Homotopy II: Exam

M2 Fundamental Mathematics

Duration: 3 hours. Printed or handwritten notes are allowed. Electronic devices are forbidden. The exam is 2 pages long. Write in French or English and justify your answers.

Exercise A. Uniqueness of lifts

Let $\mathcal{M}$ be a model category and let $A, Y \in \mathcal{M}$ be two objects. The category $\mathcal{M}_{A,Y}$ has as objects triples $(X, f\colon A \to X, g\colon X \to Y)$, and $\mathrm{Hom}_{\mathcal{M}_{A,Y}}\big((X,f,g),(X',f',g')\big) \coloneqq \{\, h\colon X \to X' \mid hf = f',\ g'h = g \,\}$.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & & \downarrow p \\ B & \xrightarrow[g]{} & Y \end{array}$$

1. Prove that $\mathcal{M}_{A,Y}$ is a model category with fibrations, cofibrations, and weak equivalences being the same as in $\mathcal{M}$.
2. Consider a commutative square as on the side, where i is a cofibration and p is an acyclic fibration. Prove that any two lifts $l, l'\colon B \to X$ (that fit in the commutative square) are homotopic when seen as morphism in $\mathcal{M}_{A,Y}$. (Hint: factor $B \cup_A B \to B$ using MC5.)

Exercise B. Sharp morphisms and right properness

Let $\mathcal{M}$ be a model category. A morphism $p\colon X \to Y$ is called *sharp* when, for any commutative diagram as displayed on the side, if both squares are pullbacks ($A = A' \times_{B'} B$, $A' = B' \times_Y X$) and j is a weak equivalence, then i is a weak equivalence.

$$\begin{array}{ccccc} A & \xrightarrow{i} & A' & \xrightarrow{f} & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ B & \xrightarrow[j]{} & B' & \xrightarrow[g]{} & Y \end{array}$$

1. Prove that every fibration is sharp if and only if $\mathcal{M}$ is *right proper*, i.e., the pullback of a weak equivalence along a fibration is a weak equivalence.

Consider the category $I = \{0 \to 2 \leftarrow 1\}$ and equip $\mathcal{M}^I = \mathrm{Fun}(I, \mathcal{M})$ with the injective model structure (weak equivalences and cofibrations are defined object-wise).

2. Prove that a diagram $\{X \to Z \leftarrow Y\} \in \mathcal{M}^I$ is fibrant if and only if Z is fibrant and both maps in the diagram are fibrations.

For the next two questions, let us assume that $\mathcal{M}$ is right proper.

3. Prove that for $\{X \to Z \leftarrow Y\} \in \mathcal{M}^I$, if $X \to Z$ a fibration and X, Y, Z are fibrant, then the pullback $X \times_Z Y$ is weakly equivalent to the homotopy pullback (= holim_I of the diagram).
4. Prove that the same conclusion holds if we assume that $X \to Z$ is sharp rather than a fibration.

Let $\mathcal{M} = \mathrm{Ch}_{\geq 0}(\mathbb{Z})$, with the projective model structure. Let us moreover equip $\mathcal{M}^I = \big(\mathrm{Ch}_{\geq 0}(\mathbb{Z})\big)^I$ with the injective model structure of diagrams as above.

5. Let $\{X \to Z \leftarrow Y\}$ be a diagram of $\mathbb{Z}$-modules such that $X \to Z$ is surjective. Prove that $\ker(X \to Z)$ is isomorphic to $\ker(X \times_Z Y \to Y)$.
6. Prove that $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is right proper (hint: use the five lemma).
7. Let $d \geq 1$ be an integer, let M be a $\mathbb{Z}$-module, and let $\Sigma^d M$ be M viewed as a chain complex concentrated in degree d. Compute the homotopy limit of the diagram $\{0 \to \Sigma^d M \leftarrow 0\}$.

Let $\mathcal{M} = s\mathcal{S}et$ be endowed with the usual model structure. Let $\pi\colon \Lambda^2_1 \to \Delta^1$ be the unique simplicial map which is given on vertices by $\pi(0) = 0, \pi(1) = \pi(2) = 1$.

8. Prove that π is **not** a Kan fibration.
9. Construct a map $\sigma\colon \Delta^1 \to \Lambda^2_1$ such that $\pi\sigma = \mathrm{id}_{\Delta^1}$ and $\sigma\pi$ is homotopic to the identity of Λ^2_1.
10. ★ Prove that π is sharp.

Exercise C. Model category of equivalence relations

Let $\mathcal{E}q$ be the category whose objects are pairs $(X, \sim)$ where X is a set and $\sim$ is an equivalence relation on X, and whose morphisms are maps which preserve equivalence, i.e.:

$$\mathrm{Hom}_{\mathcal{E}q}\big((X, \sim_X), (Y, \sim_Y)\big) \coloneqq \{\, f : X \to Y \mid \forall x, x' \in X,\ x \sim_X x' \Longrightarrow f(x) \sim_Y f(x') \,\}.$$

We will often allow ourselves the notational shortcut $X = (X, \sim_X)$, $Y = (Y, \sim_Y)$, etc.

1. Prove that the categorical product is given by $(X, \sim_X) \times (Y, \sim_Y) = (X \times Y, \sim_{X\times Y})$, where:
$$(x, y) \sim_{X\times Y} (x', y') \Leftrightarrow (x \sim_X x' \text{ and } y \sim_Y y').$$
2. Let $A = \{a, b, c\}$ with $a \sim b \nsim c$; $B = \{x, y\}$ with $x \sim y$; and $C = \{u, v\}$ with $u \nsim v$. Let $f : C \to A$ be given by $f(u) = b$, $f(v) = c$, and $g : C \to B$ be given by $g(u) = x$ and $g(v) = y$. Prove that in the pushout $A \cup_C B$, one has $a \sim c$. (A picture can help.)

For $X \in \mathcal{E}q$ and $x \in X$, we let $[x] = \{\, x' \in X \mid x' \sim_X x \,\}$ and $(X/\sim) \coloneqq \{\, [x] \mid x \in X \,\}$. For any $X, Y \in \mathcal{E}q$, a morphism $f : X \to Y$ in $\mathcal{E}q$ is called a:

- *Cofibration* if $f : X \to Y$ is injective as a map of sets.
- *Fibration* if, for all $x \in X$, the restriction $f|_{[x]} : [x] \to [f(x)]$ is surjective.
- *Weak equivalence* if the induced map on the quotient $f_* : (X/\sim) \to (Y/\sim)$ is bijective.

3. Let $j : \{0\} \to (\{0, 1\}, \sim)$ with $0 \sim 1$. Prove that a morphism is a fibration if, and only if, it has the right lifting property against j. (You may not yet assume that $\mathcal{E}q$ is a model category.)
4. Let $i_0 : \emptyset \to \{0\}$ and let $i_1 : (\{0, 1\}, \sim_1) \to (\{0, 1\}, \sim_2)$ where $0 \nsim_1 1$ and $0 \sim_2 1$. Prove that a morphism is an acyclic fibration if, and only if, it has the right lifting property against i_0 and i_1.
5. Prove that $\mathcal{E}q$ is a cofibrantly generated model category, with generating cofibrations $\mathcal{I} = \{i_0, i_1\}$ and generating acyclic cofibrations $\mathcal{J} = \{j\}$.

Let an equivalence relation $\approx$ on $\mathrm{Hom}_{\mathcal{E}q}(X, Y)$ be defined, for $f, g : X \to Y$, by:

$$f \approx g \Leftrightarrow \big(\forall x \in X, \quad f(x) \sim_Y g(x)\big).$$

In what follows, we will denote by $[X, Y]$ the hom-set equipped with this equivalence relation.

6. Prove that two morphisms f, g are homotopic in $\mathcal{E}q$ if and only if $f \approx g$.
7. Prove that the functor $\pi : \mathcal{E}q \to \mathcal{S}et$, given on objects by $X \mapsto X/\sim_X$, induces an equivalence of categories $\mathrm{Ho}(\mathcal{E}q) \simeq \mathcal{S}et$.
8. Prove that the pullback of a weak equivalence along a fibration is a weak equivalence.
9. ★ Prove that the pushout of a weak equivalence along a cofibration is a weak equivalence.
10. Prove that there is an isomorphism in $\mathcal{E}q$, natural in $A, X, Y \in \mathcal{E}q$:
$$\big[A, [X, Y]\big] \cong [A \times X, Y].$$

Let $i : A \to B$ be a cofibration and $p : X \to Y$ be a fibration (in $\mathcal{E}q$). Consider the "pullback-corner":

$$(i^*, p_*) : [B, X] \to [A, X] \times_{[B,Y]} [A, Y].$$

11. Prove that (i^*, p_*) is a fibration in $\mathcal{E}q$.
12. Prove that this fibration is acyclic if either one of the morphisms i or p is acyclic.