Homotopy II: Exam

M2 Fundamental Mathematics

Duration: 3 hours. Printed or handwritten notes are allowed. Electronic devices are forbidden. The exam is 2 pages long. Write in French or English and justify your answers.

Exercise A. Uniqueness of lifts

Let $\mathcal{M}$ be a model category and let $A, Y \in \mathcal{M}$ be two objects. The category $\mathcal{M}_{A,Y}$ has as objects triples $(X, f: A \to X, g: X \to Y)$, and $\mathrm{Hom}_{\mathcal{M}_{A,Y}}\big((X,f,g),(X',f',g')\big) \coloneqq \{\, h: X \to X' \mid hf = f',\ g'h = g \,\}$.

1. Prove that $\mathcal{M}_{A,Y}$ is a model category with fibrations, cofibrations, and weak equivalences being the same as in $\mathcal{M}$.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & & \downarrow p \\ B & \xrightarrow[g]{} & Y \end{array}$$

This is Theorem 7.6.5 in:

Hirschhorn, P.: *Model Categories and their Localizations*. Mathematical Surveys and Monographs 99. American Mathematical Society, Providence, RI (2003). DOI:10.1090/surv/099.

Note that there was a mistake in the handed-out version of the exam (corrected during the exam). One needs to fix some morphism $\phi: A \to Y$ and look at the subcategory of $\mathcal{M}_{A,Y}$ consisting of objects (X, f, g) such that $gf = \phi$. I've given full points for the question even if the proof of MC1 was missing. Sorry about that.

2. Consider a commutative square as on the side, where i is a cofibration and p is an acyclic fibration. Prove that any two lifts $l, l': B \to X$ (that fit in the commutative square) are homotopic when seen as morphism in $\mathcal{M}_{A,Y}$. (Hint: factor $B \cup_A B \to B$ using MC5.)

The proof can be found in Section 7.6.12 of the book. A little argument is needed to explain why we can choose a left homotopy ((B, i, pf) is cofibrant because i is a cofibration).

Exercise B. Sharp morphisms and right properness

A general reference for this exercise is Section 2 of:

Rezk, C.: *Fibrations and homotopy colimits of simplicial sheaves*. arXiv:math/9811038. Section 2.

Let $\mathcal{M}$ be a model category. A morphism $p: X \to Y$ is called *sharp* when, for any commutative diagram as displayed on the side, if both squares are pullbacks ($A = A' \times_{B'} B$, $A' = B' \times_Y X$) and j is a weak equivalence, then i is a weak equivalence.

$$\begin{array}{ccccc} A & \xrightarrow{i} & A' & \xrightarrow{f} & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ B & \xrightarrow[j]{} & B' & \xrightarrow[g]{} & Y \end{array}$$

1. Prove that every fibration is sharp if and only if $\mathcal{M}$ is *right proper*, i.e., the pullback of a weak equivalence along a fibration is a weak equivalence.

Consider the category $I = \{0 \to 2 \leftarrow 1\}$ and equip $\mathcal{M}^I = \mathrm{Fun}(I, \mathcal{M})$ with the injective model structure (weak equivalences and cofibrations are defined object-wise).

2. Prove that a diagram $\{X \to Z \leftarrow Y\} \in \mathcal{M}^I$ is fibrant if and only if Z is fibrant and both maps in the diagram are fibrations.

Seen in class. One needs to prove that $\{X \to Z \leftarrow Y\}$ has the RLP with respect to any acyclic cofibration (which are defined pointwise). Note that it is not enough to construct the lift pointwise, as this may not produce a morphism of diagrams. Instead, one should start by constructing the lift to Z, then draw a commutative square to construct a lift to X (or Y) which makes a morphism of diagrams.

For the next two questions, let us assume that $\mathcal{M}$ is right proper.

3. Prove that for $\{X \to Z \leftarrow Y\} \in \mathcal{M}^I$, if $X \to Z$ a fibration and X, Y, Z are fibrant, then the pullback $X \times_Z Y$ is weakly equivalent to the homotopy pullback (= holim_I of the diagram).
4. Prove that the same conclusion holds if we assume that $X \to Z$ is sharp rather than a fibration.

There is a little subtlety here. If in question 3 you argued that $X \times_Z^h Y = X \times_Z Y'$ where $Y' \to Z$ is a replacement of $Y \to Z$ by a fibration and used the fact that fibrations are sharp, then you have a little more work to do here. Indeed, to compute the homotopy pullback, you need to replace the whole thing by a fibrant diagram, so $X \to Z$ also needs to be replaced.

Let $\mathcal{M} = \mathrm{Ch}_{\geq 0}(\mathbb{Z})$, with the projective model structure. Let us moreover equip $\mathcal{M}^I = \left(\mathrm{Ch}_{\geq 0}(\mathbb{Z})\right)^I$ with the injective model structure of diagrams as above.

5. Let $\{X \to Z \leftarrow Y\}$ be a diagram of $\mathbb{Z}$-modules such that $X \to Z$ is surjective. Prove that $\ker(X \to Z)$ is isomorphic to $\ker(X \times_Z Y \to Y)$.

The astute ones will have noted that the surjectivity assumption is not necessary. Let $f: X \to Z$ and $g: Y \to Z$, then:

$$\ker(X \times_Y Z \to Y) = \{\, (x, y) \mid f(x) = g(y), y = 0 \,\} = \{\, x \in X \mid f(x) = 0 \,\} = \ker(X \to Z).$$

6. Prove that $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is right proper (hint: use the five lemma).
7. Let $d \geq 1$ be an integer, let M be a $\mathbb{Z}$-module, and let $\Sigma^d M$ be M viewed as a chain complex concentrated in degree d. Compute the homotopy limit of the diagram $\{0 \to \Sigma^d M \leftarrow 0\}$.

According to the above, and since $\Sigma^d M$ is fibrant, we just need to replace one of the two maps by a fibration. A canonical way is to take the cone

$$C(\Sigma^d M) = \left(\Sigma^d M \oplus \Sigma^{(d-1)} M, d(x, y) = (y, 0)\right).$$

Then $0 \times_{\Sigma^d M}^h 0 = C(\Sigma^d M) \times_{\Sigma^d M} 0 = \Sigma^{d-1} M$. Note that if $d = 0$, then $0 \to \Sigma^0 M$ is already a fibration so the homotopy pullback is the classical pullback, i.e., the null chain complex.

Let $\mathcal{M} = sSet$ be endowed with the usual model structure. Let $\pi: \Lambda_1^2 \to \Delta^1$ be the unique simplicial map which is given on vertices by $\pi(0) = 0, \pi(1) = \pi(2) = 1$.

8. Prove that π is **not** a Kan fibration.

Define a square:

$$\begin{array}{ccc} \Lambda_1^1 & \overset{f}{\to} & \Lambda_1^2 \\ \downarrow & & \downarrow p \\ \Delta^1 & \underset{=}{\to} & \Delta^1 \end{array}$$

By $f(1) = 2$, and the bottom map is the identity. If a lift $l: \Delta^1 \to \Lambda_1^2$ existed, it would correspond to a 1-simplex of $x = l(0 \to 1) \in \Lambda_1^2$ such that $d_0(x) = 2$ and $\pi\left(d_1(x)\right) = 0$, i.e., an edge from vertex 0 to vertex 2. Such an edge does not exist, so there is no lift.

9. Construct a map $\sigma: \Delta^1 \to \Lambda_1^2$ such that $\pi\sigma = \mathrm{id}_{\Delta^1}$ and $\sigma\pi$ is homotopic to the identity of Λ_1^2.

Just take $\sigma(0) = 0$ and $\sigma(1) = 1$ (and $\sigma(0 \to 1) = 0 \to 1 \in \Lambda_1^2$.

10. ★ Prove that π is sharp.

The above shows that π is deformation retraction. Taking the pullback of π along any map remains a deformation retraction, so one can could by (MC2).

Exercise C. Model category of equivalence relations

This exercise is taken from:

Larusson, F.: *The homotopy theory of equivalence relations*. arXiv:math/0611344v2.

Let $\mathcal{E}q$ be the category whose objects are pairs $(X,\sim)$ where X is a set and $\sim$ is an equivalence relation on X, and whose morphisms are maps which preserve equivalence, i.e.:

$$\mathrm{Hom}_{\mathcal{E}q}\big((X,\sim_X),(Y,\sim_Y)\big) := \{\, f: X \to Y \mid \forall x, x' \in X,\ x \sim_X x' \Longrightarrow f(x) \sim_Y f(x') \,\}.$$

We will often allow ourselves the notational shortcut $X = (X,\sim_X)$, $Y = (Y,\sim_Y)$, etc.

For the first two questions, it's not possible to assume from the beginning that the underlying set of the (co)product is the (co)product of the underlying set; a proof is needed.

1. Prove that the categorical product is given by $(X,\sim_X) \times (Y,\sim_Y) = (X \times Y, \sim_{X\times Y})$, where:
$$(x,y) \sim_{X\times Y} (x',y') \Leftrightarrow (x \sim_X x' \text{ and } y \sim_Y y').$$
2. Let $A = \{a,b,c\}$ with $a \sim b \nsim c$; $B = \{x,y\}$ with $x \sim y$; and $C = \{u,v\}$ with $u \nsim v$. Let $f: C \to A$ be given by $f(u) = b$, $f(v) = c$, and $g: C \to B$ be given by $g(u) = x$ and $g(v) = y$. Prove that in the pushout $A \cup_C B$, one has $a \sim c$. (A picture can help.)

For $X \in \mathcal{E}q$ and $x \in X$, we let $[x] = \{\, x' \in X \mid x' \sim_X x \,\}$ and $(X/\sim) := \{\, [x] \mid x \in X \,\}$. For any $X, Y \in \mathcal{E}q$, a morphism $f: X \to Y$ in $\mathcal{E}q$ is called a:

- *Cofibration* if $f: X \to Y$ is injective as a map of sets.
- *Fibration* if, for all $x \in X$, the restriction $f|_{[x]}: [x] \to [f(x)]$ is surjective.
- *Weak equivalence* if the induced map on the quotient $f_*: (X/\sim) \to (Y/\sim)$ is bijective.

3. Let $j: \{0\} \to (\{0,1\},\sim)$ with $0 \sim 1$. Prove that a morphism is a fibration if, and only if, it has the right lifting property against j. (You may not yet assume that $\mathcal{E}q$ is a model category.)

Arguing that f is a fibration and that j is an acyclic cofibration and that fibrations lift against acyclic cofibrations is insufficient. We don't know yet that we have a model category! Same deal for the next question.

4. Let $i_0: \emptyset \to \{0\}$ and let $i_1: (\{0,1\},\sim_1) \to (\{0,1\},\sim_2)$ where $0 \nsim_1 1$ and $0 \sim_2 1$. Prove that a morphism is an acyclic fibration if, and only if, it has the right lifting property against i_0 and i_1.

Note that even if $f: X \to Y$ is surjective, it's possible that f is not a fibration. For $x \in X$, every element of $[f(x)]$ has a preimage... But this preimage may not belong to $[x]$! A counterexample is right in the question: i_1 is surjective, but it is not a fibration, as e.g., $i_1: [0] = \{0\} \to [i_1(0)] = \{0,1\}$ is not surjective.

5. Prove that $\mathcal{E}q$ is a cofibrantly generated model category, with generating cofibrations $\mathcal{I} = \{i_0, i_1\}$ and generating acyclic cofibrations $\mathcal{J} = \{j\}$.

Since we don't know that we have a model category, this needs to be proved. Thanks to what we've done in the previous questions, almost all the hypotheses of the theorem on existence of a cofibrantly generated model structure are verified.

Let an equivalence relation $\approx$ on $\mathrm{Hom}_{\mathcal{E}q}(X,Y)$ be defined, for $f,g: X \to Y$, by:

$$f \approx g \Leftrightarrow \big(\forall x \in X,\quad f(x) \sim_Y g(x)\big).$$

In what follows, we will denote by $[X,Y]$ the hom-set equipped with this equivalence relation.

6. Prove that two morphisms f,g are homotopic in $\mathcal{E}q$ if and only if $f \approx g$.

Since all objects are fibrant and cofibrant, you may choose a left or right homotopy, but this needs to be said.

7. Prove that the functor $\pi: \mathcal{E}q \to \mathcal{S}et$, given on objects by $X \mapsto X/{\sim_X}$, induces an equivalence of categories $\mathrm{Ho}(\mathcal{E}q) \simeq \mathcal{S}et$.
8. Prove that the pullback of a weak equivalence along a fibration is a weak equivalence.
9. ★ Prove that the pushout of a weak equivalence along a cofibration is a weak equivalence.
10. Prove that there is an isomorphism in $\mathcal{E}q$, natural in $A, X, Y \in \mathcal{E}q$:
$$\big[A, [X, Y]\big] \cong [A \times X, Y].$$

Let $i: A \to B$ be a cofibration and $p: X \to Y$ be a fibration (in $\mathcal{E}q$). Consider the "pullback-corner":

$$(i^*, p_*): [B, X] \to [A, X] \times_{[B,Y]} [A, Y].$$

11. Prove that (i^*, p_*) is a fibration in $\mathcal{E}q$.
12. Prove that this fibration is acyclic if either one of the morphisms i or p is acyclic.