Configuration Spaces and Graph Complexes

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Configuration Spaces

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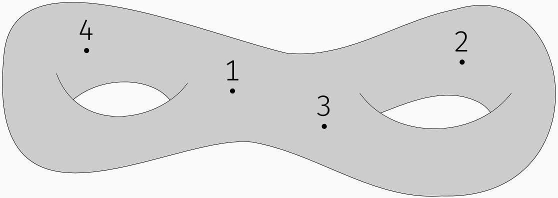
$$\mathrm{Conf}_k(M) := \{(x_1, \ldots, x_k) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



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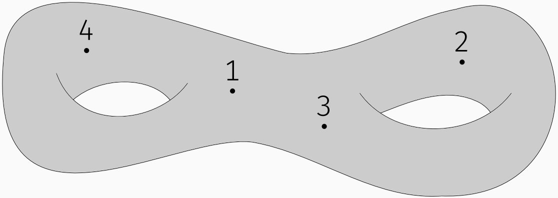

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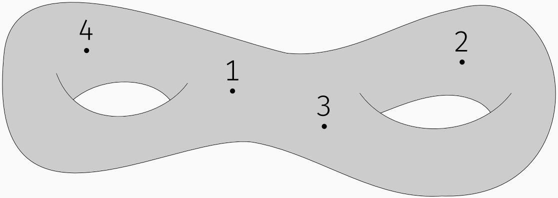

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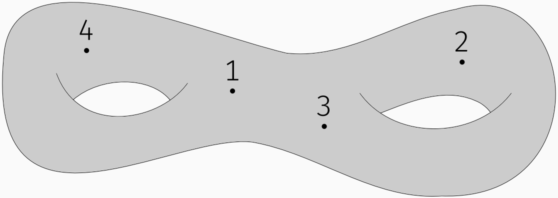

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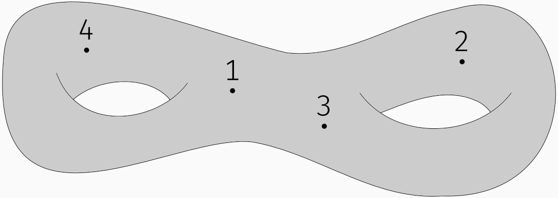

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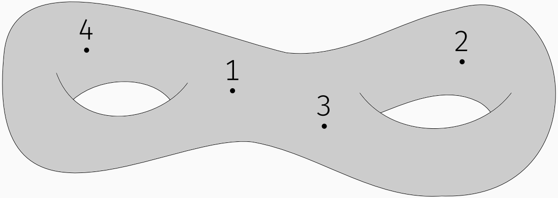

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General context: Homotopy Theory

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Real homotopy theory: up to homotopy and "modulo torsion".
$\to$ Sullivan's theory (1977): real homotopy type of M is determined by the algebra of de Rham forms $\Omega^*_{\mathrm{dR}}(M)$.

An open problem

Question

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Closed manifolds

Longoni–Salvatore (2005): example of $M \sim N$ s.t. $\mathrm{Conf}_k(M) \not\sim \mathrm{Conf}_k(N)$,

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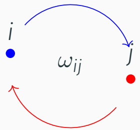
Simply connected closed manifolds

Homotopy invariance is still open!

M is locally $\mathbb{R}^n \to$ presentation of $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ due to Arnold and Cohen:



- Generators: ω_{ij}, $1 \leq i \neq j \leq k$
- Relations:

$$\omega_{ij}^2 = \omega_{ji} - (-1)^n \omega_{ij} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$$

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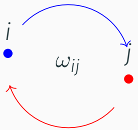

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Theorem (Arnold 1969)

Formality: $H^*(\mathrm{Conf}_k(\mathbb{C})) \sim_{\mathbb{C}} \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_k(\mathbb{C}))$, $\omega_{ij} \mapsto \mathrm{d}\log(z_i - z_j)$.

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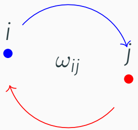

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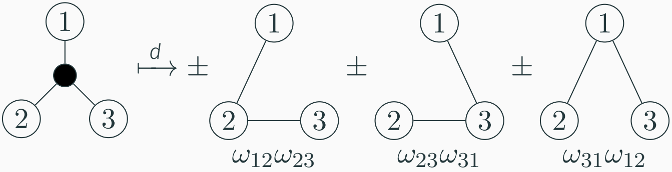
Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \sim_{\mathbb{R}} \Omega^*_{\mathrm{dR}}(\mathrm{Conf}_k(\mathbb{R}^n))$ for all k and all $n \geq 2$.

[Kontsevich] $\mathtt{Graphs}_n(k)$:



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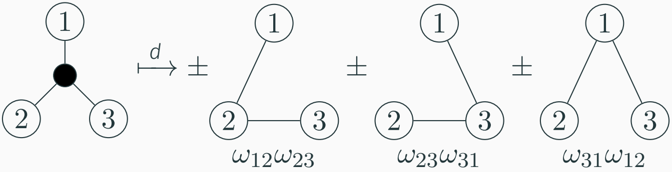
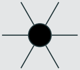


Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n);\mathbb{R}) \xleftarrow{\sim} \mathtt{Graphs}_n(k) \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_k(\mathbb{R}^n))$$

$\omega_{ij} \longleftarrow\!\!\!\shortmid$  $\longmapsto$ explicit representatives

$0 \longleftarrow\!\!\!\shortmid$  $\longmapsto$ "explicit" integrals

Compactification

Problem: Conf_k is not compact – why does $\int$ converge?

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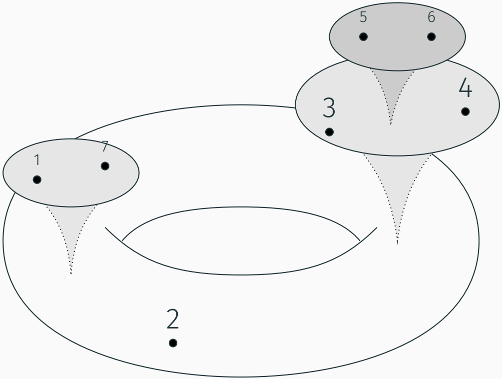
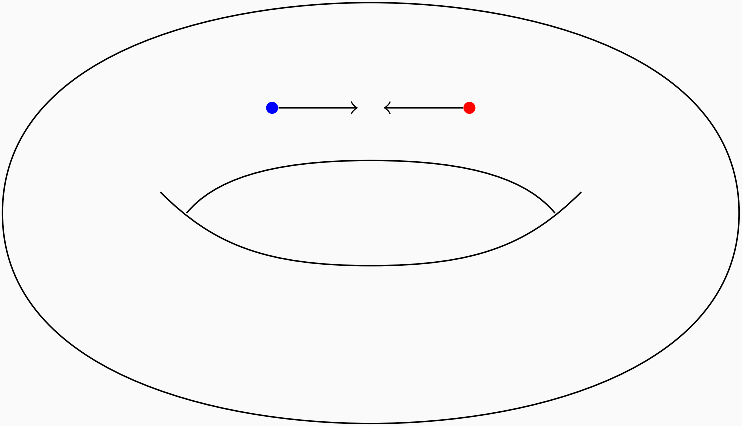
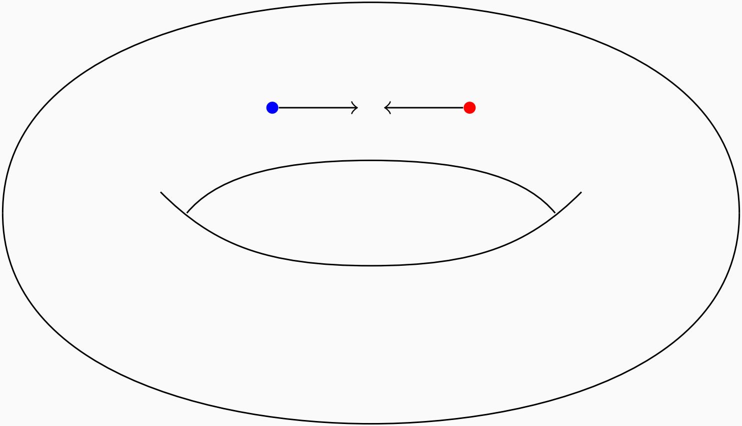
Fulton–MacPherson compactification $\mathrm{Conf}_k(M) \xrightarrow{\sim} \mathsf{FM}_M(k)$





Animation #1



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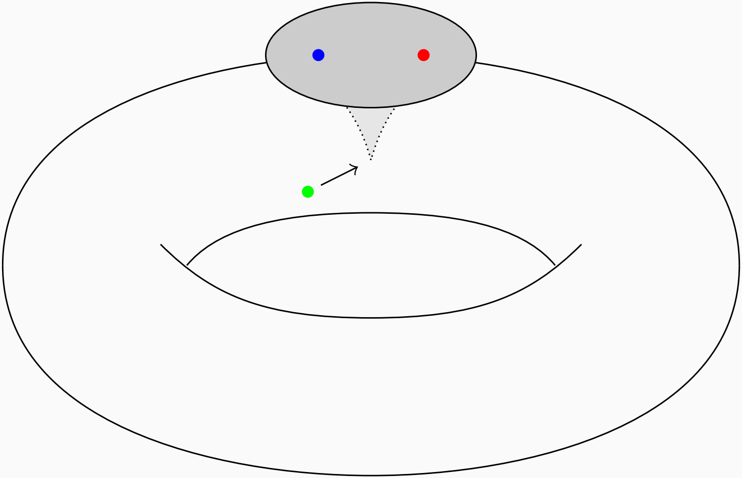

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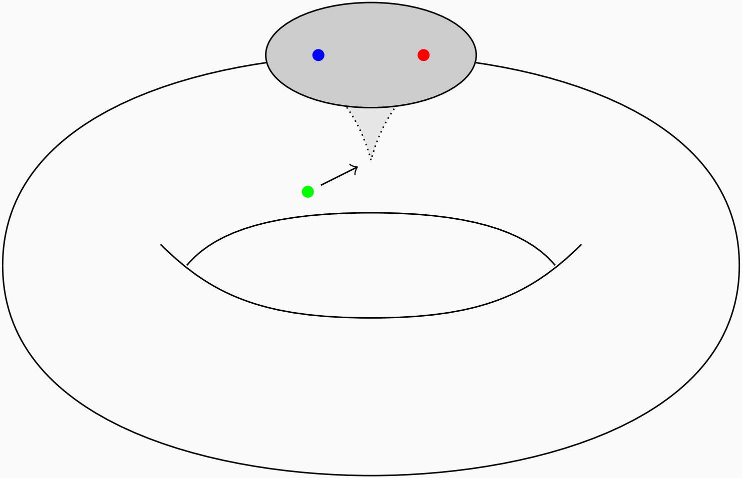

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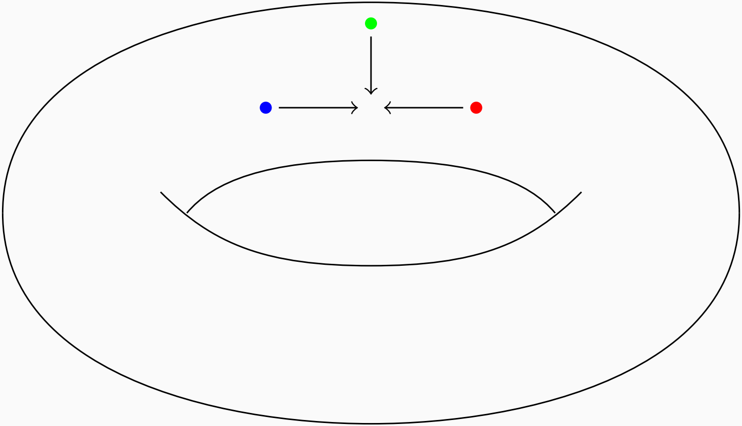

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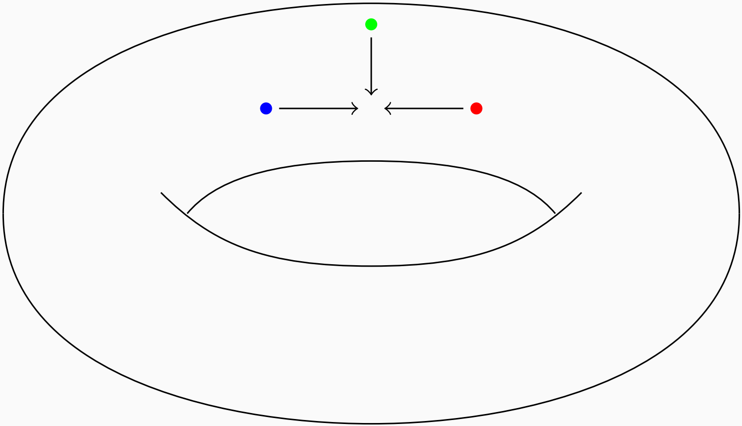

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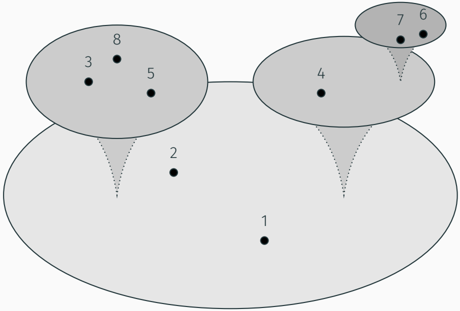
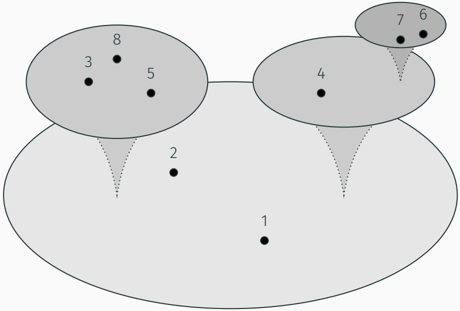


We also have $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\sim} \mathrm{Conf}_k(\mathbb{R}^n)/(\mathbb{R}^n \rtimes \mathbb{R}_{>0}) \xhookrightarrow{\sim} \mathsf{FM}_n(k)$



(+ normalization because $\mathbb{R}^n$ is not compact)

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- $k \geq 3$: more complicated.

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- **2015** [Cordova Bulens] model of $\mathrm{Conf}_2(M)$ if $\dim M = 2m$

First part of the theorem

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Let M be a simply connected smooth closed manifold. Then $\mathsf{G}_A(k)$ is a model over $\mathbb{R}$ of $\mathrm{Conf}_k(M)$ for all $k \geq 0$.

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We can "compute everything" (over $\mathbb{R}$) for $\mathrm{Conf}_k(M)$.

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Let M be a simply connected smooth closed manifold. Then $\mathsf{G}_A(k)$ is a model over $\mathbb{R}$ of $\mathrm{Conf}_k(M)$ for all $k \geq 0$.

Corollaries

$M \sim_{\mathbb{R}} N \implies \mathrm{Conf}_k(M) \sim_{\mathbb{R}} \mathrm{Conf}_k(N)$ for all k.

We can "compute everything" (over $\mathbb{R}$) for $\mathrm{Conf}_k(M)$.

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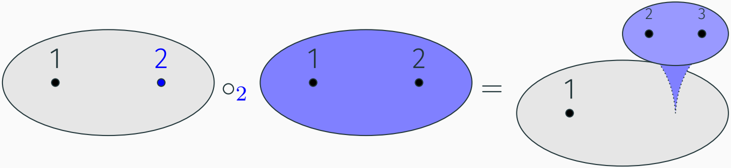
$\mathsf{FM}_n = \{\mathsf{FM}_n(k)\}_{k\geq 0}$ is an operad: we can "insert" a configuration into another:



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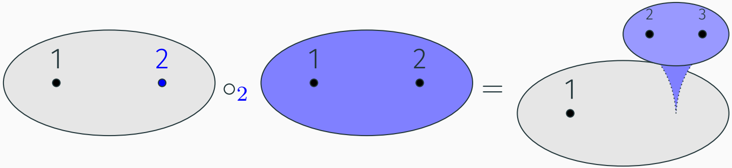

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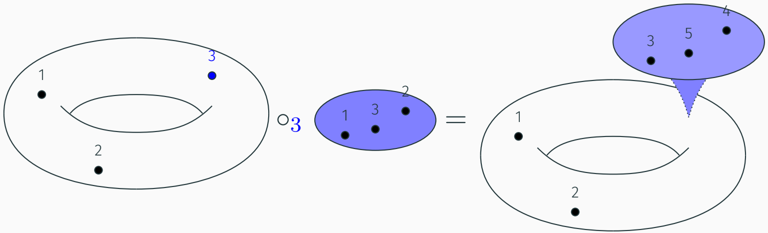
Remark

Equivalent in homotopy to the "little disks operad".

M parallelized $\implies$ $\mathsf{FM}_M = \{\mathsf{FM}_M(k)\}_{k\geq 0}$ is a right FM_n-module: we can insert an infinitesimal configuration into a configuration of M:



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By some abstract nonsense:

Proposition

$\chi(M) = 0 \implies \mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k \geq 0}$ is a right $H^*(\mathsf{FM}_n)$-comodule.

Theorem (I. 2016)

M: simply connected smooth closed manifold, $\dim M \geq 4$

$$\begin{array}{ccccc} \mathsf{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R & \overset{\sim}{\dashrightarrow} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M) \\ \circlearrowleft^{\dagger} & & \circlearrowleft^{\dagger} & & \circlearrowleft^{\ddagger} \\ H^*(\mathsf{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n) \end{array}$$

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$A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(M)$

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Upshot

We have a model for each $\mathrm{Conf}_k(M)$ + richer structure if we consider all of them together.

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Manifolds with boundary: homotopy invariance + generalization of Lambrechts–Stanley model (+ more!) under good conditions.

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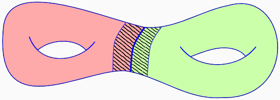

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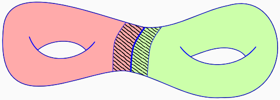

Work in progress j/w Campos, Ducoulombier, Willwacher

Model for framed configurations of points: get a module structure even if the manifold is not parallelized.

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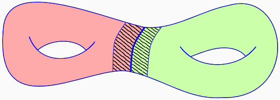

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Model for framed configurations of points: get a module structure even if the manifold is not parallelized.

Allows to compute spaces of embeddings of manifolds and/or factorization homology for more general manifolds (see next slide).

Applications

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Thank you for your attention!

These slides, links to papers: `https://idrissi.eu`