# Hamburg 2021

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Configuration spaces of Surfaces

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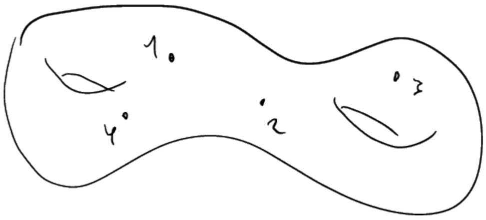
Joint work with Ricardo Campos and Thomas Willwacher

M: manifolds

$\mathrm{Conf}_M(r) = \{(x_1, \dots, x_r) \in M^r \mid \forall i \neq j \quad x_i \neq x_j\}$



$B_r$ :  $= \pi_1(\underbrace{\mathrm{Conf}_{D^2}(r)/\Sigma_r}_{K(\pi,1)})$

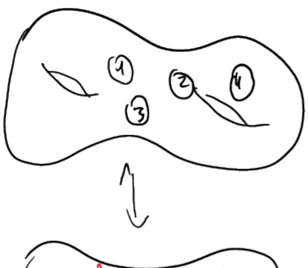
Goodwillie - Weiss manifold calculus / embedding calculus $\quad \mathrm{Emb}(M,N) = \{f : M \hookrightarrow N\}$

$\mathrm{Emb}(M,N) \subset \prod_{n \geq 0} \mathrm{Map}(\mathrm{Conf}_M(n), \mathrm{Conf}_N(n))$

$f \longmapsto (f_n)_{n \geq 0}$

$R\mathrm{Map}_{D_m^{fr}}(D_M^{fr}, D_N^{fr})$

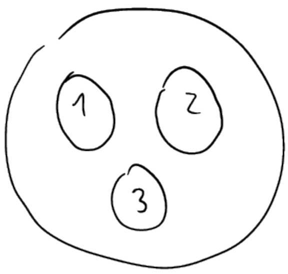
 $f_3(X_t) \to f_1(\lim X_t \in \mathrm{Conf}_M(1))$



$D_M^{fr}(r) = \mathrm{Emb}((D^m)^{\sqcup r}, M) \simeq \mathrm{Conf}_M^{fr}(r) \to \mathrm{Fr}_M^{\times r}$

$\downarrow \qquad \downarrow$

$\mathrm{Conf}_M(r) \to M^r$

$D_{D^m}^{fr} = D_m^{fr}$  has an operadic structure

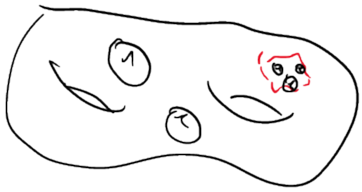
$D^{fr}(k) \times D^{fr}(l) \to D^{fr}(k+l-1)$  $\circ_3$

$$D_m^{fr}(k) \times D_m^{h}(l) \xrightarrow[o_i]{} D_m^{h}(k+l-1)$$

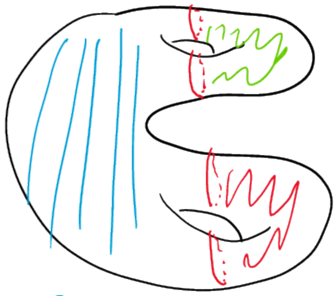
$1 \leq i \leq k$

$o_3$

$$D_M^{h}(k) \times D_m^{f}(l) \longrightarrow D_M^{h}(k+l-1)$$



How to compute the homotopy type of $D^h_{\Sigma_g}$ where $\Sigma_g$ is a surface?



$S^1 \times \mathbb{R}$

$S^1 \times \mathbb{R}$

$S^2 \setminus (D^2)^{\sqcup 2g}$

Whenever $M = M' \cup_{N \times \mathbb{R}} M''$

↳ collar

$\Rightarrow D^h_{N\times\mathbb{R}}$ is a monoid up to homotopy

$\Rightarrow D^h_M$ is a left module over this monoid

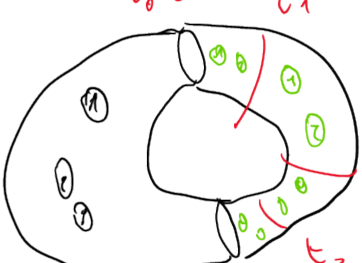
$D^h_{M''}$ is a right module

$$D^h_M \simeq D^h_{M'} \overset{\mathbb{L}}{\underset{D^h_{N\times\mathbb{R}}}{\otimes}} D^h_{M''}$$

$$\Rightarrow D^h_{\Sigma_g} \simeq \overset{\mathbb{L}}{\underset{(D^h_{S^1\times\mathbb{R}})^{\otimes g}}{\bigotimes}} D^h_{S^2 \setminus (D^2)^{\sqcup g}}$$

$$D^{h}_{\Sigma_g}(n) \simeq \coprod_{\vec{n}} \coprod_{f : S \to B(\vec{n})} D^h_{S^2 \setminus \sqcup^g D^2}(f^{-1}(0)) \times \prod_{\alpha=1}^{g} D^h_{S^1\times\mathbb{R}}(f^{-1}(\alpha,1)) \times \cdots \times D^h_{S^1\times\mathbb{R}}(\alpha, n_\alpha) \times \Delta^{n_\alpha}$$

$\vec{n} = (n_1, \dots, n_g)$ $\quad \sum n_i = n$ $\quad B(\vec{n}) = \{0, (1,1), \dots, (1, n_1), \dots, (g,1), \dots (g, n_g)$



$t_0 = 0$

$t_1$

$t_2$

$t_3$

$t_{n_\alpha+1}$

$\underline{t} \in \Delta^4$

Rational homotopy types of $\rightsquigarrow$ Simply connected CDGAs

simply co[illegible]

Rational homotopy types of simply connected spaces $\rightsquigarrow$ Simply connected CDGAs

$$X \longmapsto \Omega^*_{PL}(X)$$
$$\langle A\rangle \longleftarrow A$$

$S^1 \times \mathbb{R} = D^2 \setminus *$ $\qquad S^2 \setminus \bigcup^{2g} D^2 = D^2 \setminus \bigcup^{2g-1} *$

$\mathrm{Conf}_{M\setminus *}(n) \hookrightarrow \mathrm{Conf}_M(n+1) \xrightarrow[p_{n+1}]{} M$ Fadell-Neuwirth fibration

$$D^{fr}_{D_2} = D_2^{fr}$$

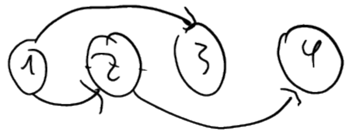
Thm [Kontsevich, Tamarkin] The operad $D_2$ is formal:
$$\Omega^*_{PL}(D_2) \simeq H^*(D_2;\mathbb{Q})$$

[K] $H^*(D_m) \xleftarrow[\pi]{\sim} \mathrm{Graphs}_m \xrightarrow[\omega]{\sim} \Omega^*(D_m)$

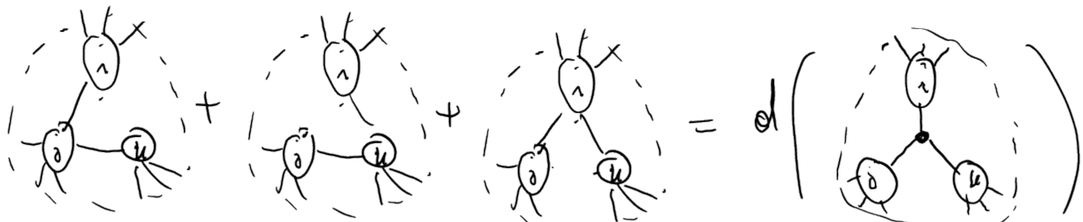
$$H^*(D_m(n)) = S(\omega_{ij})_{1\le i\ne j\le n} \Big/ \left( \omega_{ij} = (-1)^m \omega_{ji},\ \omega_{ij}^2 = 0,\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0 \right)$$

$\deg \omega_{ij} = m-1$

$\rightsquigarrow \quad \omega_{i_1j_1} \cdots \omega_{i_dj_d}$

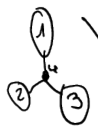


$\omega_{12}\omega_{24}\omega_{13}$



$\omega(i - j) = p_{ij}^*(\mathrm{vol}_{S^{m-1}})$

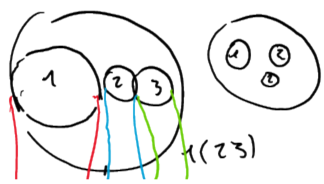
$p_{ij} : D_m(n) \to D_m(2) \simeq S^{m-1}$

$\omega\Big($  $\Big) = \int_4 \omega_{14}\omega_{24}\omega_{34}$

[T] $H^*(D_2) \xleftarrow{\sim} C^*_{CE}(t) \xrightarrow[\eta]{} \Omega^*_{PL}(\langle C^*_{CE}(t)\rangle) \xrightarrow{\Phi} \Omega^*(N.PaB) \simeq \Omega^*(D_2)$

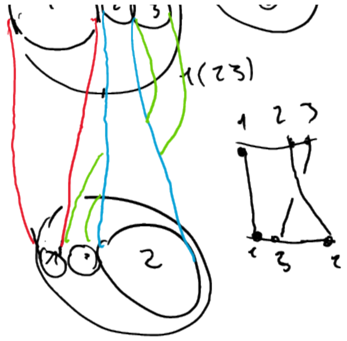
$t$ = Drinfeld-Kohno Lie algebra

$C^*_{CE}$ = Chevalley-Eilenberg complex



$C^*_{\mathfrak{g}}$ = Chevalley-Eilenberg complex

$\Phi$ : Drinfeld associator




Thm (CIW) We have a model of $D^{fr}_{\Sigma_g}$ given by

$$G^{fr}_{\Sigma_g}(n) = \frac{S(\omega_{ij})_{1\le i\ne j\le n} \otimes S(\alpha^i_1 \dots \alpha^i_g, \beta^i_1 \dots \beta^i_g)_{1\le i\le n} \otimes S(\theta_i)_{1\le i\le n}}{\left(\text{Arnold relation for } \omega_{ij},\ \alpha^i_k\omega_{ij} = \alpha^j_k\omega_{ij},\ \alpha^i_k\beta^i_k = \alpha^i_{k'}\beta^i_{k'},\ \beta^i_k\omega_{ij} = \beta^j_k\omega_{ij},\ \theta_i\omega_{ij} = \theta_j\omega_{ij}\right.}$$

$d\omega_{ij} = \Delta_{ij}$ "diagonal class" $\quad d\theta_i = (2-2g)\cdot\alpha^i_1\beta^i_1$

pf $\quad G^{fr}_{\Sigma_g} \xleftarrow[k]{\sim} \mathrm{Graphs}^{fr}_{\Sigma_g} \xrightarrow{\omega} \hat{\bigotimes}_{H^*(..)} H^*(..) \simeq \hat{\bigotimes}_{\Omega^*(D^{fr}_{S^1,\mathbb{R}})} \Omega^*(D^{fr}_{S^1 \times \mathbb{R}}) \, \Omega^*(D_{\Sigma_g})$

T + generalization of Ševera to $D^{fr}_2$ + cyclic formality

$\omega$ is built by "integrals"

↳ explicit "volume form" in arity 2

$$\mathrm{GRT} = h\mathrm{Aut}(D_2^{fr}) \xleftarrow{U} \mathrm{GRT}_g = h\mathrm{Aut}(D^{fr}_{\Sigma_g}, D^{fr}_2)$$

$\exists$ for $g=1$

$\mathrm{Emb}(\Sigma_g, \mathbb{R}^n) \quad n \geq 5$

$D^{fr}_n$