(Non)-Formality of Swiss-Cheese operads

① Motivation: embedding calculus

Goal: $\mathrm{Emb}(M,N) = \{ f : M \hookrightarrow N \}$ $\qquad m = \dim M,\ n = \dim N$

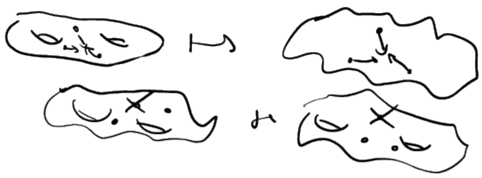
What is an embedding?
- continuous map
- injective
- injective on tangent spaces

Approximate using $\mathrm{Conf}_r(M) = \{ (x_1, \dots, x_r) \in M^r \mid \forall i \neq j,\ x_i \neq x_j \}$

$\mathrm{Conf}_r^{fr}(M) = \mathrm{Fr}_M^{\times r} \times_{M^{\times r}} \mathrm{Conf}_r(M) = \{ (x_1, \underbrace{\xi_{1,1} \dots \xi_{1,m}}_{\text{basis } T_{x_1}M}; \dots; x_r, \underbrace{\xi_{r,1}, \dots, \xi_{r,m}}_{\text{basis } T_{x_r}M}) \}$

$\mathrm{Emb}(M,N)$ approx by subspace of $\prod_{r \geq 0} \mathrm{Map}_{\Sigma_r}(\mathrm{Conf}_r^{fr}(M), \mathrm{Conf}_r^{m\text{-}fr}(N))$

- near points $\longmapsto$ near points
- sub-config $\mapsto$ sub-config
- $SO(m)^r$ - equivariance



Can be made precise using OPERADS

$\Rightarrow$ fatten points into disks: (Boardman-Vogt, May, Getzler)

$\mathcal{D}_m^{fr}(r) \subset \mathrm{Emb}(\coprod^r \mathbb{D}^m, \mathbb{D}^m)$ "standard embeddings"

$\hookrightarrow$ multivariable "operations", composition gives a structure of operad

$$\mathcal{D}_m(k) \times \mathcal{D}_m(l) \to \mathcal{D}_m(k+l-1)$$

$\mathcal{D}_M^{fr}(r) = \mathrm{Emb}(\coprod^r \mathbb{D}^m, M)$

$\hookrightarrow$ operadic "right module" using precomposition of embeddings

Post-composition by $f : M \hookrightarrow N$ is compatible w/ pre composition

$\rightarrow$ morphism of operadic right modules

Thm [GW, AT, T, BW] If $n - m \geq 3$, then

$$\mathrm{Emb}(M,N) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{D}_m^{fr}\text{-}\mathrm{RMod}}(\mathcal{D}_M^{fr}, \mathcal{D}_N^{fr})$$

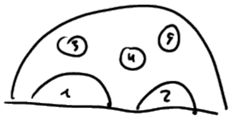
$\mathrm{Emb}(-,-)$ … $\mathcal{D}_m^{fr}$-RMod (…)

Rk More familiar: $\mathcal{D}_m \subset \mathcal{D}_m$ "unframed disks"

What about manifolds with boundary?

$\hookrightarrow$ use 2-colored configurations

$\mathcal{SC}_m(k,\ell) \subset \mathcal{D}_m(k+\ell)$



k terrestrial
ℓ aerial

+ two compositions: $\mathcal{SC}_m(k,\ell) \times \mathcal{D}_m(\ell') \longrightarrow \mathcal{SC}_m(k,\ell+\ell'-1)$

$\mathcal{SC}_m(k,\ell) \times \mathcal{SC}_m(k',\ell') \longrightarrow \mathcal{SC}(k+k'-1,\ell+\ell')$

Rk Can also do a framed one

Rk In Voronov's original version, $\mathcal{SC}_m^{vor}(0,\ell) = \emptyset$

② Homology of $\mathcal{D}_m, \mathcal{SC}_m, \mathcal{SC}_m^{vor}$

Thm (Cohen) For $m \geq 2$, $H_*(\mathcal{D}_m) = \mathrm{Pois}_m$ is generated by

$\mu_m \in H_0(\mathcal{D}_m(2))$, $\lambda_m \in H_{m-1}(\mathcal{D}_m(2))$, $\eta \in H_0(\mathcal{D}_m(0))$

Associative + comm, Jacobi + (antisym), unit + central

$\rightarrow$ very nice: basis, resolutions …

Thm (Voronov) $H_*(\mathcal{SC}_m^{vor})$ gen by:

- $(\mu_m, \lambda_m, \eta_m)$
- $(\mu_{m-1}, \lambda_{m-1}, \eta_{m-1})$
- $\alpha \in H_0(\mathcal{SC}(1,1))$ central action of the $\mathcal{D}_m$-alg on the $\mathcal{D}_{m-1}$-alg

Thm (Hoefel) $H_*(\mathcal{SC}_m)$: replace α by $f \in H_0(\mathcal{SC}_m(0,1))$: central morphism

③ Rational homotopy theory

$X \simeq_{\mathbb{Q}} Y$ if $\exists\ X \xleftarrow{\sim_{\mathbb{Q}}} \cdots \xrightarrow{\sim_{\mathbb{Q}}} Y$, iso on $H^*(-;\mathbb{Q})$

In general, $H^*(X;\mathbb{Q})$ is not enough to recover X up to $\simeq_{\mathbb{Q}}$

(Sullivan, …) Model: CDGA $\simeq_{\mathbb{Q}} \Omega^*(X)$

Thm For nilp ft spaces, $\underset{\text{up to} \simeq_{\mathbb{Q}}}{\text{model}} \leftrightarrow \underset{\text{up to} \simeq_{\mathbb{Q}}}{\text{space}}$

Can be upgraded to operads: $\Omega_\sharp^* P$ is a cooperad in CDGA (Fresse)

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Thm [Kontsevich[1999], ...] $\mathcal{D}_m$ is formal: $\Omega^*_\# \mathcal{D}_m \simeq_{\mathbb{Q}} H^*(\mathcal{D}_m; \mathbb{Q})$

Steps of the proof:

- compactify $\mathrm{Conf}_n(\mathbb{R}^m)$ into the ASFM operad [points as close] $\simeq \mathcal{D}_m$ [Salvatore]
- resolve $H^*(\mathcal{D}_m)$ using the operad of graphs Graphs_m

  basis of $H^*(\mathcal{D}_m)$: graphs mod 3T rel°

  add new vertices + diff to kill this rel°
- map $\mathrm{Graphs}_m \longrightarrow \Omega^*_{PA}(\mathcal{D}_m)$ using integrals
- key point: many integrals vanish (dim count + ad hoc argument)

Thm [Livernet[2015], ...] $\mathcal{SC}_m$ is not formal

Proof: uses Massey products: $\langle p; q_1, q_2 \rangle_\Pi = [x \circ_2 q_2 - \pm y \circ_1 q_1]$ $\quad dx = p \circ_1 q_1$, $dy = p \circ_2 q_2$

here, $\langle \underbrace{\mu - \mu \cdot (12)}_{\lambda'_1}; f, f \rangle_\Pi = [f \cdot \lambda_2] \neq 0$ for $m = 2$ (diff for $m > 2$)

half of x

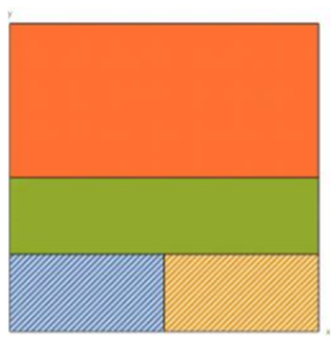
st $dx = \lambda'_1 \circ_1 f$

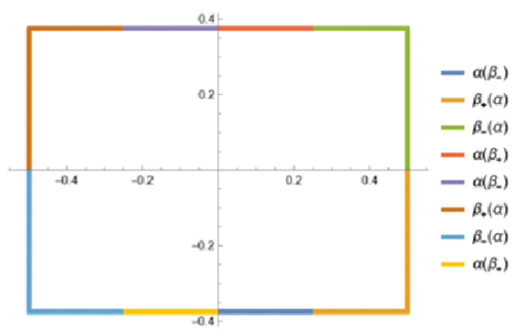
$\Rightarrow$ decomposes $f \circ \lambda_1$ into two half circles

Thm [Vieira 2018] $\mathcal{SC}_2^{\mathrm{vor}}$ is not formal

Uses β^1_+:  , β^1_- analogous

$\Rightarrow$ combine to form almost a Massey product



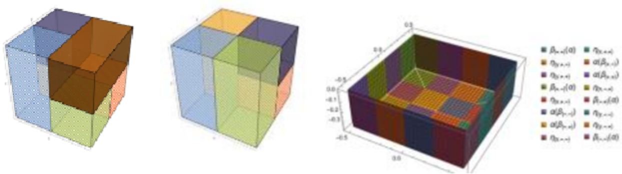


4 New results

Thm [I - Vieira 2023] SC_m^{vor} is not formal for any n

Idea: for $m=3$:

- $2\times 2\times 2$ new chains built out of product of $\beta_I^1, \beta_{\bar I}^1$, direction
- 2×2 new higher dim $\beta^2_{I,I}$
- gluing chains



$\hookrightarrow$ glue them together to get two hemispheres

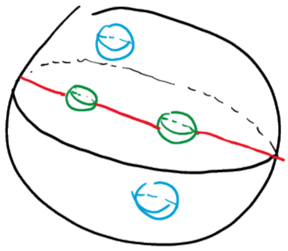
$\rightsquigarrow \alpha(\mu_4, \lambda) \neq 0$

For general m, obstruction in arity $(2^{m-1}, 2)$ of degree $m-1$

$\Rightarrow$ open questions:

- is $SC_m^{(vor)}$ formal below some degree over $\mathbb{Q}, \mathbb{F}_p$?
- is SC_m^{vor} formal if arity is truncated? maybe $< (m-1, 2)$
- action of the symm gp in the wrong place $\rightarrow$ new kind of Massey prod?

Higher codim SC operad: $CD_{m,n}$



Rk $\neq$ from Willwacher's $ESC_{m,n}$

Why $CD_{m,n}$? Algebras!

$D_m \circlearrowright \Omega^m X$ [S, BV, M]

$SC_m \circlearrowright (\Omega^m(X,A), \Omega^m X)$ [HLS, V]

$CD_{m,n} \circlearrowright (\Omega^{m,n}(X,A), \Omega^n X)$

Prop $H_*(CD_{m,n})$ is gen by $H_*(D_m)$, $H_*(D_n)$, and $f + \varepsilon\, \delta$

f central mor, δ: central f-deriv

Thm [I 2022] $CD_{m,n}$ is formal over $\mathbb{R}$ for $n-m \geq 2$

Pf Similar to Kontsevich's

Graphs w/ bicol vtx & bicol edges

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Obstruction to formality = vanishing of $\int$

$\rightarrow$ reduce to the terrestrial part, then use result on $H^\bullet(HGC_{m,n})$

split the H^* into terr $\otimes$ aerial

<u>Open Q</u> :

- recog principle for $\mathcal{C}D_{mn}$?
- corners? (WIP)
- use $\mathcal{C}D_{mn}$ to build models for $\mathrm{Conf}(N, M)$?