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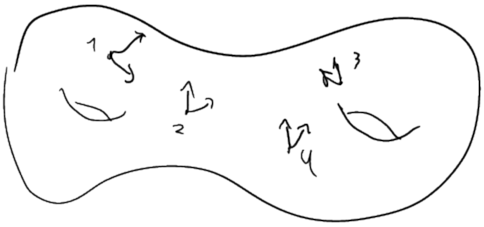
Configuration spaces of Surfaces

MIT Topology Seminar

Najib Idrissi - Université de Paris & IMJ-PRG
Joint work with Ricardo Campos and Thomas Willwacher

$M$: manifold $\quad \mathrm{Conf}_M(r) = \{x \in M^r \mid \forall i \neq j,\ x_i \neq x_j\}$

$M = \Sigma_2$




$B_r$ : braid group on $r$ strands

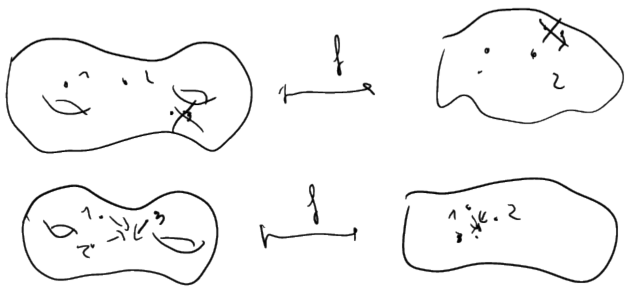

$\pi_1\big(\underbrace{\mathrm{Conf}_{D^2}(r)/\Sigma_r}_{K(\pi,1)}\big)$

Goodwillie-Weiss (manifold / embedding) calculus $\quad \mathrm{Emb}(M,N) = \{f : M \hookrightarrow N\}$

$$\mathrm{Emb}(M,N) \subset \prod_{r\geq 0} \mathrm{Map}\big(\mathrm{Conf}^{fr}_M(r), \mathrm{Conf}^{fr}_N(r)\big)$$

$$f \longmapsto (f_r)_{r\geq 0}$$

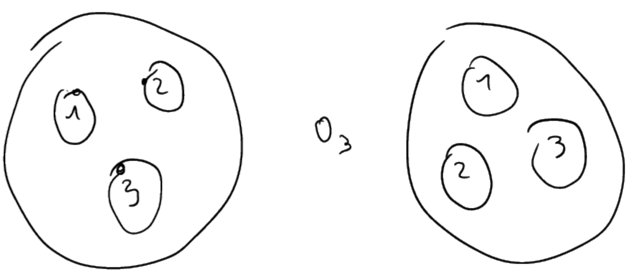
$$\begin{array}{ccc} \mathrm{Conf}^{fr}_M(r) & \longrightarrow & Fr_M^{\,r} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Conf}_M(r) & \longrightarrow & M^r \end{array}$$




$$D^{fr}_M(r) = \mathrm{Emb}\Big(\coprod^r \mathbb{D}^n, M\Big) \ni (f_i)$$
$$\sim\downarrow \qquad\qquad \downarrow$$
$$\mathrm{Conf}^{fr}_M(r) \ni (f_i(0), df_i(0)(\dots))$$

$D^{fr}_{\mathbb{D}^n} =: D^{fr}_n$ : framed little disks operad

$$D^{fr}_n(k) \times D^{fr}_n(l) \longrightarrow D^{fr}_n(k+l-1)$$




$D^{fr}_M$ is a right module over $D^{fr}_n$ $\qquad D^{fr}_M(k) \times D^{fr}_n(l) \longrightarrow D^{fr}_M(k+l-1)$

$D^n_M$ is a right module over $D_n$ $\qquad D^n_M(k) \times D_n(\ell) \to D^n_M(k+\ell-1)$

$f : M \hookrightarrow N \Rightarrow$ induces a morphism of right $D^{fr}_{\dim M}$ - modules

Thm [Boavida - Weiss, Turchin, Sinha ....]

$\mathrm{Emb}(M,N) \simeq \mathbb{R}\mathrm{map}_{D^{fr}_{\dim M}}(D^{fr}_M, D^{fr}_N)^{\mathbb{Q}}$ if $\dim N - \dim M \geq 3$

## Rational htpy theory

$f : X \to Y$ is a rational equivalence if $\pi_* f \otimes_{\mathbb{Z}} \mathbb{Q}$ is an iso

Thm [Sullivan] There is an equ

- simply connected finite type spaces up to $\mathbb{Q}$ equ
- simply connected $\underbrace{\text{commutative dg-algebras}}_{\text{CDGA}}$ up to $\underbrace{\text{quasi-isomorphism}}$

finite type

$A^0 = \mathbb{Q}$ $\quad \dim H^i(A) < \infty$

$A^1 = 0$

$\mathrm{Top} \rightleftarrows \mathrm{CDGA}$

$X \longmapsto \Omega^*_{PL}(X)$ = piecewise linear forms on $X$

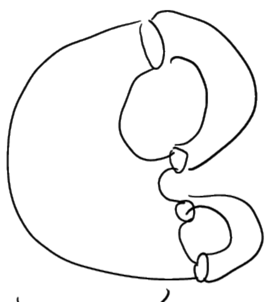
Goal $\Omega^*(D^{fr}_{\Sigma_g}) \circlearrowleft \Omega^*(D^{fr}_2)$ ?

$\mathrm{Conf}^{fr}_{M,*}(n-1) \hookrightarrow \mathrm{Conf}^{fr}_M(n)$

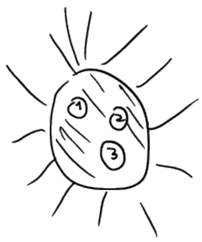
$\downarrow p_1$ fiber bundle

$\mathrm{Fr}_M$

$\Sigma_g$



$S^2 \setminus \sqcup^{2g} D^2 \quad (S^1 \times \mathbb{R})^{\sqcup g}$ } $\mathbb{D}^2 \setminus \{$some points$\}$

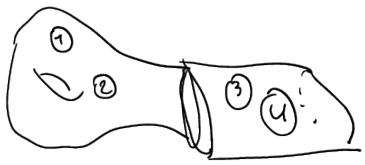


$\cdot\, D^{fr}_{N,\partial}$ $\to$ monoid (up to homotopy)

- $D^{fr}_{N\times\mathbb{R}}$ → monoid (up to homotopy)

$$D^{fr}_{N\times\mathbb{R}}(k)\times D^{fr}_{N\times\mathbb{R}}(l)\longrightarrow D^{fr}_{N\times\mathbb{R}}(k+l)$$

- if $\partial M = N$ then $D^{fr}_M$ is a right module over this monoid

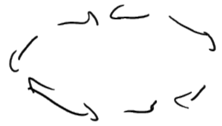


- if $\partial M = \partial \bar{M}' = N$

$$D^{fr}_{M\cup_N M'} \simeq D^{fr}_M \underset{D^{fr}_{N\times\mathbb{R}}}{\overset{\mathbb{L}}{\otimes}} D^{fr}_{M'}$$

Thm [Kontsevich, Tamarkin $_{n=2}$] The operad $D_2$ is formal

i.e. $H^*(D_2;\mathbb{Q})\simeq \Omega^*_{PL}(D_2)$

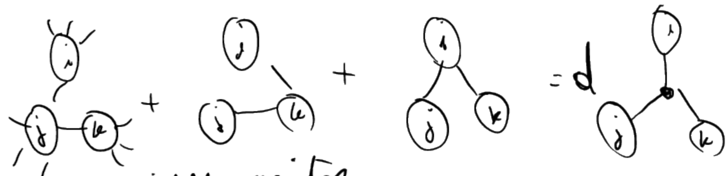


[K] $H^*(D_n) \xleftarrow{\sim} \mathrm{Graphs}_n \xrightarrow{\sim} \Omega^*(D_n)$

$$H^*(D_n(n)) = S(\underset{\deg n-1}{w_{ij}})_{1\le i\ne j\le n} \Big/ \left( \begin{array}{l} w_{ji} = (-1)^n w_{ij} \\ w_{ij}^2 = 0 \\ w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij} \end{array} \right)$$

$w_{i_1 j_1} \cdots w_{i_k j_k}$

$w_{13}\, w_{43}\, w_{24}$



[T] $H^*(D_2) \xleftarrow{\sim} C^*_{CE}(t) \xrightarrow{\Phi} \Omega^*(N.P.B) \simeq \Omega^*(D_2)$

$\Phi$ = Drinfeld associator

$\Uparrow$

Drinfeld-Kohno Lie algebra

$\mapsto e^{1/2\, t_{12}}$

$\mapsto \Phi(t_{12}, t_{23})$

[Ševera] $D_2^h$ is formal
(Giansiracusa - Salvatore)

Thm [CIW] $D_2^h$ is formal as a cyclic operad

**Thm** [CIW] We have an explicit model for $D^{fr}_{\Sigma_g}$ :

$$G^{k}_{\Sigma_g}(r) = \frac{S(\omega_{ij})_{1\le i\neq j\le r} \otimes S(\alpha^i_1 \cdots \alpha^i_g\, \beta^i_1 \cdots \beta^i_g)_{1\le i\le r} \otimes S(\theta_i)_{1\le i\le r}}{\left(\begin{array}{l}\text{Previous relations for } \omega_{ij} \\ \alpha^i_k \beta^i_k = \alpha^i_l \beta^i_l\end{array}\right.}$$

$$\alpha^i_k \omega_{ij} = \alpha^j_k \omega_{ij}$$

$$\beta^i_k \omega_{ij} = \beta^j_k \omega_{ij}$$

$$\theta_i \omega_{ij} = \omega_{ij} \theta_j$$

$$d\omega_{ij} = \text{diagonal class } \Delta_{ij} \qquad d\theta_i = (2-2g)\,\mathrm{vol}^i$$

$$\text{Pf } G^{k}_{\Sigma_g} \longleftarrow \text{Graphs}^{k}_{\Sigma_g} \xrightarrow{\ \omega\ } \bigotimes \cdots \simeq \Omega^*(D^{k}_{\Sigma_g})$$

$$C^{CE}_*\left(H^{2-*}(\Sigma_g) \otimes \mathrm{Lie}_n[1-n]\right)$$

$$H^* \longrightarrow \Omega^*$$