt{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n) \end{array}$$

† When $\chi(M) = 0$

‡ When M is framed

$$A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(M)$$

Configuration Spaces of Manifolds with Boundary

Poincaré–Lefschetz duality models

Now: $\partial M \neq \varnothing \implies H^*(M) \cong H_{n-*}(M, \partial M)$ for M oriented

Poincaré–Lefschetz duality pair $(B \xrightarrow{\lambda} B_\partial)$:

- $(B_\partial, \varepsilon_\partial)$ Poincaré duality CDGA of dimension $n-1$; (models ∂M, $\int_{\partial M}$)
- B: fin. type connected CDGA; (models M)
- $\lambda : B \twoheadrightarrow B_\partial$: surjective CDGA morphism; (models $\partial M \hookrightarrow M$)
- $\varepsilon : B^n \to \mathbb{R}$ s.t. $\varepsilon(dy) = \varepsilon_\partial(\lambda(y))$; (models $\int_M(-)$ & Stokes formula)
- if $K = \ker \lambda$, then $\theta : B \to K^\vee[-n]$, $b \mapsto \varepsilon(b \cdot -)$ is a surjective quasi-isomorphism. ($K \simeq \Omega^*(M, \partial M)$)

In this case, $A := B / \ker \theta$ is a model of M, and $\theta : A \xrightarrow{\cong} K^\vee[-n]$

Existence & example of PLD models

Example

If $M = N \setminus \{*\}$ with N closed: take P a Poincaré duality model of N

$$B = (P \oplus \mathbb{R}v_{n-1}, dv = \mathrm{vol}_P) \twoheadrightarrow B_\partial = H^*(S^{n-1}) = (\mathbb{R} \oplus \mathbb{R}v_{n-1}, d = 0)$$

Proposition

If M is simply connected, ∂M is simply connected, and $\dim M \geq 7$, then $(M, \partial M)$ admits a PLD model.

Remark

Also true if M admits a “surjective pretty model”, cf. theorems of Cordova Bulens and Cordova Bulens–Lambrechts–Stanley.

The "naïve" dg-module G_A

Given a PLD model (B, B_∂) and $A = B/\ker\theta$, can build $\mathsf{G}_A(k)$ as before.

Theorem

$$\dim H^i(\mathrm{Conf}_k(M)) = \dim H^i(\mathsf{G}_A(k))$$

Idea of proof

Combine:

- Techniques of Lambrechts–Stanley to compute homology of spaces of the type $M^k \setminus \bigcup_{i\neq j} \Delta_{ij}$;
- Techniques of Cordova Bulens–L–S to compute homology of $M = N \setminus X$ where N is a closed manifold and $X \subset N$ is a sub-polyhedron.

The actual model

In general, $\mathsf{G}_A(k)$ is not actually a CDGA model for $\mathrm{Conf}_k(M)$.

Motivation

$$M = S^1 \times (0,1) \cong \mathbb{R}^2 \setminus \{0\} \implies \mathrm{Conf}_2(M) \simeq \mathrm{Conf}_3(\mathbb{R}^2)$$

Then $A = H^*(M) = \mathbb{R} \oplus \mathbb{R}\eta$. In $\mathsf{G}_A(2)$, relation $(1 \otimes \eta)\omega_{12} = (\eta \otimes 1)\omega_{12}$.
But in $\mathrm{Conf}_3(\mathbb{R}^2)$, Arnold relation: $(1 \otimes \eta)\omega_{12} = (\eta \otimes 1)\omega_{12} \pm (\eta \otimes \eta)$.

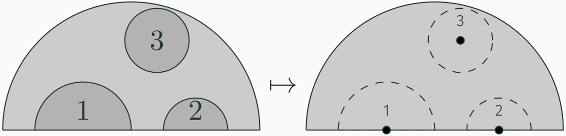
$\implies$ must define a "perturbed model" $\tilde{\mathsf{G}}_A(k)$

Proposition

Isomorphism of dg-modules $\mathsf{G}_A(k) \cong \tilde{\mathsf{G}}_A(k)$.

Swiss-Cheese & graphs

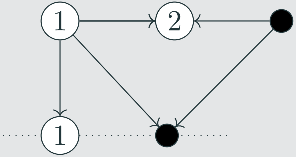
M looks like $\mathbb{H}^n$ (locally) $\implies$ Swiss-Cheese operad



$\mathsf{SC}_2^{\mathfrak{o}}(2,1)$ $\qquad$ $\mathrm{Conf}_{2,1}(\mathbb{H}^2)$

Theorem (Willwacher 2015)

Model $\mathsf{SGraphs}_n$ for $\mathsf{SFM}_n = \overline{\mathrm{Conf}_{\bullet,\bullet}(\mathbb{H}^n)} \simeq \mathsf{SC}_n$:



$\in \mathsf{SGraphs}_n(1,2)$

Theorem for manifolds with boundary

Using similar techniques:

Theorem

For M a smooth, compact manifold of dimension at least ≥ 7, M and ∂M simply connected:

$$\begin{array}{ccccc} \tilde{\mathsf{G}}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R^{\mathbf{z}_\varepsilon} & \overset{\sim}{\dashrightarrow} & \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\varnothing, -)) \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft^{\dagger} \\ H^*(\mathsf{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n) \end{array}$$

Moreover: model $\mathtt{SGraphs}_{R,R_\partial}^{c_M,\mathbf{z}_\varphi^S}(k,l)$ of $\mathsf{SFM}_M(k,l)$, compatible with the (co)action of $\mathtt{SGraphs}_n$ / SFM_n

Fin de la présentation