Models for configuration spaces of manifolds

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1 Introduction

1.1 Configuration spaces

$$\mathrm{Conf}_X(r) = \{(x_1, \dots, x_r) \in X^r \mid \forall i \neq j,\ x_i \neq x_j\}.$$

Numerous uses: • braid groups; • iterated loop spaces; • moduli spaces of complex curves; • manifold calculus; • Gelfand–Fuks cohomology $H^*_{cont}(\Gamma_c(M, TM))$; • stable splitting of $\mathrm{Map}(A, X)$; • particles in motion in physics; • motions planning…

For these applications, knowing the homotopy type of these configuration spaces is important.

Conjecture. *If $M \simeq N$ are two closed simply connected manifolds, then $\mathrm{Conf}_M(r) \simeq \mathrm{Conf}_N(r)$.*

1.2 Operadic structures

For many applications it is essential to also know the operadic structures of configuration spaces.

Definition. Operad: $\mathsf{Disk}_n(r) = \mathrm{Emb}(\bigsqcup^r \mathbb{R}^n, \mathbb{R}^n)$, $\mathsf{Disk}_n \simeq D_n \rtimes O(n)$

Definition. Right module: $\mathsf{Disk}_M(r) = \mathrm{Emb}(\bigsqcup^r \mathbb{R}^n, M)$, $\mathsf{Disk}_M(r) \simeq \mathrm{Conf}^{fr}_M(r)$.

Remark. Variants: $\mathsf{Disk}^{fr}_n \simeq D_n$, $\mathsf{Disk}^{fr}_M(r) \simeq \mathrm{Conf}_M(r)$.

1.3 Rational homotopy theory

Focus on rational homotopy type of spaces.

Thanks to Sullivan's theory, purely algebraic: $M \simeq_{\mathbb{Q}} N \iff \Omega^*_{PL}(M) \simeq \Omega^*_{PL}(N)$.

The same theory has been developed for operads and for right modules over operads (even if Ω^*_{PL} is lax, not colax).

2 Closed manifolds

2.1 Building block: $\mathbb{R}^n$

Well-known [Arnold, Cohen]: $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) = S(\omega_{ij})/(\dots)$.

Theorem (Arnold)**.** *$\mathrm{Conf}_{\mathbb{R}^2}(r)$ is formal: $H^*(\mathrm{Conf}_{\mathbb{R}^2}(r)) \simeq \Omega^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$, $\omega_{ij} \mapsto d\log(z_i - z_j)$*

Two questions: true for higher n? True for the operad?

Theorem (Tamarkin, Kontsevich, Lambrechts–Volić, Fresse–Willwacher, Petersen, Boavida–Horel)**.** *The little n-disks operad is formal: $H^*(\mathsf{Disk}^{fr}_{\mathbb{R}^n}) \simeq \Omega^*(\mathsf{Disk}^{fr}_{\mathbb{R}^n})$.*

Kontsevich's proof:

$$H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) \leftarrow \mathsf{Graphs}_n(r) \to \Omega^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$$

Key: set all internal components to zero thanks to computations of integrals over configuration spaces.

2.2 Lambrechts–Stanley model

Model conjectured by Lambrechts–Stanley (based on earlier works by Cohen–Taylor, Bendersky–Gitler, Kriz...). Closed manifold M, model A that satisfies Poincaré duality on the nose $\implies \mathsf{G}_A(r) = \big(A^{\otimes r} \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))/..., d\big)$

Theorem. *For any smooth simply connected closed manifold M, for any Poincaré duality model A, $\mathsf{G}_A(r)$ is a real model of $\mathrm{Conf}_M(r)$. For $n \geq 4$ and framed manifolds, compatible with operadic structure.*

Corollary (I, CW)**.** *Real homotopy invariance of $\mathrm{Conf}_{(-)}(r)$ for this class of manifolds.*

For $n \geq 3$ this is easy (only spheres). For $n \geq 4$, the proof is inspired by Kontsevich's proof. Roughly, build a zigzag:

$$\mathsf{G}_A \leftarrow \mathsf{Graphs}_R \to \Omega^*(\mathrm{Conf}_M).$$

Key point: check that internal component get sent to zero ($\approx$ triviality of Chern–Simons invariants on such manifolds).

2.3 Framed configurations

What to do when M is not framed?

$$\mathrm{Conf}_M(r) = \{(x_1, \dots, x_r, B_1, \dots, B_r) \in \mathrm{Conf}_M(r) \times Fr_M^r \mid B_i : \text{ basis of } T_{x_i}M\}$$

Already difficult for the operad:

Theorem (Ševera $n = 2$, Giansiracusa–Salvatore $n = 2$; Moriya odd n; Khoroshkin–Willwacher)**.** *The framed little disks operads $\mathsf{Disk}_{\mathbb{R}^n}$, $\mathsf{Disk}^{or}_{\mathbb{R}^n}$ are formal for even n; $\mathsf{Disk}^{or}_{\mathbb{R}^n}$ is not formal for odd n.*

The case $n = 2$ is simpler because one can find explicit 1-form vol_{S^1} in $\Omega^*(SO(2))$. Idea of the KW proof: build a graph complex model $\mathsf{BGraphs}_n$ (graphs decorated by cohomology of $BSO(n)$) that depends on integrals m (seen as a MC element in a certain deformation complex). Use obstruction theory to show trivial in even case, nontrivial in odd case.

Adapt this for closed manifolds

Theorem (CW $n = 2$; CDIW)**.** *Graphical model $\mathsf{BGraphs}_M$ for Disk^{or}_M for a smooth oriented closed manifold M, given by graphs decorated by $H^*(M)$ and by $H^*(BSO(n))$.*

Problem: non-explicit Maurer–Cartan element (= value of internal components).

3 Manifolds with boundary

3.1 Gluing

M: compact manifold with boundary $\partial M = N$

Then $\mathsf{Disk}_{N \times \mathbb{R}}$ is a (homotopy) algebra in the category of right Disk_n-modules (see *picture*) and Disk_M is a (homotopy) module over this algebra, AKA a "boundary module".

Proposition. *If $\partial M = \partial M' = N$ then $\mathsf{Disk}_{M \cup_N M'} = \mathsf{Disk}_M \otimes_{\mathsf{Disk}_{N \times \mathbb{R}}} \mathsf{Disk}_{M'}$.*

This can be used to find models "inductively".

3.2 Graphical models

Theorem (CILW). *Graphical model* $\mathsf{aGraphs}_N$ *for* $\mathsf{Disk}^{or}_{N\times\mathbb{R}}$*, compatible with the right module and the algebra stuctures. Only depends on the real homotopy type of* N*, without conditions.*

Oriented graphs decorated by a model of N, right comodule structure as usual, coalgebra structure : cut graph in two, replace edge by half of diagonal class.

Theorem (CILW). *Graphical model* $\mathsf{mGraphs}_M$ *for* Disk^{or}_M*, compatible with the right* Disk_n *module and the* $\mathsf{Disk}_{N\times\mathbb{R}}$ *module stuctures. Only depends on the real homotopy type of* M *if* $\dim M \geq 4$ *and* M *is simply connected.*

Similar description, cut graphs in two and multiply by a lift of half the diagonal class.

Remark. If $\dim M \leq 3$ or M is not simply connected: depends on non-explicit integrals.

3.3 Perturbed LS model

Theorem (CILW). *Quotient of* $\mathsf{aGraphs}_M$ *is a small, Lambrechts–Stanley-like model for configuration spaces of* M*. Uses a Poincaré–Lefschetz duality model of* $(M, \partial M)$*. Valid if* M *and* ∂M *are simply connected and* $\dim M \geq 7$*; also true if* $4 \leq \dim M \leq 6$ *if we use* $H^*(M)$ *as the model.*

4 Surfaces

4.1 Splitting

Any oriented surface Σ_g can be split as union of handles: *picture.*

Let $A = \mathsf{Disk}^{or}_{S^1\times\mathbb{R}}$ (algebra in right Disk^{or}_2 modules) and $M = \mathsf{Disk}_{S^2\setminus\bigsqcup^{2g} D^2}$ (boundary module). Then $\mathsf{Disk}^{or}_{\Sigma_g}$ is a kind of iterated Hoschild complex $\bigotimes_A^{(1,2)\dots(2g-1,2g)} M$.

To modelize this algebraically, we need to know a model of $\mathsf{Disk}^{or}_{S^2\setminus\bigsqcup^{2g} D^2}$ and a model of $\mathsf{Disk}^{or}_{S^1\times\mathbb{R}}$. In both cases, they can be deduced from a model of Disk^{or}_2 *as a cyclic operad* to modelize the action on the right or the left.

4.2 Cyclic formality of E_2

Theorem (CIW, extendending Ševera, Giansiracusa–Salvatore). *The framed little disks operad* Disk^{or}_2 *is formal as a cyclic operad.*

Recall Tamarkin's proof of the formality of E_2:

$$H^*(E_2) \leftarrow C^*_{CE}(\mathfrak{t}) \rightarrow \Omega^*(N_\bullet PaB).$$

Here $\mathfrak{t}$ is the Drinfeld–Kohno Lie algebra and PaB is the operad of parenthesized braids. The left map is a quotient map. The right map is built out of a Drinfeld associator:

$$N_\bullet PaB \xrightarrow{\Phi} B_\bullet \mathbb{G}\hat{\mathbb{U}}\mathfrak{t} \hookrightarrow \sim MC_\bullet(\mathfrak{t}) = \langle C^*_{CE}(\mathfrak{t})\rangle$$

This was adapted by Ševera to show the rational formality of E_2^{fr}:

$$N_\bullet PaRB \xrightarrow{\Phi} B_\bullet \mathbb{G}\hat{\mathbb{U}}\mathfrak{t}^R \hookrightarrow \sim MC_\bullet(\mathfrak{t}^R) = \langle C^*_{CE}(\mathfrak{t}^R)\rangle$$

where $\mathfrak{t}^R$ is the framed Drinfeld–Kohno algebra (add new generators t_{ii}, $R \mapsto e^{t_{11}/2/}$.

We just check that this is compatible with the cyclic structure and that when applying Ω^* the result is a quasi-isomorphism.

Corollary. *The algebra* $\mathsf{Disk}^{or}_{S^1\times\mathbb{R}}$ *is formal, the module* $\mathsf{Disk}^{or}_{S^2\setminus\dots}$ *is formal (consider fibers).*

A model of $\mathsf{Disk}^{or}_{\Sigma_g}$ is thus the iterated Hochschild complex $\bigotimes_{BV_1^\vee}^{(1,2)\dots(2g-1,2g)} BV^\vee_{(g,g)}$.

4.3 The model

Model: framed version of the Lambrechts–Stanley model

$$\mathsf{G}^{fr}_{S^2}(r) = \left(H^*(\Sigma_g)^{\otimes r} \otimes BV^\vee(r)/(\dots), d\omega_{ij} = \Delta_{ij}, d\theta_i = 2\nu_i\right)$$

Theorem (CIW). *This is a model for* $\mathsf{Disk}^{or}_{\Sigma_g}$ *as a right* Disk^{or}_2 *module.*

Sketch of proof:

$$\mathsf{G}^{fr}_{S^2} \xleftarrow[\text{quot}]{\sim} \mathsf{BGraphs}^{triv}_{\Sigma_g} \xrightarrow{\omega} \bigotimes_{BV_1^\vee}^{(1,2)\dots(2g-1,2g)} BV^\vee_{(g,g)} \simeq \Omega^*(\mathsf{Disk}^{or}_{\Sigma_g}).$$

The first map is a quotient map and is a quasi-iso by combinatorial argument. The last equivalence follows from cyclic formality of Disk^{or}_2.

The middle map is the key. We find an explicit "propagator" in the iterated Hochschild complex and we define a combinatorial "fiber integral" map. The definition is then formally analogous to Kontsevich's formality map.

4.4 Interpretation

Thanks to Felder's results, the above theorem implies that the partition function on Σ_g is gauge trivial. Hence the Poisson σ model on Σ_g can be explicitly computed. We can also view this as a section of $GRT_1 \to GRT$ (but not for higher genus due to a lack of functoriality in the choice of base point).