Préparation exposé Strasbourg 30/10/2025

① Application intended

Homological stability: $\dots \to G_n \to G_{n+1} \to \dots$ very often $H_k(G_n) \xrightarrow{\simeq} H_k(G_{n+1})$ for $n >> k$

can also be improved $H_k(G_n; M_n) \xrightarrow{\simeq} H_k(G_{n+1}; M_{n+1})$ in many cases

ⓐ $\mathrm{Aut}(F_n)$ $\quad F_n = \langle x_1 \dots x_n \rangle$, $\varphi \in \mathrm{Aut}(F_n) \to \varphi' \in \mathrm{Aut}(F_{n+1})$, $\varphi'(x_i) = \begin{cases} \varphi(x_i) & \text{if } i \leq n \\ x_{n+1} & \text{if } i = n+1 \end{cases}$

$H_k(F_n; \mathbb{Z} \text{ or } \mathbb{Q})$ stabilize (Hatcher–Vogtmann 98)

$\mathbb{Q} \to$ to zero (Galatius 10)

$H(n) = H_1(F_n; \mathbb{Q}) \circlearrowleft \mathrm{Aut}(F_n)$, $K^{pq}(n) = H(n)^{\otimes p} \otimes H^*(n)^{\otimes q}$ (use retraction for H^*)

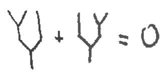
$\to H^*(\mathrm{Aut}(F_n); K^{pq}(n))$ stab (RWW17)

$H^*(\mathrm{Aut}(F_\infty); K^{pq}(\infty))$ $\to = 0$ for $p = 0$ [DV15], [RW18]
$\to$ computed by [V18], [RW18]
$\to$ conjecture by Djament

Thm [L25] $H^*(\mathrm{Aut}(F_\infty); K^{pq}(\infty)) = \begin{cases} P_{pq} \otimes \mathrm{sgn}_p \otimes \mathrm{sgn}_q & \text{if } * = p - q \\ 0 & \text{o/w} \end{cases}$ $\quad n \geq 2* + p + q + 3$

$P_{pq} = \mathbb{Q}\langle \lambda \vdash p$, at least q parts, labeled by $\underline{q}$; the rest unlabeled$\rangle$

$\mathcal{H} = \{H^*(\mathrm{Aut}(F_\infty); K^{pq}(\infty))\}_{p,q \geq 0}$ is a wheeled prop [KV23], $\subset \mathrm{Com}[1]$
$\to$ equal to wheeled prop assoc to $\mathrm{Com}[1]$, gen by $h_1 \in \mathcal{H}(2,1)_1^1$, $(h_1 \otimes 1) \circ h_1 + (1 \otimes h_1) \circ h_1 = 0$



↳ how does $\mathcal{H}(p,q)$ decompose as a rep of $\Sigma_p \times \Sigma_q$?

ⓑ IA_n identity on abelianization $\quad IA_n = \ker(\mathrm{Aut}(F_n) \to GL_n(\mathbb{Z}))$

⚠ Does not stabilize, $H^*(IA_\infty; \mathbb{Q}) = \lim H^*(IA_n)$

But! $GL_n(\mathbb{Z}) \circlearrowleft H^*(IA_n; \mathbb{Q})$; do we have representation stability ?

eg $H^1(IA_n; \mathbb{Q}) = \Lambda^2 H^*(n) \otimes H(n)$ [Kaw05]

Thm ~~Conj~~ (Katada) $H_A^*(IA_n) = \mathrm{im}\left(\Lambda H^1(IA_n) \xrightarrow{\cup} H^*(IA_n)\right)$

$\overset{?}{\simeq} \Lambda^*(\Lambda^2 H^\vee(n) \otimes H(n)) / IH_n$ for $n >> *$

$IH_n = \left(\sum_{i=1}^{n} (f_1 \wedge f_2 \otimes e_i) \wedge (e_i^\# \wedge f_3 \otimes v) - (f_3 \wedge f_1 \otimes e_i) \wedge (e_i^\# \wedge f_2 \otimes v) \right)$

(Lin24) if $H^*(IA_\infty)$ is fdim in a range $n >> *$, then $H^*(IA_\infty)^{alg} \simeq \mathbb{Q}[y_4, y_8, \dots] \otimes H_A^*(IA_\infty)$
cf HK23
$|y_i| = i$, invar under $GL_n(\mathbb{Z})$

$\rightsquigarrow \mathcal{H}' \subset \mathcal{H}$ no partition of size 1 $\to$ the multiplicity of an irrep in $H_A^*(IA_\infty) \circlearrowleft GL_\infty$ is equal to the mult of the corresponding irrep in $\mathcal{H}'$

↳ how does $\mathcal{H}'$ decompose as a rep of $\Sigma_p \times \Sigma_q$?

(done by Kawazumi 2005 for $d = 1$
Pettet 2005 for $d = 2$
Katada 2022 for $d = 3$

② Symmetric functions & rep th of Σ_n

$\Lambda_k = \mathbb{Z}[x_1, \dots, x_k]^{\Sigma_k}$ $\Lambda = \lim \Lambda_k$ Λ^n = homogeneous of degree n,

$\Lambda = \bigoplus_n \Lambda^n \subset \hat{\Lambda} = \prod_n \Lambda^n$ degree completion

Classical elements: $e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}$, $h_n = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}$, $p_n = \sum_i x_i^n$

$\{e_n\}, \{h_n\}$: mult basis for Λ $\{p_n\}$: mult basis for $\Lambda \otimes \mathbb{Q}$

For $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ a partition, let $a_\alpha = \det(x_i^{\alpha_j}) \Rightarrow s_\lambda = \frac{a_{\delta+\lambda}}{a_\delta}$, $\delta = (n-1, \dots, 0)$

$\rightarrow \{s_\lambda\}$ is an additive basis of Λ

$R(\Sigma) = \bigoplus_{n \geq 0} R(\Sigma_n)$ representation ring of Σ_*, sum = $\oplus$ of reps, product $V \cdot W = \mathrm{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} (V \otimes W)$

given M ~~$\in R(\Sigma)$~~ (e.g. a symmetric sequence)

$ch(M) \in \Lambda \otimes \mathbb{k}$ est tq $ch(M)|_{\Lambda_n} = \mathrm{tr}\left(F_M(\mathbb{k}^n) \xrightarrow{\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}} F_M(\mathbb{k}^n)\right)$, $F_M(V) = \bigoplus_m (M(m) \otimes V^{\otimes m})_{\Sigma}$

also a graded version $ch(M) \in \Lambda((\hbar)) \otimes \mathbb{k}$, degree d part $\rightarrow$ mult by $(-\hbar)^d$

ex $M(2) = \mathrm{triv}_2$, $M(n) = 0 \Rightarrow F_M(V) = S^2 V$ spanned by sym tensors

$F_M(\mathbb{k}^n) = \langle e_i e_j \rangle_{i \leq j} \rightarrow$ eigenvalues of $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ are $\{x_i x_j\}_{i \leq j} \Rightarrow ch(M) = h_2$

Thm $R(\Sigma) \cong \Lambda((\hbar))$ decategorification

Involution $\omega(h_n) = e_n$ $\omega(s_\lambda) = s_{\lambda^t}$, $\omega(p_n) = (-1)^{n-1} p_n$

$\rightarrow$ corresponds to $M \longmapsto \{M(n) \otimes \mathrm{sgn}_n\}_{n \geq 0}$

Composition product $M \circ N = \bigoplus_{m \geq 0} M(m) \otimes_{\Sigma_m} N^{\otimes m} = \left\{ \bigoplus_{m, n_1 + \dots + n_m = k} M(m) \otimes_{\Sigma_m} \mathrm{Ind}_{\Sigma_{n_1} \times \dots \times \Sigma_{n_m}}^{\Sigma_k} N(n_1) \otimes \dots \otimes N(n_m) \right\}_{k \geq 0}$

$\hookrightarrow$ on sym f°, corresponds to plethysm: for $f, g \in \Lambda$, let $g = \sum g_\alpha x^\alpha$, $g_\alpha \geq 0$

introduce vars z st $\prod_i (1 + z_i t) = \prod_\alpha (1 + x^\alpha t)^{g_\alpha}$

$\Rightarrow (f \circ g)(x_1, \dots) = f(z_1, \dots)$

to fully define, use: $g \longmapsto p_n \circ g$ is an endo for n fixed, $f \longmapsto f \circ g$ is an endo for g fixed

- $p_n \circ g = g(x_1^n, \dots) = g \circ p_n$, $p_n \circ p_m = p_{nm}$ $\omega(f \circ g) = \begin{cases} f \circ \omega(g) & |g| \text{ even} \\ \omega(f) \circ \omega(g) & |g| \text{ odd} \end{cases}$

Extends to $\hat{\Lambda} \times \hat{\Lambda}^{\geq 1}$ and to $\hat{\Lambda}((\hbar))$ $f \circ (\hbar g) = \hbar^{|f|} f \circ g$

$g(0)$ in positive $\hbar$-degree

Thm $ch(M \circ N) = ch(M) \circ ch(N)$

ex $u\mathcal{P}ois_d = H(E_d) = u\mathcal{C}om \circ \mathcal{L}ie_d$ $ch(u\mathcal{C}om) = \sum_{n \geq 0} h_n = \exp\left(\sum_{n \geq 1} \frac{p_n}{n}\right)$

$ch(\mathcal{L}ie) = \sum_{n \geq 1} \frac{-\mu_n}{n} \log(1 - p_n) = \sum_{n, k \geq 1} \frac{\mu_n}{n} \cdot \frac{p_n^k}{k}$ (Witt's formula) $\Rightarrow ch(\mathcal{L}ie_d) = (-1)^d \hbar^{1-d} \sum \frac{\mu_m}{m} \log(1 + (-1)^d \hbar^{(d-1)m} p_m)$

$\Rightarrow \log ch(u\mathcal{P}ois_d) = \left(\sum \frac{p_k}{k}\right) \circ (\dots) = (-1)^d \sum_{m,k} \hbar^{(1-d)k} \frac{\mu_m}{mk} \log(1 + (-1)^d \hbar^{(d-1)mk} p_{mk}) = (-1)^d \sum_m \sum_{n \mid m} \hbar^{(1-d)\frac{m}{n}} \frac{\mu_n}{m} \log(1 + (-1)^d \hbar^{(d-1)m} p_m)$

$d = 1$, $\sum_{n \mid m} \mu_n = 0 \Rightarrow ch(u\mathcal{P}ois_1) = \frac{1}{1 - p_1} = \sum p_1^m = ch(u\mathcal{A}ss)$ $d = 2 \rightarrow$ Koszul($\mathcal{C}om$) est acyclique

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③ Bisymmetric functions & rel operads

$\Lambda_{x,y} = \Lambda_x \otimes \Lambda_y$, $\hat{\Lambda}_{x,y}$, $\hat{\Lambda}_{x,y}((t))$ etc

Relative plethysm: $(\bar{f}, f) \bar{\circ} (\bar{g}, g) = (\bar{f} \bar{\circ} (\bar{g}, g), f \circ g) \in \Lambda_{x,y} \times \Lambda_y$

$g = \sum g_\beta y^\beta$, $\bar{g} = \sum \bar{g}_{\alpha\beta} x^\alpha y^\beta \rightsquigarrow \prod (1+z_i t) = \prod_{\alpha,\beta} (1 + x^\alpha y^\beta t)^{\bar{g}_{\alpha,\beta}}$ $\quad \prod_j (1+w_j t) = \prod_\beta (1+y^\beta t)^{g_\beta}$

$\rightarrow (\bar{f} \circ (\bar{g}, g))(x \ldots, y \ldots) = \bar{f}(z \ldots, w \ldots)$

$\bar{f} \mapsto \bar{f} \circ (\bar{g}, g)$ is an endo, $f(y) \bar{\circ} (\bar{g}, g) = f \circ g(y)$

$p_m(y) \bar{\circ} (\bar{g}, g) = g(y_1^m, \ldots) = (g \circ p_m)(y)$ $\quad p_m(x) \bar{\circ} (\bar{g}, g) = \bar{g}(x_1^m, \ldots; y_1^m, \ldots) = \bar{g} \bar{\circ} (p_m(x), p_m(y))$

+ associativity $\quad$ ⚠ Different from Koike's plethysm $f(x) \circ_K \bar{g} = f(x) \bar{\circ} (\bar{g}, 0)$

Relative operad (P, Q) $\quad P$: operad, Q: operad in right P-modules ex $\{\mathrm{Hom}(X^n, X)\}, \{\mathrm{Hom}(X^n \otimes Y^m, Y)\}$ End_X

$\Leftrightarrow$ monoid for $(Q, P) \bar{\circ} (Q', P') = (Q \bar{\circ} (Q', P'), P \circ P')$

$$Q \bar{\circ} (Q', P') = \bigoplus_{m,n} \left(Q(m,n) \otimes Q'^{\otimes m} \otimes P'^{\otimes n} \right)_{\Sigma_m \times \Sigma_n}$$

$ch(Q) \in \hat{\Lambda}_{x,y}((t))$, proj on $(\Lambda_{x,y})_{r,s}$ = trace of $F_Q(\mathrm{diag}(x_i), \mathrm{diag}(y_i))$

Thm iso of rings $R(\Sigma_* \times \Sigma_*) \cong \hat{\Lambda}_{x,y}((t))$

ex $\bar{M}(m,n) = \begin{cases} \mathbb{Q}[\Sigma_n], & m = n \\ 0 & \text{o/w} \end{cases} \Rightarrow ch(\bar{M}) = \sum_{\lambda \vdash m} S_\lambda(x) S_\lambda(y) =: R_m(x,y)$

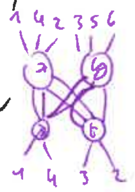
Thm ch preserves $\bar{\circ}$: $ch(\bar{M} \bar{\circ} (\bar{N}, N)) = ch \bar{M} \bar{\circ} (ch \bar{N}, ch N)$

$ch(\mathcal{A}ss\,d) = ch(\mathcal{C}\mathrm{oins}_{d-1})(x) \bar{\circ} (p_1(x) + ch(\mathcal{A}ss_{d-1}))$

④ Props, properads

Symmetric bimodule $M = \{M(m,n)\}$, $\Sigma_m \times \Sigma_n^{op} \circlearrowright M(m,n)$ $\quad$ out in $\quad \{\mathrm{Hom}(X^n, X^m)\} = \mathrm{End}_X$

$$M \boxtimes N(m,n) = \bigoplus_{N \geq 1} \left(\bigoplus_{a,b} \bigoplus_{\bar{k}, \bar{\lambda}, \bar{j}, \bar{\ell}} \mathbb{K}[\Sigma_m] \otimes_{\Sigma_{\bar{\ell}}} M(\bar{\ell}, \bar{k}) \otimes_{\Sigma_{\bar{k}}} \mathbb{K}[\Sigma_N] \otimes_{\Sigma_{\bar{j}}} N(\bar{j}, \bar{\lambda}) \otimes_{\Sigma_{\bar{\lambda}}} \mathbb{K}[\Sigma_n] \right) / \sim$$



⚠ Not associative, not unital $((I \boxtimes I) \boxtimes M)(2,2) = \mathbb{Q}^2$ $\quad M = M(2,2) = \mathbb{Q}$

$(I \boxtimes (I \boxtimes M))(2,2) = \mathbb{Q}$

$S(M) = \bigoplus (M^{\otimes n})_{\Sigma_n}$ saturation $= I \boxtimes M$ if $M(0, \ldots) = 0$ $\quad$ ex $S(I) = I \boxtimes I =$

$= M \boxtimes I$ if $M(\ldots, 0) = 0$ $\quad (I \boxtimes I)(m,n) = \begin{cases} \mathbb{Q}[\Sigma_n] & m = n \\ 0 & \text{o/w} \end{cases}$

Prop = sym bimod + $\gamma: P \boxtimes P \longrightarrow P$ $\Leftrightarrow \begin{cases} \text{monoid for horizontal} \\ P \boxtimes' P \longrightarrow P, \quad \boxtimes': \text{single vertex on each level} \end{cases}$

$M \boxtimes_c N$: same but the graphs must be connected $\rightarrow$ associative, unital $\quad$ Properad = monoid

$S(M \boxtimes_c N) = M \boxtimes N$ $\quad \mathcal{C}om^c_c \bar{\circ} (M, 0) = S(M)$, $\mathcal{C}om^c_c(\mathcal{A}, 0) = \mathcal{L}ie_{\ldots}$, Odd

$ch(\mathcal{C}om^c_c) = \sum_{m \geq 1} h_m(x) = \exp\left(\frac{p_m(x)}{m}\right) - 1 \in \hat{\Lambda}_{x,y} \Rightarrow S(\bar{f}) = \sum_{m \geq 1} h_m(x) \bar{\circ} (\bar{f}(x,y), 0)$ if $\bar{f}(0,0)$ is in positive t-degree

$ch(S(M)) = S(ch\,M)$

ex $ch(S(I)) = \sum_{m \geq 1} h_m(x) \bar{\circ} (h_1(x) h_1(y), 0) = \underbrace{\prod_{i,j} (1 - x_i y_j)^{-1}}_{=: R_m(x,y)} - 1$ $\rightarrow$ recover a formula

4) Character of box product

Scalar product on Λ : $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$, $z_\lambda = \prod_i i^{m_i} m_i!$ m_i = number of occurr of i in λ

→ extend to $\Lambda_{x,y}$: $\langle p_\lambda(x) p_{\lambda'}(y), p_\mu(x) p_{\mu'}(y) \rangle = \langle p_\lambda, p_\mu \rangle \cdot \langle p_{\lambda'}, p_{\mu'} \rangle$

→ can define the adjoint of a map $T : \Lambda_{x,y} \to \Lambda_{x,y}$ ⇝ $T^\perp : \Lambda_{x,y} \to \Lambda_{x,y}$

for $f \in \Lambda_{x,y}$, let $f^\perp : \Lambda_{x,y} \to \Lambda_{x,y}$ adj of mult by f

ex $p_m^\perp(f) = m \cdot \frac{\partial f}{\partial p_m}(p_1, p_2, \dots)$ if $f = f(p_1, p_2, \dots)$

⇒ define the box product : $f \boxtimes g(x,y) = \sum_{m \geq 1} R_m(x',y')^\perp \left(S(f)(x,y') \cdot S(g)(x',y) \right) \Big|_{x'=y'=0}$

where $R_m^\perp = \sum_{\lambda \vdash m} z_\lambda^{-1} p_\lambda(x)^\perp p_\lambda(y)^\perp$

Thm $ch(M \boxtimes N) = ch M \boxtimes ch N$

lemma $ch(V)^\perp(ch(W)) = ch\left(Hom_{\Sigma_m \times \Sigma_n} \left(V, Res^{\Sigma_p \times \Sigma_q}_{\Sigma_m \times \Sigma_{p-m} \times \Sigma_n \times \Sigma_{q-m}} W \right) \right)$

$\Sigma_m \times \Sigma_m^{op} \curvearrowright V$, $\Sigma_p \times \Sigma_q^{op} \curvearrowright W$, $m \leq p$, $m \leq q$

$E = \sum_{n \geq 0} h_n(x) = \exp\left(\sum \frac{p_n(x)}{n} \right)$ $L = \sum_{k \geq 1} \frac{\mu_k}{k} \log(1 + p_k(x))$ $(E-1) \circ L = h_1 = L \circ (E-1)$

$(E(x)-1) \bar{\circ} (f, 0) = S(f)$

$f \boxtimes_c g = L \bar{\circ} (f \boxtimes g, 0)$ ⇒ $ch(M \boxtimes_c N) = ch M \boxtimes_c ch N$

5) Applications

$\mathcal{H} = \{ H^*(Aut(F_\infty); K^{p,q}) \}_{p,q \geq 0}$ is the wheeled prop assoc to Com [1]

[Kim 24] $\mathcal{H} \cong \omega \mathcal{P}$ where $\mathcal{P}(q,p) = \mathbb{Q}\{P(q,p)\}[p-q]$, $P(q,p) = \{ \lambda \vdash \underline{p} \mid$ at least q parts, q are labelled by $1 \dots q$, rest are unlabelled $\}$

$Q(q,p) = \begin{cases} triv_p & \text{if } q \in \{0,1\}, p \geq 1 \\ 0 & o/w \end{cases}$ ⇒ $\mathcal{P}(q,p) = S(Q)(q,p)[p-q]$

let $\Psi : \hat{\Lambda}_{x,y} \to \hat{\Lambda}_{x,y}((t))$, $h_q(x) \mapsto (-t)^{-q} h_q(x)$, $h_q(y) \mapsto (-t)^p h_p(y)$, $\Psi(f) = f \bar{\circ} (-t^{-1} h_1(x), -t h_1(y))$

⇒ $ch(\mathcal{H}) = \omega \Psi \, ch(S(Q)) = \omega \Psi \left(\sum_{m \geq 1} h_m(x) \bar{\circ} \left(\sum_{p \geq 1} h_p(y) + h_p(y) h_1(x) \right) \right)$

[Kim 25] $H^*_A(IA_\infty; \mathbb{Q})$ ⇝ $\mathcal{H}' \subset \mathcal{H}$ maximal non-unital sub-prop assoc to $\mathcal{H}$

⇒ $ch(\mathcal{H}') = \omega \Psi \, ch(S(Q'))$ where $Q'(q,p) = \begin{cases} triv_p & \text{if } q=0, p \geq 1 \text{ or } q=1, p \geq 2 \\ 0 & o/w \end{cases}$

$ch\, Q' = \sum_{p \geq 1} h_p(y) + \sum_{p \geq 2} h_p(y) h_1(x)$

Used some Mathematica code to do the computation (have to truncate $\mathcal{H}$)

$H^1_A(IA_\infty) = V_{1,1^2} \oplus V_{\emptyset,1}$, $H^2 = V_{2,21^2} \oplus V_{1^2,2^2} \oplus V_{1^2,1^4} \oplus V_{1,21} \oplus V_{1,1^3}^{\otimes 2} \oplus V_{\emptyset,1^2}^{\otimes 2}$

etc, matches previous computations