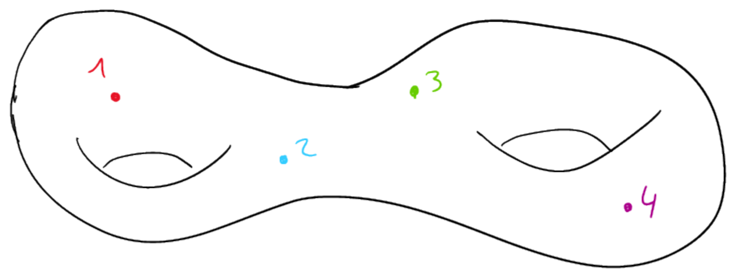
M : manifold of dimension n

$\rightsquigarrow \quad \mathrm{Conf}_M(r) := \{(x_1, \dots, x_r) \in M^r \mid \forall i \neq j,\ x_i \neq x_j\}$



Classical object in algebraic topology!

Braid groups

$\tau \in B_r \mapsto$ path in $\mathrm{Conf}_{D^2}(r)$

$\mathrm{Conf}_{D^2}(r) \simeq K(PB_r, 1)$

More generally, $\mathrm{Conf}_\Sigma(r) \simeq K(\dots, 1)$

surface braid group

Iterated loop spaces

$\Omega^n X := \{\gamma : D^n \to X \mid \gamma(\partial D^n) = \{*\}\}$

Has an algebraic structure based on operads built out of Conf_{D^n} (Boardman-Vogt, May)

Goodwillie - Weiss manifold calculus

Goal : compute $\mathrm{Emb}(M, N) = \{ f : M \hookrightarrow N \} \subset \mathrm{Map}(M, N)$

Under good conditions, approximated by a subspace of

$$\prod_{n \geq 0} \mathrm{Map}(\mathrm{Conf}_M(n), \mathrm{Conf}_N(n))$$

Gelfand - Fuks cohomology

Characteristic classes of foliations live in

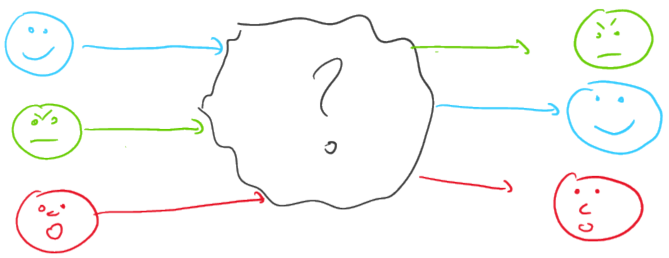
$$H^*_{GF}(M) := H^*_{cont}(\Gamma_c(M, TM))$$

Lie algebra

[Cohen - Taylor] computed by a spectral sequence involving Conf_M

Motion planning

Want to move robots / drones / trains / ... at the same time in some space



$\Longleftrightarrow$ find a section of

$$\begin{tikzcd} \mathrm{Map}([0,1], \mathrm{Conf}_M(n)) \arrow[r] & \mathrm{Conf}_M(n) \times \mathrm{Conf}_M(n) \arrow[l, bend right, dashed, "\exists ? \sigma"'] \\ \gamma \arrow[r, mapsto] & (\gamma(0), \gamma(1)) \end{tikzcd}$$

The minimum number of domains of continuity of σ is called topological complexity and depends on the homotopy type of Conf_M (cf Farber)

Homotopy type

We want the homotopy type of configuration spaces conjecture (long standing) $\forall M, N$ simply connected closed manifolds,

$M \simeq N \implies \mathrm{Conf}_M(r) \simeq \mathrm{Conf}_N(r)$

Rk · Wrong for open manifolds: $\mathbb{R} \simeq \{0\}$ but $\underbrace{\mathrm{Conf}_{\mathbb{R}}(2)}_{\simeq S^0} \not\simeq \underbrace{\mathrm{Conf}_{\{0\}}(2)}_{= \emptyset}$

· For non simply connected manifolds: counter-example of Longoni-Salvatore

$L_{p,q}$ lens space $\quad \mathrm{Conf}_{L_{7,1}}(2) \not\simeq \mathrm{Conf}_{L_{7,2}}(2)$

$S^3/(\mathbb{Z}/p\mathbb{Z})$

Some evidence

· $H_*(\mathrm{Conf}_M(r))$ Bödigheimer-Cohen-Taylor, Bendersky-Gitler
(w/ some hypotheses: dimension, coefficients)

· $\Omega\, \mathrm{Conf}_M(r)$ Levitt $\quad (\implies \pi_* \mathrm{Conf}_M)$

· $\Sigma^\infty \mathrm{Conf}_M(r)$ Aouina-Klein

Rational homotopy

Rational homotopy equiv: $f: M \to N$ s.t. $\pi_* f \otimes \mathbb{Q}$ isomorphism

$M \simeq_{\mathbb{Q}} N$ if $M \xleftarrow{\sim_{\mathbb{Q}}} \cdot \xrightarrow{\sim_{\mathbb{Q}}} \cdots \xleftarrow{\sim_{\mathbb{Q}}} N$

Sullivan's theory: $M \simeq_{\mathbb{Q}} N \iff \Omega^*(M) \simeq_{CDGA} \Omega^*(N)$

de Rham,
piecewise linear...

$\rightsquigarrow$ model of M: CDGA $\simeq \Omega^*(M)$

can compute everything about RHT from a model

Goal Find a model of Conf_M from a model of M

Closed manifolds

Basic building block: $\mathbb{R}^n$

THM [Arnold, Cohen] $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ has the presentation:

Generators: ω_{ij} for $1 \leq i \neq j \leq r$, $\deg \omega_{ij} = n-1$

↪ counts

Relations: $\omega_{ij}^2 = 0$

$\omega_{ji} = (-1)^n \omega_{ij}$

$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$

THM [Arnold] $\mathrm{Conf}_{\mathbb{R}^2}(r) = \mathrm{Conf}_{\mathbb{C}}(r)$ is formal, ie $\Omega^*(\mathrm{Conf}_{\mathbb{C}}(r)) \simeq H^*(\mathrm{Conf}_{\mathbb{C}}(r))$

proof:
$$H^*(\mathrm{Conf}_{\mathbb{C}}(r)) \xrightarrow{\sim} \Omega^*_{dR}(\mathrm{Conf}_{\mathbb{C}}(r); \mathbb{C})$$
$$\omega_{ij} \longmapsto d\log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}$$

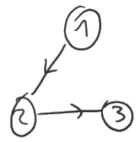
No hope of adapting this proof for higher n.
But!

THM [Kontsevich, Lambrechts-Volić] $\mathrm{Conf}_{\mathbb{R}^n}(r)$ is formal $\forall n, r$

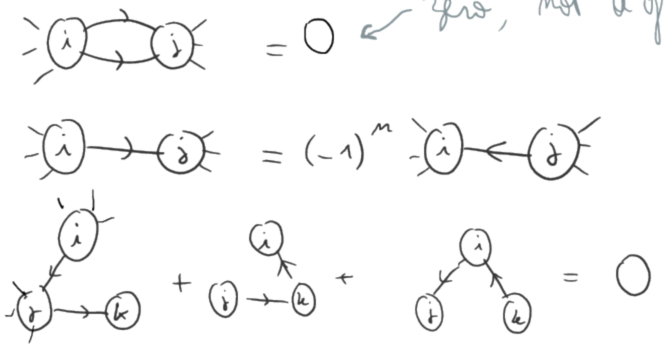
$\Rightarrow$ the RHT of $\mathrm{Conf}_{\mathbb{R}^n}(r)$ can be computed from $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$
("homotopy is a formal consequence of cohomology")

Idea of the proof

We want $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) \xleftarrow{\sim} \cdots \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$

View $\Gamma \in H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ as linear combinations of graphs on r vertices : eg $\omega_{12}\,\omega_{23} =$  $\in H^*(\mathrm{Conf}_{\mathbb{R}^n}(3))$

modulo local relations :

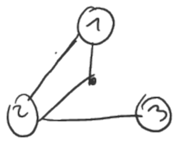


zero, not a graph!

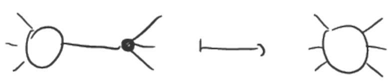
$\Longrightarrow$ replace the last relation by a differential !

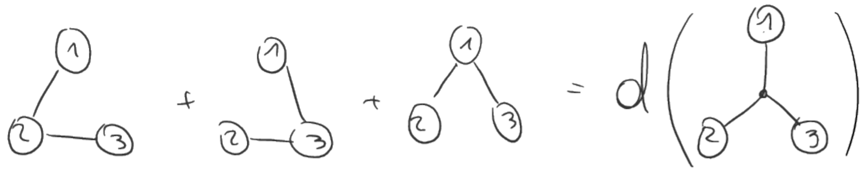
$H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) \xleftarrow[p]{\sim} \mathrm{Graphs}_n(r) \xrightarrow[I]{\sim} \Omega^*(\mathrm{Conf}_{\mathbb{R}}(r))$

In Graphs_n : r external vertices + undistinguishable internal vertices

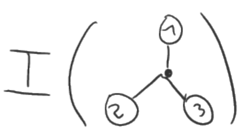


$\in \mathrm{Graphs}_n(3)$

diff : $\sum$ 

ex 

$I : \mathrm{Graphs}_n(r) \to \Omega^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ given by integrals along fibers

ex : 

$= \int_{\mathrm{Conf}_{\mathbb{R}^n}(4) \to \mathrm{Conf}_{\mathbb{R}^n}(3)} \omega_{14} \wedge \omega_{24} \wedge \omega_{34}$

$$\frac{1}{\ } \left(\quad \textcircled{2} \quad \textcircled{3} \right) = \int_{\mathrm{Conf}_{\mathbb{R}^n}(4) \to \mathrm{Conf}_{\mathbb{R}^n}(3)}$$

Key point: must mod out by components w/ just internal vertices

$\hookrightarrow$ must check that $I(\cdots\cdots \quad) = 0$

Lambrechts - Stanley model

M : oriented closed manifold

A : model of M that satisfies Poincaré duality on the nose $A^k \otimes A^{m-k} \to \mathbb{R}$ non degen, $a \otimes b \mapsto \varepsilon(ab)$

$\rightsquigarrow$ $G_A(n)$: model inspired by $\mathrm{Conf}_M(n) = M^n \setminus \bigcup_{i \neq j} \Delta_{ij}$

- generators : $A^{\otimes n}$ + the ω_{ij} from before
- relations : same relations for ω_{ij}
  - symmetry : $p_i^*(a)\, \omega_{ij} = p_j^*(a)\, \omega_{ij}$
- differential : d_A + $d\omega_{ij}$ kills Poincaré dual of $\Delta_{ij} = \{x_i = x_j\}$

ex $G_A(0) = \mathbb{R}$ model of $\mathrm{Conf}_0(M) = \{\varnothing\}$ empty configuration

$G_A(1) = A$ model of M by hypothesis

$$G_A(2) = \left(A^{\otimes 2} \oplus A^{\otimes 2} \otimes \omega_{12} / (a \otimes 1 \otimes \omega_{12} = 1 \otimes a \otimes \omega_{12}) \,,\ d\omega_{12} = \Delta_A = \sum a_i \otimes a_i^\vee \right)$$

$$\cong (A^{\otimes 2} \oplus A \otimes \omega_{12} \,,\ d\omega_{12} = \Delta_A) \quad \overset{\cdot}{\underset{\mathbb{R}}{\otimes}} A$$

$$= \mathrm{cone}\left(A \xrightarrow{\cdot \Delta_A} A^{\otimes 2} \right) \simeq A^{\otimes 2} / (\Delta_A) \quad \text{model (?) of } M^2 \setminus \Delta$$

THM [I., see also Campos-Willwacher]

M : simply connected closed smooth manifold
A : any Poincaré duality model of M
$\Rightarrow G_A(n) \simeq \Omega^*(\mathrm{Conf}_M(n)) \ \forall n$

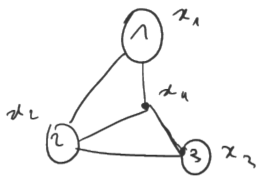
Cor $M \simeq_{\mathbb{R}} N \implies \mathrm{Conf}_M(n) \simeq_{\mathbb{R}} \mathrm{Conf}_N(n)$

Cor $M \simeq_{\mathbb{R}} N \Rightarrow \mathrm{Conf}_M(n) \simeq_{\mathbb{R}} \mathrm{Conf}_N(n)$

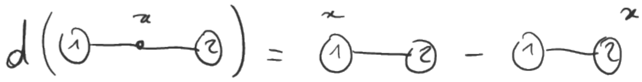
Proof

Inspired by Kontsevich's ideas: $G_A(n) \leftarrow \mathrm{Graphs}_R(n) \longrightarrow \Omega^*(\mathrm{Conf}_M(n))$

consider decorated graphs where R is some resolution of A



ex



Key point: need $I(\dots, \boxtimes) = 0$

$\Rightarrow$ we need $\dim M \geq 4$ and $\pi_1 M = 0$ for the arguments (just count degrees of graphs + reduce to trivalent case)

$\dim M \leq 3 \Rightarrow M \in \{*, S^2, S^3\} \Rightarrow$ different methods

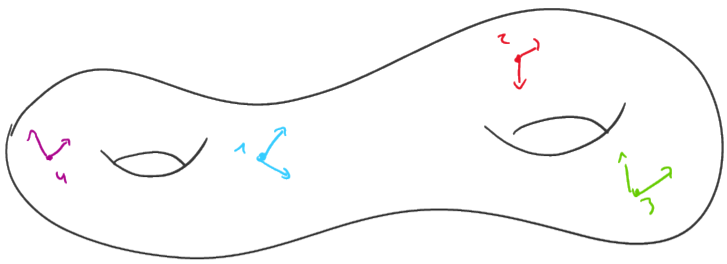
$\hookrightarrow$ Poincaré conjecture!

Bonus: $\mathrm{Graphs}_R(n)$ is also a model (focus of [CW])

$\Rightarrow$ free as an algebra, good for homological algebra

Framed configurations

$$\mathrm{Conf}^{fr}_M(n) = \left\{ (x_1, \dots, x_n, \xi_1, \dots, \xi_n) \;\middle|\; \begin{array}{l} (x_1, \dots, x_n) \in \mathrm{Conf}_M(n) \\ \xi_i : \text{oriented basis of } T_{x_i}M \end{array} \right\}$$



$\in \mathrm{Conf}^{fr}_{\Sigma_2}(4)$

⇒ useful for applications, but more complicated (already for $M = \mathbb{R}^n$)

THM [Campos - Ducoulombier - I - Willwacher]

Graphical model $\mathsf{Graphs}^{fr}_{M}(n)$ for (oriented) $\mathrm{Conf}^{fr}_{M}(n)$

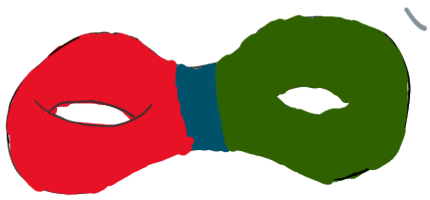
↪ graphs decorated by model of M & $H^*(BSO(n))$

Problems: · depends on non-explicit integrals: cannot get rid of 

· no homotopy invariance (yet?)

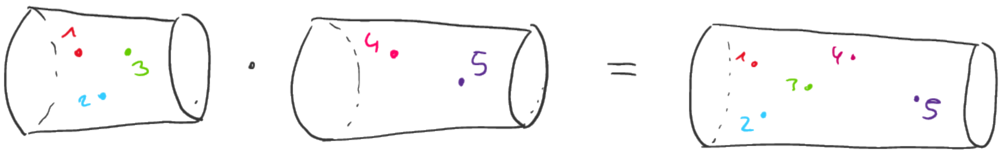
Manifolds w/ boundary

Goal: $M = M' \cup_{N\times\mathbb{R}} M''$

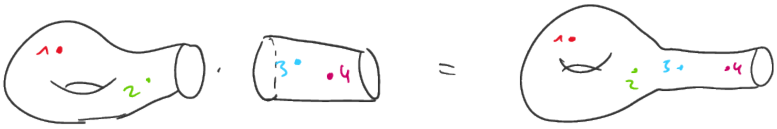


⇒ Want to compute configuration space "by induction"

$\mathrm{Conf}_{N\times\mathbb{R}}$ is a monoid (up to homotopy)



$\mathrm{Conf}_{M'}$ is a left module, $\mathrm{Conf}_{M''}$ is a right module (up to homotopy)



Formula: $\mathrm{Conf}_M \simeq \mathrm{Conf}_{M'} \otimes^{\mathbb{L}}_{\mathrm{Conf}_{N\times\mathbb{R}}} \mathrm{Conf}_{M''}$

THM (Campos - I - Lambrecht - Willwacher)
Graphical model for the monoid $\mathrm{Conf}_{N\times\mathbb{R}}$
+ homotopy invariance

Rk no condition on N! crossing w/ $\mathbb{R}$ simplifies a lot
(eg Raptis - Salvatore : $\mathrm{Conf}_2(- \times X)$ htpy inv for contractible $X \neq *$)

THM [CILW] Graphical model for the module $\mathrm{Conf}_{M'}$. $(M', \partial M')$
Homotopy invariance for $\dim M \geq 4$ + $\pi_1 M = 0$
(otherwise, cannot get rid of $\boxtimes$ ⇝ depends on integrals)

THM [CILW] Lambrecht - Stanley - like model for Conf_M for $\dim M \geq 4$ & $\pi_1 M = 0$
(decorated by a Poincaré - Lefschetz duality model of $(M, \partial M)$)

Surfaces

Only simply connected surface : S^2
→ what about the others?

We view $\Sigma_g = \left(S^2 \setminus (D^2)^{\sqcup 2g}\right) \cup \left(S^1 \times \mathbb{R}\right)^{\sqcup g}$

eg $\Sigma_2 =$ 

⇝ Conf_{Σ_g} can be expressed as a kind of iterated tensor product

We need · $\mathrm{Conf}_{S^1\times\mathbb{R}}$ as a monoid

· $\mathrm{Conf}_{S^2\setminus(D^2)^{\sqcup 2g}}$ as a module w/ g left actions & g right actions

↪ to deal w/ left/right, we need to model orientation reversal on $\mathrm{Conf}_{S^1\times\mathbb{R}}$

+ we do everything framed

THM [Campos-I-Willwacher] $\mathrm{Conf}^{fr}_{S^2\setminus(D^2)^{\sqcup 2g}}$ and $\mathrm{Conf}^{fr}_{S^1\times\mathbb{R}}$ + algebraic structures are formal

Proof:

Both $S^2\setminus(D^2)^{\sqcup 2g}$ and $S^1\times\mathbb{R}$ are $\mathbb{R}^2\setminus\{\text{points}\}$

→ start from $\mathrm{Conf}_{\mathbb{R}^2}$ (formal)

→ use the fibration $\mathrm{Conf}^{fr}_{M\setminus *}(r) \hookrightarrow \mathrm{Conf}^{fr}_{M}(r+1) \to \mathrm{Fr}_M$

to get to $\mathrm{Conf}^{fr}_{\mathbb{R}^2\setminus\{pts\}}$ inductively

from $\mathrm{Conf}^{fr}_{\mathbb{R}^2}(r) = \mathrm{Conf}_{\mathbb{R}^2}(r)\times SO(2)^r$

We get a "Hochschild complex"

$$\mathrm{Conf}^{fr}_{\Sigma_g} \simeq \widehat{\bigotimes}_{\mathrm{Conf}^{fr}_{S^1\times\mathbb{R}}} \mathrm{Conf}^{fr}_{S^2\setminus(D^2)^{\sqcup 2g}}$$

THM [CIW] Rational model for $\hat{\lrcorner}$:

→ $H^*(BSO(2))$

THM [CiW] Rational model for $\lrcorner$:

$$G^{fr}_{\Sigma_g}(r) = \left(\frac{H^*(\Sigma_g)^{\otimes r} \otimes \overset{H^*(BSO(2))}{S(\theta_i, \omega_{ij})}}{\begin{pmatrix}\text{Arnold relations}\\ \text{Symmetry relations} \\ \text{for } H^*(\Sigma_g) \ \& \ \theta_i\end{pmatrix}} \; , \; d\omega_{ij} = \Delta_{ij} \; , \; d\theta_i = (2-2g)\mathrm{vol}_i \right)$$

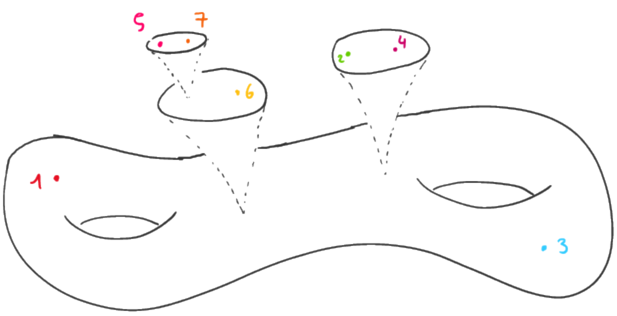
Proof :

$$G^{fr}_{\Sigma_g}(r) \xleftarrow[\text{Combinatorics}]{\sim} \underset{\text{graphs decorated by } H^*(\Sigma_g) \ \& \ H^*(BSO(2))}{\mathrm{BVGraphs}_{\Sigma_g}} \xrightarrow[\text{“Integrals” à la Kontsevich}]{\sim} \underbrace{\widehat{\bigotimes}^{(1,1)..(g,g)}_{BV_1^\vee} BV^\vee_{g,g}}_{\text{Cohomology of big } \hat{\otimes} \text{ above}} \underset{\text{general theory (Künneth)}}{\simeq} \Omega^*(\mathrm{Conf}^{fr}_{\Sigma_g})$$

Operads

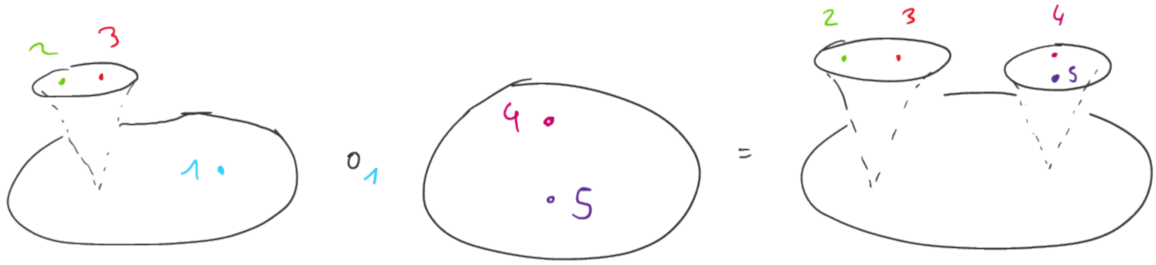
To ensure that integrals converge $\Rightarrow$ we need to compactify!

Add a boundary to $\mathrm{Conf}_M(r)$ that contains configurations with infinitesimally close points

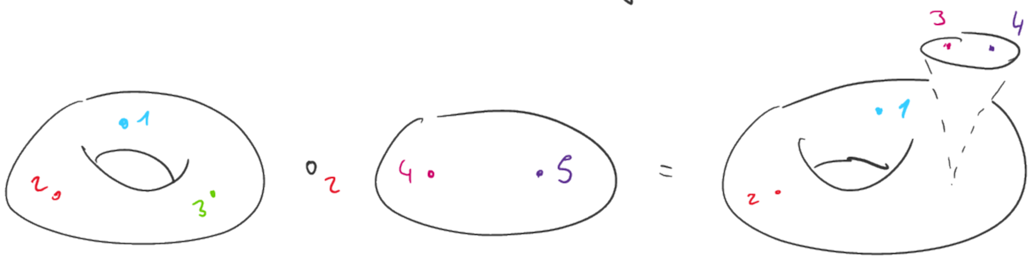


$\Longrightarrow$ new algebraic structures appear:

$\mathrm{Conf}_{\mathbb{R}^n}$: operad



Conf_M : operadic right module / $\mathrm{Conf}_{\mathbb{R}^n}$ (if M framed)



This operadic structure appears in previous results:

- Kontsevich formality
- G_A has a right comodule structure, reflects structure of Conf_M
- Graphs_M^{fr} too (over $\mathrm{Conf}^{fr}_{\mathbb{R}^n}$ $\to$ model of Khoroshkin-Willwacher)
- $G^{fr}_{\Sigma_g}$ too, needs Tamarkin formality of $\mathrm{Conf}_{\mathbb{R}^2}$ + Ševera's formality of $\mathrm{Conf}^{fr}_{\mathbb{R}^2}$

Applications of operadic structures:

- Goodwillie-Weiss manifold calculus
- Factorisation homology

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Thanks for your attention!