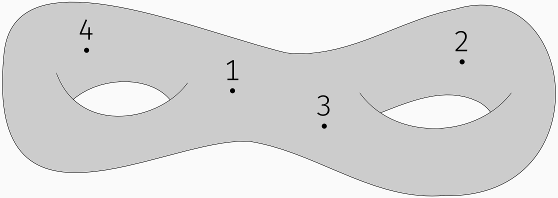
Configuration Spaces of Manifolds with Boundary

Najib Idrissi j/w R. Campos, P. Lambrechts, T. Willwacher

Graph Complexes, Configuration Spaces and Manifold Calculus @ UBC

Recollections

$$\mathrm{Conf}_k(M) := \{(x_1, \ldots, x_k) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



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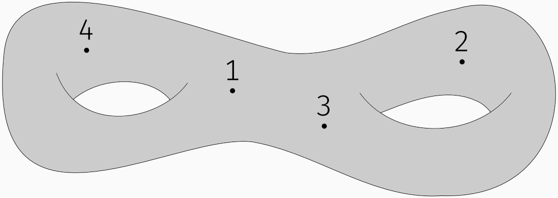

Question

Does the homotopy type of M determine the homotopy type of $\mathrm{Conf}_k(M)$? How to compute the homotopy type of $\mathrm{Conf}_k(M)$?

Theorem (Arnold, Cohen)

$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = S(\omega_{ij})/(\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij},\, \omega_{ij}^2,\, \omega_{ji} - \pm\omega_{ij})$

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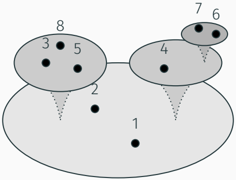

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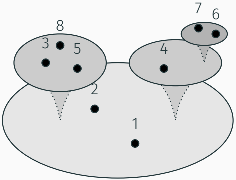

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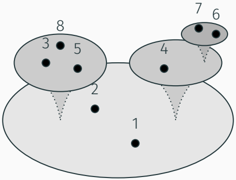

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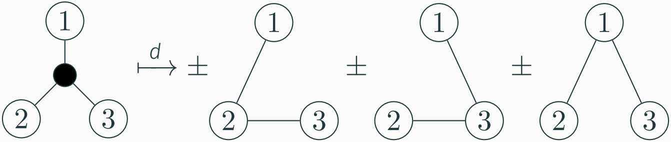

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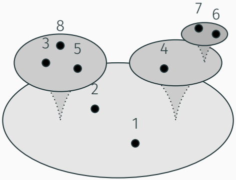

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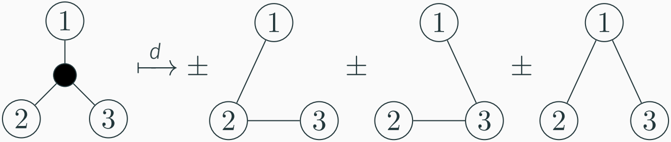

Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

$H^*(\mathsf{FM}_n) \xleftarrow{\sim} \mathtt{Graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$ as Hopf cooperads.

Recollections: $\mathrm{Conf}_k(M)$ for M closed

M: smooth, simply connected, closed n-manifold

$\to$ compactification FM_M of $\mathrm{Conf}_\bullet(M)$

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$A = S(\tilde{H}^*(M))$ or a cofibrant model of $M \implies A$-decorated graphs

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Corollary

For smooth closed simply connected manifolds,

$$M \simeq_{\mathbb{R}} N \implies \mathrm{Conf}_k(M) \simeq_{\mathbb{R}} \mathrm{Conf}_k(N).$$

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Generalizations for manifolds with boundary in three directions:

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Remark: we do everything in the fiberwise setting, so the operadic comodule structures exist in all cases. For simplicity I only state the parallelized case.

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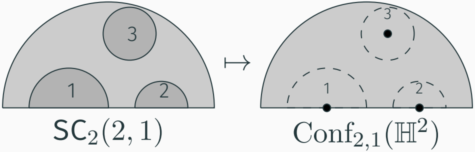

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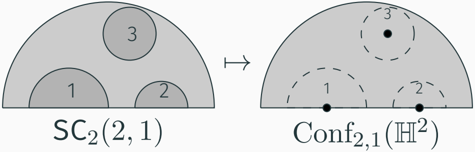
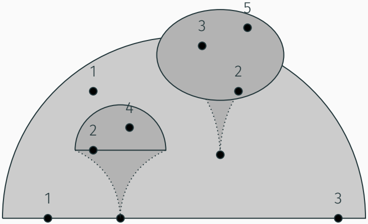


Compactify $\mathrm{Conf}_{\bullet,\bullet}(\mathbb{H}^n)/\mathbb{R}^{n-1} \rtimes \mathbb{R}_{>0} \implies \mathsf{SFM}_n$



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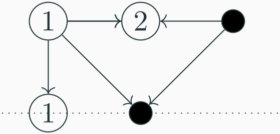

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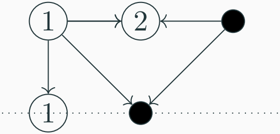

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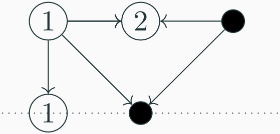

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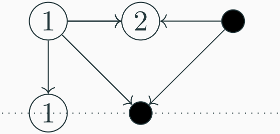
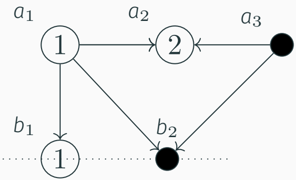


Theorem (Willwacher 2015)

$\mathtt{SGraphs}_n$ is a model for $\mathsf{SFM}_n = \overline{\mathrm{Conf}_{\bullet,\bullet}(\mathbb{H}^n)} \simeq \mathsf{SC}_n$.

$A = S(\tilde{H}^*(M) \oplus H^*(M, \partial M))$ and $A_\partial = S(\tilde{H}^*(\partial M)) \implies$ bicolored graphs:



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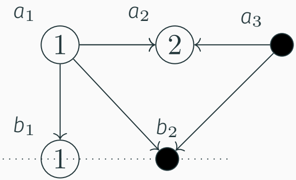

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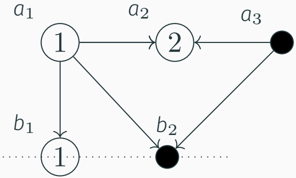

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Corollary

For smooth, simply connected, compact manifolds with boundary of dimension ≥ 5, the real homotopy type of SFM_M (incl. SFM_n-module structure) only depends on the real homotopy type of $(M, \partial M)$.

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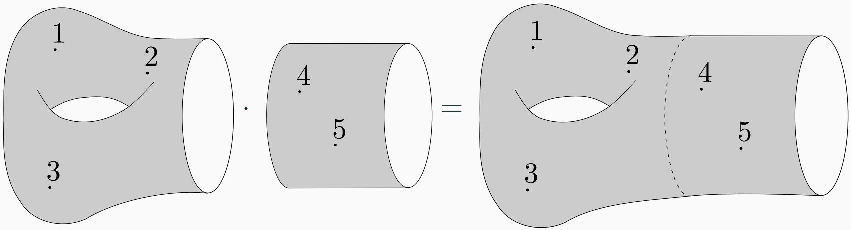

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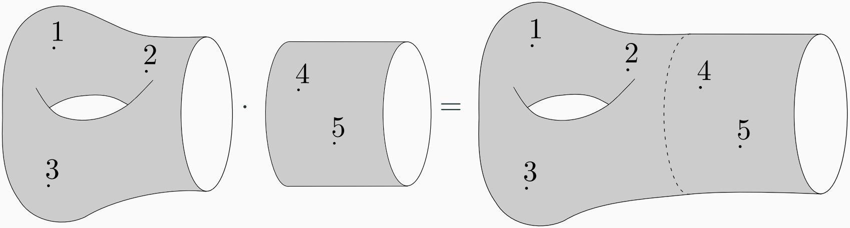

$\rightarrow$ useful to get the homotopy type of Conf_k "inductively":

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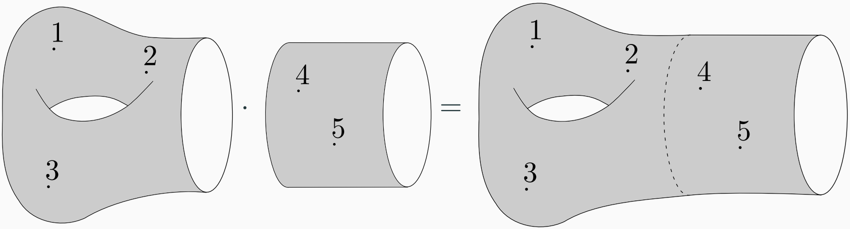

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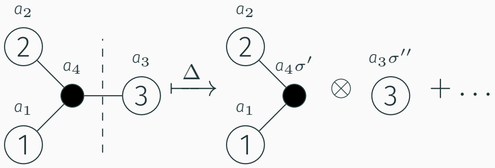
We can compactify configuration spaces and strictify this structure:

$$\mathtt{aFM}_{\partial M}(k) = \overline{\mathrm{Conf}_k(\partial M \times \mathbb{R})/\mathbb{R}_{>0}}, \qquad \mathtt{mFM}_M(k) = \overline{\mathrm{Conf}_k(M)}.$$

For $A_\partial = S(\tilde{H}(M)) \implies$ coalgebra in $\mathtt{Graphs}_n$-comodules $\mathtt{aGraphs}_{A_\partial}$:

- coalgebra: graph cutting



Configurations in a collar

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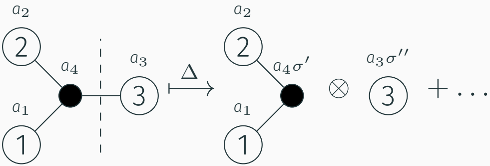

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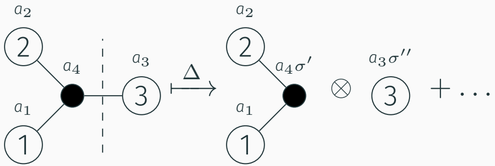

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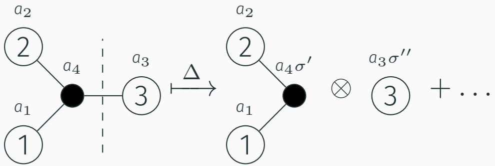

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Corollary

For a closed smooth $(n-1)$-manifold $N = \partial M$, the real homotopy type of N determines the real homotopy type of $\mathtt{aFM}_N$.

$\mathtt{mGraphs}_A$: $\mathtt{aGraphs}_{A_\partial}$-comodule in Hopf $\mathtt{Graphs}_n$-comodules given by A-labeled graphs

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Configurations in the interior

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Corollary

For a smooth compact manifold with boundary M of dimension ≥ 4, the real homotopy type of $(M, \partial M)$ determines the real homotopy type of $\texttt{mFM}_M = \overline{\mathrm{Conf}_\bullet(M)}$.

The Lambrechts–Stanley Model

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Obtain a smaller model for configuration spaces.

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Poincaré duality CDGA (P, ε):

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- $\varepsilon : P^n \to \mathbb{R}$ such that $\varepsilon \circ d = 0$;
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Theorem (Lambrechts–Stanley 2008)

M: simply connected + closed $\implies \exists (P, \varepsilon)$ Poincaré duality model:

$$P \xleftarrow{\sim} A \xrightarrow{\sim} \Omega^*(M).$$

Lambrechts–Stanley model (Intuition: $\mathrm{Conf}_k(M) = M^k \setminus \bigcup_{i \neq j}\{x_i = x_j\}$)

$$\mathsf{G}_P(k) := \left(\frac{P^{\otimes k} \otimes H^*(\mathrm{Conf}_k(\mathbb{R}^n))}{p_i^*(x)\omega_{ij} = p_j^*(x)\omega_{ij}}, d\omega_{ij} = p_{ij}^*(\Delta_P) \right),$$

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Theorem (Lambrechts–Stanley 2008)

$H^i(\mathsf{G}_P(k)) \cong_{\Sigma_k\text{-Vect}} H^i(\mathrm{Conf}_k(M))$.

Lambrechts–Stanley model (Intuition: $\mathrm{Conf}_k(M) = M^k \setminus \bigcup_{i \neq j}\{x_i = x_j\}$)

$$\mathsf{G}_P(k) := \left(\frac{P^{\otimes k} \otimes H^*(\mathrm{Conf}_k(\mathbb{R}^n))}{p_i^*(x)\omega_{ij} = p_j^*(x)\omega_{ij}}, d\omega_{ij} = p_{ij}^*(\Delta_P) \right),$$

Examples: $\mathsf{G}_P(0) = \mathbb{R}$ ✓, $\mathsf{G}_P(1) = P$ ✓, $\mathsf{G}_P(2) \simeq P^{\otimes 2}/(\Delta_P)$.

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Theorem (I.)

M smooth, closed, simply connected manifold, $\dim M \geq 4 \implies$

$$\mathsf{G}_P(k) \xleftarrow{\sim} \mathtt{Graphs}_A \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M),$$

compatible with $H^*(\mathsf{FM}_n) \xleftarrow{\sim} \mathtt{Graphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$ if parallelized.

Poincaré–Lefschetz duality

$\partial M \neq \varnothing \implies H^*(M)$ is paired with $H^{n-*}(M, \partial M)$

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Poincaré–Lefschetz duality pair $(B \xrightarrow{\lambda} B_\partial, \varepsilon, \varepsilon_\partial)$:

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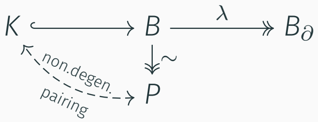
$\implies P^k \otimes K^{n-k} \to \mathbb{R}$, $x \otimes y \mapsto \varepsilon(xy)$ is non-degenerate for all k.



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Example and existence

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Remark

We can use PLD pairs instead of $S(\tilde{H}(M) \oplus H(M, \partial M))$ and $S(\tilde{H}(\partial M))$ in all the graph models.

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Then $P = H^*(M) = \mathbb{R} \oplus \mathbb{R}\eta$.

- in $\mathsf{G}_P(2)$: $(1 \otimes \eta)\omega_{12} = (\eta \otimes 1)\omega_{12}$.
- in $\mathrm{Conf}_3(\mathbb{R}^2)$ (Arnold): $(1 \otimes \eta)\omega_{12} = (\eta \otimes 1)\omega_{12} \pm (\eta \otimes \eta)$.

The actual model

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For $\dim M \leq 6$, we can define $\tilde{\mathsf{G}}_{H^*(M)}(k) = \mathtt{mGraphs}_{H^*(M)}(k)/(\text{int. vtx.})$, still a model but less explicit if $\dim M \leq 3$.

Thank you for your attention!

These slides, links to papers: `https://idrissi.eu`