Motivation

$$\mathrm{Conf}_M(n) = \{(x_1 \dots x_n) \mid i \neq j \Rightarrow x_i = x_j\}$$

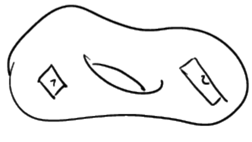
1) GW calculus

Approximate $\mathrm{Emb}(M, N)$ by a subspace of $\prod_{n \geq 0} \mathrm{Map}_{\Sigma}(\mathrm{Conf}_M(n), \mathrm{Conf}_N(n))$

2 conditions : near $\longmapsto$ near
subconf $\longmapsto$ subconf

How to encode these conditions precisely " up to homotopy "

$\Rightarrow$ use operads !

$\mathcal{C}_m(n) = \{$  $\}$, $\mathcal{C}^{h}_M(n) = \{$  $\}$

operad structure = composition   right-module structure = pre-composition

Post-composition by $f : M \hookrightarrow N$ is clearly compatible w/ precomposition

$$\Rightarrow \mathrm{Emb}^h(M, N) \xrightarrow[\text{natural}]{\exists} \mathrm{RMap}_{\mathcal{C}_m}(\mathcal{C}^h_M, \mathcal{C}^{m-h}_N)$$

[Boavida-Weiss, Arone-Turchin, Turchin...] equiv if $n-m \geq 3$

$\Rightarrow$ how to deal with $\mathrm{Emb}_\partial(M, N)$?

$\rightarrow$ use the Swiss-Cheese operad

$SC_m(n,s) = \{$  $\}$ $SC^h_M(n,s) =$ 

$\rightarrow$ want to study $\mathrm{RMap}_{SC_m}(SC^h_M, SC^{m-h}_N)$

2) Factorization homology

Idea : charged particles on M, collision $\rightarrow$ charges are multiplied

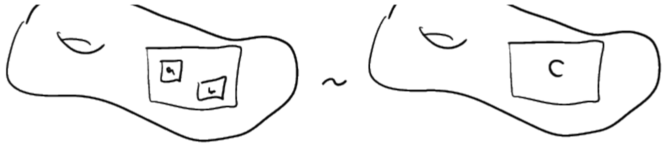
$a, b \in A$

What kind of structure for A ?

If just abelian gp $\Rightarrow$ higher Hochschild homology, only detect $H_*(M)$

$\rightarrow$ want " homotopy commutative " multiplication

$\Rightarrow$ 

$\Rightarrow$ 

where $c =$  in the $\mathcal{C}_2$-algebra structure of A

[Francis] equiv to topol chiral homology def by Lurie (cf also chiral homology BD, blob homology MW, any sp comm alg S)

$\mathcal{C}_M^{fr} \circ^{\mathbb{L}}_{\mathcal{C}_m} A$

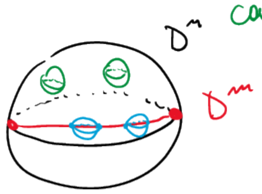
Can also define for mfd w/ boundary $\Rightarrow SC_M \circ^{\mathbb{L}}_{SC_m} A$

3) Stratification

More general than mfds w/ boundary $\longrightarrow$ stratified mfd

In this talk, stratif of the form $(L \subset M)$, L: closed submfd of codim ≥ 2, M: closed

case of interest: knot $\subset S^3$

$\Rightarrow$ operad $CD_{m,n}$ 

Real homotopy type

We can rationalize: $\underbrace{M^{\mathbb{Q}} \circ^{\mathbb{L}}_{P^{\mathbb{Q}}} A^{\mathbb{Q}}}_{\text{always works}}$, $\underbrace{\mathrm{RMap}_{P^{\mathbb{Q}}}(M^{\mathbb{Q}}, N^{\mathbb{Q}})}_{\text{needs some connectivity}}$

$\Rightarrow$ want to know about $\mathcal{C}_m^{\mathbb{Q}}$, $SC_m^{\mathbb{Q}}$, $\mathcal{C}_M^{\mathbb{Q}}$, $SC_M^{\mathbb{Q}}$, $CD_{m,n}^{\mathbb{Q}}$ ...

Closed case: the "basic" case

Thm [K[03], T[03], LV[14], P[14], FW[20], BH[22]] The operad $\mathcal{C}_m$ is formal for all m:

$\Omega^*(\mathcal{C}_m) \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^*(\mathcal{C}_m)$ over $\mathbb{Q}$

Most relevant for us: the pf of K / LV

Steps:

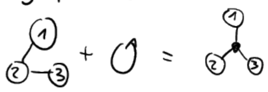
(GC) Resolve $H^*(\mathcal{C}_m)$ using graph complexes $\mathrm{Graph}_m \xrightarrow{\sim} H^*(\mathcal{C}_m)$

$H_*(\mathcal{C}_m) = \mathrm{Com} \circ \mathrm{Lie}_m = \mathrm{Pois}_m$

basis $= \sqcup$ Lie words, resolve Jacobi / 3T relation

$H^*(\mathcal{C}_m) = \mathrm{Com} \circ \mathrm{Lie}_m = \mathrm{Pois}_m$

basis = $\sqcup$ lie words, resolve Jacobi / 3T relation by adding new graphs w/ "internal vertices



Ⓒ Compactify $\mathrm{Conf}_{\mathbb{R}^m}(r) \rightsquigarrow$ Axelrod - Singer - Fulton - MacPherson

allow points infinitesimally close together
stratif of boundary $\leftrightsquigarrow$ operad structure

Ⓘ Map $\mathrm{Graphs}_m \longrightarrow \Omega^*(\mathcal{C}_m)$ using integrals

Ⓥ Key point : many integrals vanish

Ⓓ In principle, def over $\mathbb{R}$, but descent for formality

$\Rightarrow$ template for finding "models"

$\Rightarrow$ Works for closed mfd that are simply conn, dim ≥ 4

Adaptations required :

- the model isn't $H^*(\mathrm{Conf}_M) \to$ LS model

GC : decorated graphs

I : need to find representative in $\Omega^*(FM_M)$: "propagator" / Pontryagin - Chern class

V : degree counting argument &  $= 0$

D : doesn't work. Alternative approach by Willwacher (2023) for parallelized mfd closed $\to$ obstruction theory

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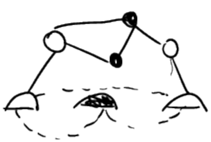
Swiss - cheese

works for $CD_{m,n}$

Adaptations required :

GC : 4 kinds of vertices
2 kinds of edges



+ {order on ter vert} $\xrightarrow{\sigma} \mathbb{Q}$

if $m = 1$

$\underline{m=1}$: bottom part : iso to ( ) + {lie word on ter vert} $\xrightarrow{\lambda} \mathbb{Q}$

$\to$ kill lie - disconnected graphs

$$p : \mathbb{R}^n \longrightarrow (\mathbb{R}^m)^\perp = 0 \times \mathbb{R}^{n-m}$$

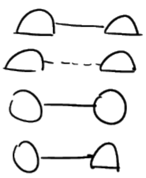
C. the compactification keeps :

$$\theta_{ij}(x) = \frac{x_i - x_j}{\|x_i - x_j\|}, \quad \delta_{ijk}(x) = \frac{\|x_i - x_j\|}{\|x_i - x_k\|}, \quad \alpha_v^{(-)} = \frac{p(x_v)}{\|p(x_v)\|}$$

$$e_{vv'} = \frac{\|p(x_v)\|}{\|p(x_{v'})\|}, \quad \sigma_{ijv} = \frac{\|x_i - x_j\|}{\|p(x_v)\|}$$

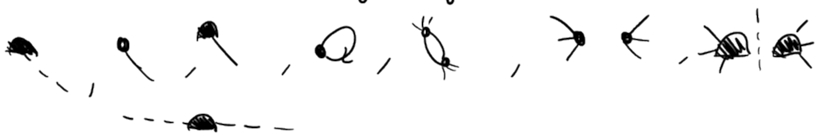
$\Rightarrow$ operad, stratif = operad structure

I: take pullbacks of volume forms of spheres



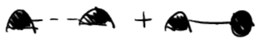
$\mapsto 0$
$\mapsto p^*(\varphi_m)$
$\mapsto p^*(\psi_{m,n})$
$\mapsto p^*(\psi^{\partial}_{m,m})$

V: vanish on many things



(unless $m = 1$, [graph] $\neq 0$)

⚠ Not enough: need to reduce to



(or ... if $m=1$)

↳ reduce again to this MC elt

- $c \in$ tree bound GC
- this GC is $\simeq HGC_{m,n}$
- vanish in degree > -1 for $n - m \geq 2$

D: unknown so far? but I guess descent isn't too difficult

Future: $\mathrm{Conf}_{N,M,N}$? Difficult: missing Poincaré duality / Lefschetz duality

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Doesn't work for SC_n. Point of failure: (V)!

We cannot get the integrals to vanish on enough graphs

$\to$ good reason: SC_n is <u>not</u> formal

Two proof: Livernet (2015), Willwacher (2017)

Livernet: existence of non-trivial Massey product in SC_n

$$\lambda = \begin{cases} a - a.(c_1 c_2) & n = 2 \\ \mathrm{vol}_{n-2} & n > 2 \end{cases} \in H_{n-2}(SC_n(2,0))$$

$$f \in H_0(SC_n(0,1))$$

$\Rightarrow \{f, \lambda, f\}$ is non trivial Massey product, prevents formality

$\Rightarrow$ can only do "half" of the template: $S\mathrm{Graphs}_n \xrightarrow{\sim} \Omega^*(SC_n)$ [Willwacher 2015]

⇒ can only do "half" of the template : $\mathrm{SGraphs}_n \xrightarrow{\sim} \Omega^*(SC_n)$ [Willwacher 2015]

but $\mathrm{SGraphs}_n$ has many nontrivial terms in the differential

Adaptation for $(M, \partial M)$: decorated graphs that model $\mathrm{Conf}_{M, \partial M, \partial M}$ [CILW, 2023]

if $\dim \partial M \geq 4$, $\pi_{\leq 1} M = \pi_{\leq 1} \partial M = 0$ ⇒ homotopy invariance

→ careful degree counting in $KGC_M \rtimes (A \otimes KGC_n)$

Non-formality SC_n^{vor} [IV 2023]

Voronov's original version: $SC_n^{vor}(0,-) = \emptyset$

$SC_n^{vor} \subset SC_n$ is a suboperad of SC_n

non-formality of SC_n $\not\Rightarrow$ non-formality of SC_n^{vor}

Intuitively, $H_*(SC_n^{vor})$-alg : $(A, B, \alpha : A \otimes B \to B)$

$H_*(SC_n)$-alg : $(A, B, f : A \to B)$

Rk $H_*(SC_2^{vor}) = \langle \lambda_2, \mu_2, \mu_1, \alpha \rangle / \left(\text{quadratic} : \begin{array}{l} \mu_2 \text{ assoc/Com} \\ \lambda_2 \text{ Lie} \\ \mu_1 \text{ assoc} \\ \alpha : \text{action} \end{array} \right)$

$H_*(SC_2) = \langle \lambda_2, \mu_2, \mu_1, f \rangle / \left(\begin{array}{l} \mu_2 : \text{assoc Com} \\ \lambda_2 : \text{Lie} \\ \mu_1 : \text{assoc} \\ f : \text{morphism} \to \text{cubic!} \end{array} \right)$

↳ can be made quadratic-linear by adding the generator α

& the rel° $\begin{cases} \mu_1(f(-), -) = \alpha(-,-) = \mu_1(-, f(-)) \\ \alpha(f(-), -) = f \mu_2(-,-) \end{cases}$

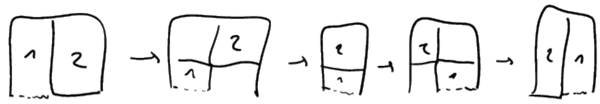
[Hoefel–Livernet] $H_0(SC_2^{vor})$ & $H_0(SC_2)$ are Koszul

- $H_0(SC_2^{vor})^! = \mathcal{LP}(\mathfrak{g}, A, \rho)$ $\mathfrak{g}$: Lie, A : assoc, $\rho(x,-) : A \to A$ derivation, $\rho(-, a)$: Lie morphism
- $H_0(SC_2)^! \Rightarrow (\mathfrak{g}, A, \rho, \varphi)$ $\mathfrak{g}$: dg Lie ; A : dg assoc ; ρ : as above ; $\varphi : \mathfrak{g} \to A$ degree -1

$d\varphi = -\varphi d$; $\varphi([x,y]) = \pm \rho(x, \varphi(y)) \mp \rho(y, \varphi(x))$

; $d\rho(x,a) = \rho(dx, a) \pm \rho(x, da) + \varphi(x) a \mp a \varphi(x)$

Livernet's non-triv Massey product : B_+ :



B_- defined similarly

together → define a nonzero homology class in $SC_2(0,2)$

β_- defined similarly

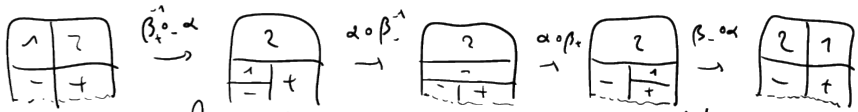
Together $\Rightarrow$ define a nonzero homology class in $SC_2(0,2)$

Problem: no access to f in SC_n^{vor}

Instead:

$\underline{n=2}$ [Vieira 2018] β_+ 

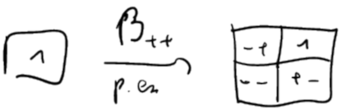
β_- : 

Together: $\eta =$ 

$\eta + \eta\cdot(12) \neq 0$ in homology, obstruction to formality

A

$\underline{n\geq 2}$: $\mu^2 =$ 

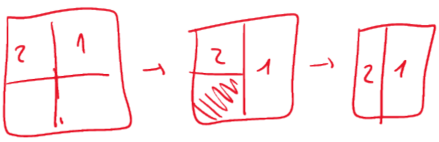
Chains $\beta_{\varepsilon_1\varepsilon_2}$ that move 

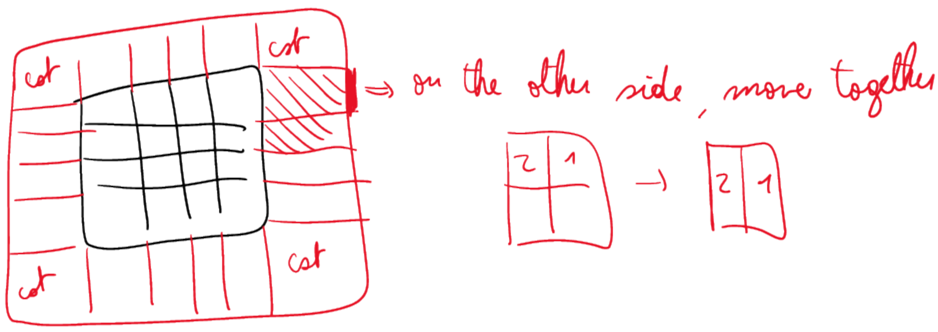
$\beta_{+-}\circ\alpha$	?	?	$\beta_{--}\circ\alpha$ $\eta_{++,\emptyset,\emptyset}$
?	$\alpha\circ\beta_{-+}$	$\alpha\circ\beta_{++}$ $\eta_{++,\{1\},\{2\},\emptyset}$	?
?	$\alpha\circ\beta_{--}$	$\alpha\circ\beta_{+-}$	?
$\beta_{++}\circ\alpha$	?	?	$\beta_{-+}\circ\alpha$

$$\beta_-^{\{1\}}\left(\mu^{\{2\}}, \beta_+^{\{2\}}\right)$$

Still not enough!

1 moves first on the boundary from above:





In total, we need $2\times 6\times 6$ 2-chains to build the obstruction

More generally, we need $2\times 2^{n-1}\times 3^{n-1}$ $(n-1)$-chains for the non-formality of SC_n^{vor}, in arity $(2^{n-1}, 2)$

Some questions.

- can we do better? Fewer chains, lower arity?
  ie is $\tau_{\leq(k,l)} SC_n^{vor}$ formal?
- can we interpret the obstruction as a Massey product?

The issue: antisymmetrization is only possible after both compositions

$\Rightarrow$ we don't have two compositions that vanish in homology...