Configurations spaces, algebraic topology and operads

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Symbols

- Important
- Question
- Theorem
- Definition
- Conjecture

Configuration spaces

Definition and applications

- Ordered configuration space of r points in M: $\mathrm{Conf}_M(r) := \left\{ (x_1, \dots, x_r) \in M^r \mid \forall i \neq j, x_i \neq x_j \right\}$.
- Applications: Braid groups; Loop spaces; Splitting of Map_c; Embedding calculus; Gelfand-Fuks cohomology; Path planning

Homotopy invariance

- Homotopy, $\mathbb{R} \simeq \{0\}$
- Conjecture: $X \simeq Y \Rightarrow \mathrm{Conf}_X(r) \simeq \mathrm{Conf}_Y(r)$, counterexamples are easy to find
- Restrictions to manifolds:
  - Obvious if $\dim M \leq 2$
  - Bödigheimer-Cohen-Taylor, Bendersky-Gitler: invariance of homology
  - Levitt: invariance of $\Omega \mathrm{Conf}_M$
  - Aouina-Klein: invariance of $\Sigma^\infty \mathrm{Conf}_M$
- Counterexample due to Longoni-Salvatore: $L_{7,1}$ vs $L_{7,2}$
- Conjecture remains for simply connected spaces

Rational homotopy theory

Homotopy groups: $\pi_*(X)$, Whitehead theorem

- Definition of rational equivalence for simply connected spaces
  - Can be generalized to nilpotent spaces of finite type
- Conjecture $\otimes \mathbb{Q}$: if $M \simeq_{\mathbb{Q}} N$ then $\mathrm{Conf}_M(r) \simeq_{\mathbb{Q}} \mathrm{Conf}_N(r)$
- General conjecture does not imply rational conjecture
- Sullivan models
  - Definitions of CDGAs & quasi-isomorphisms
  - Examples: $H^*(M)$, $A = A^0$, $\Omega^*_{dR}(M)$
  - Piecewise linear forms
    - $\mathcal{A}_\mathrm{n} := S(t_0, \dots, t_n, dt_0, \dots, dt_n) / (\sum t_i = 1, \sum dt_i = 0)$
    - Differential $\delta(t_i) = dt_i$, $\delta(dt_i) = 0$
    - Cofaces, codegeneracies:
    - $\sigma^\mathrm{i}(t_k) = \begin{cases} t_k & k < i \\ 0 & k = i \\ t_{k-1} & k > i \end{cases}$, $\quad \partial^j(t_k) = \begin{cases} t_k & k < i \\ t_k + t_{k+1} & k = i \\ t_{k+1} & k > i \end{cases}$
    - Finally, $\Omega^k_{PL}(X) := \left\{ \left(\omega_f \in \mathcal{A}^k_n \right)_{f : \Delta^n \to X} \,\middle|\, d_i(\omega_f) = \omega_{f \circ \partial^i},\ s_j(\omega_f) = \omega_{f \circ \sigma^j} \right\}$
  - Theorem $\Omega^*_{\mathrm{PL}}(X) \simeq C^*(X; \mathbb{Q})$
  - Theorem equivalence between $\mathrm{Top}\left[\mathcal{W}_{\mathbb{Q}}^{-1}\right]$ and $\mathrm{CDGA}_{\mathbb{Q}}[\mathrm{qiso}^{-1}]$.
  - Sullivan model of X = any CDGA quasi-iso to $\Omega^*_{PL}(X)$: knows everything about the rational homotopy type of X
  - There is a version of all that over the real numbers
- Refined conjecture: can find a model of $\mathrm{Conf}_M(r)$ from a model of M

Formality of $\mathrm{Conf}_{\mathbb{R}^n}(r)$

- Theorem (Arnold)

  $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r)) = S(\omega_{ij})_{1 \leq i \neq j \leq r} / \left(\omega_{ij}^2 = 0, \omega_{ji} = (-1)^n \omega_{ij}, \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0 \right)$
  - ω_{ij} is dual to two points rotating one around the other
  - $\mathrm{Conf}_{\mathbb{R}^n}(2) \simeq S^{n-1}$ and ω_{ij} is the fundamental class, first two relations follow easily
  - Third relation: Jacobi-type
  - Proof using the method of solar systems

- Interpretation using graphs
- A space X is formal if $H^*(X;\mathbb{Q})$ is a Sullivan model of X
- Examples and counterexamples
  - Spheres
  - Lie groups, H-spaces
  - Spaces of dimension $\leq 4p-2$ such that $\pi_{<p}=0$
  - Compact Kähler manifolds ($\omega = \Im h$ is closed for some Hermitian metric)
  - If S is a surface of genus 2, then $\mathrm{Conf}_S(2)$ is not formal (even though S is formal)
- Theorem (Arnold) $\mathrm{Conf}_{\mathbb{C}}(r)$ is formal for all r. Proof: $\omega_{ij} \mapsto d\log(z_i - z_j)$.
- Theorem (Kontsevich) $\mathrm{Conf}_{\mathbb{R}^n}(r)$ is formal for all $n \geq 2$ and all r.

Lambrechts–Stanley model

Model definition

- Theorem (Lambrechts-Stanley) Any manifold any a Poincaré duality model: CDGA (A,d) equipped with $\epsilon: A^n \to \mathbb{R}$ st $\epsilon \circ d = 0$ and $A^\vee[-n] \cong A$.
- Diagonal class: $\Delta_A \in A \otimes A$ such that $\forall a \in A, a = (\epsilon \otimes 1)(\Delta_A \cdot (a \otimes 1))$.
- Candidate model of $\mathrm{Conf}_M(r)$ $G_A(r) := \left(A^{\otimes r} \otimes H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))/(a_i\omega_{ij} = a_j\omega_{ij}), d\omega_{ij} = \Delta_{ij}\right)$.
- Interpretation in terms of graphs with decorations.
- Small examples: $r \in \{0,1,2\}$
- History of this model:
  - Cohen-Taylor $E^2 = G_{H^*(M)}(r) \Rightarrow H^*(\mathrm{Conf}_M(r))$
  - Kriz, Totaro: when M is a smooth projective complex manifold (implies compact Kähler)
  - Lambrechts-Stanley: case $r=2$ and $\pi_{\leq 2}M = 0$
  - Cordova Bulens: case $r=2$, $\pi_{\leq 1}M = 0$, $\dim M$ even
  - Bendersky-Gitler: dual spectral sequence (Félix-Thomas, Berceanu-Markl-Papadima)
  - Lambrechts-Stanley: $H^*(G_A(r)) = H^*(\mathrm{Conf}_M(r))$ if $\pi_{\leq 1}M = 0$ as $\mathfrak{S}_r$-modules
- Theorem Always a model if M is smooth, simply connected, dimension ≥ 4.
- Sketch of proof:
  - Resolve $G_A(r)$ using graph complexes
  - Combinatorics: $\mathrm{Graphs}_A(r) \xrightarrow{\sim} G_A(r)$
  - Compactify configuration spaces to compute integrals
  - The bundles aren't submersions: we must use PA forms
  - Some integrals vanish thanks to $\dim M \geq 4 \wedge \pi_1 M = 0$
  - $G_A(r) \simeq G_B(r)$ if $A \simeq B$

Fulton-MacPherson compactification

- Compactification of $\mathrm{Conf}_{\mathbb{R}^n}(U)$ for a finite set U
- $\theta_{ij}: \mathrm{Conf}_{\mathbb{R}^n}(U) \to S^{n-1}$, $(x_u)_{u\in U} \mapsto \dfrac{x_i - x_j}{\|x_i - x_j\|}$, $\quad \delta_{ijk}: \mathrm{Conf}_{\mathbb{R}^n}(U) \to [0,+\infty]$, $(x_u)_{u\in U} \mapsto \dfrac{\|x_i - x_j\|}{\|x_i - x_k\|}$
- $\mathrm{FM}_n(U)$ is the closure of the image in $(S^{n-1})^{r(r-1)} \times [0,+\infty]^{r(r-1)(r-2)}$ (where $r = \#U$)
- This is a smooth manifold with corners of dimension $nr - n - 1$ (or 0 if $r \leq 1$).
- Boundaries
  - Given $W \subset U$, $\partial_W \mathrm{FM}_n(U)$ is isomorphic to $\mathrm{FM}_n(U/W) \times \mathrm{FM}_n(W)$.
  - $\partial \mathrm{FM}_n(U) = \bigcup_{W\subset U, \#W\geq 2} \partial_W \mathrm{FM}_n(U)$, each part is of codimension 1, intersection of codim ≥ 2
  - Fiberwise boundary $\pi: E \to B$, $\dim \pi = k$: $\pi^\partial: \bigcup_{x\in B} \partial\pi^{-1}(\{x\}) \to B$, bundle, $\dim \pi^\partial = r - 1$.
  - Fiberwise boundary of $\pi: \mathrm{FM}_n(U) \to \mathrm{FM}_n(A)$ is given by
    $$\mathrm{FM}_n^\partial(A) := \bigcup_{\substack{\#W\geq 2 \\ \#W\cap U \leq 1 \vee U \subset W}} \partial_W \mathrm{FM}_n(U)$$

Semi-algebraic sets and PA forms

- Problem: $\mathrm{FM}_M(A \sqcup I) \to \mathrm{FM}_M(A)$ is not a submersion in general
- Solution: they are semi-algebraic fiber bundles!
- Semi-algebraic (SA) set: union of intersection of solution sets of polynomial inequalities
- SA bundle: $\pi: E \to B$ + cover $\{U_\alpha\}$ of B + SA homeos $U_\alpha \times F \cong \pi^{-1}(U_\alpha)$ compatible with π
- SA chains: $C_*^{SA}(X) := \{ f_*[\![M]\!] \mid M: \text{compact SA variety}, f: M \to X \text{ SA map} \}$.
- For ω a differential form on $\mathbb{R}^N$ with support on X, we let $\langle f_*[\![M]\!], \omega \rangle := \sum_i \int_{S_i} f^*\omega$, where $M = \bigcup_i S_i$ is a suitable

stratification.

- Minimal forms: for $f, g_i: X \to \mathbb{R}$, let $\lambda(f; g_1, \dots, g_k) \in \Omega^k_{\min}(X)$ defined by $\langle \lambda(f; g_1, \dots, g_k), \gamma \rangle \coloneqq \left\langle (f, g_1, \dots, g_k)_* \gamma, x_0\, dx_1 \wedge \dots \wedge dx_k \right\rangle$
- PA forms: formal integrals along boundaries $\pi_* \lambda \in \Omega^{(k-d)}_{\mathrm{PA}}(X)$ where $\lambda \in \Omega^k_{\min}(E)$ and $\pi: E \to X$ of rank d
- More generally, for $\Phi: X \to C_l(Y)$ strongly continuous, can define $\left\langle \int_\Phi \lambda, \gamma \right\rangle \coloneqq \langle \lambda, \Phi \ltimes \gamma \rangle$.
- Necessary: e.g. $\frac{dt}{t} \in \Omega^1_{\min}([0,1])$ but only the boundary of $\log t = (\mathrm{proj}_2)_* \left(\chi_{t_2 < t_1}\, dt_2 \right) \in \Omega^0_{\mathrm{PA}}([0,1])$
- Most important property: Stokes formula $d(\pi_* \lambda) = \pi_*(d\lambda) \pm \pi^\partial_*(\lambda_{E^\partial})$
- Theorem [HLTV] $\Omega^*_{\mathrm{PA}}(X) \simeq C^*(X; \mathbb{R})$ for a compact SA set X
- $\mathrm{FM}_M(r)$ and $\mathrm{FM}_n(r)$ are compact SA sets, projections are SA fiber bundles.

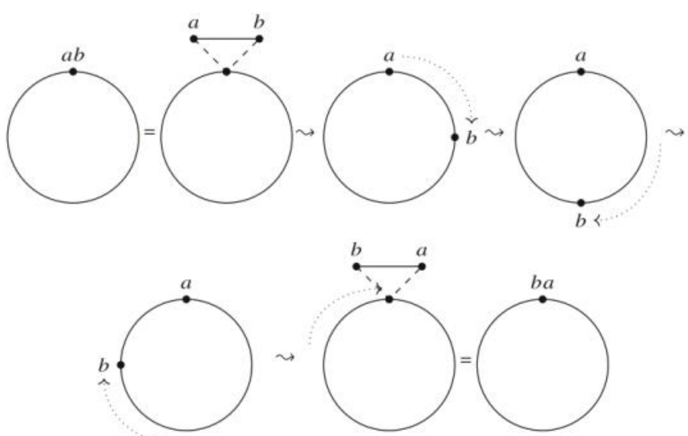
Graph complexes

- No hope of finding $G_A(r) \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_M(r))$ in general
- Standard technique: resolve $G_A(r)$, i.e., turn relations into a differential
- Also resolve A s.t. $A \xleftarrow{\sim} R \xrightarrow{\sim} \Omega^*(M)$
- $\omega_{ij}^2 = 0$ and $\omega_{ji} = \pm \omega_{ij}$ are easy to resolve: remove some generators
- For the others, use graph complexes: add internal vertices, $d = d_R + d_{\mathrm{split}} + d_{\mathrm{contr}}$
- Combinatorics: $G_A \xleftarrow{\sim} \mathrm{Graphs}_R(r)$ using filtrations ($\#E - \#V$), spectral sequences, induction on r.
- Corresponds to fiber integrals: $\omega(\Gamma) \coloneqq \int_{\mathrm{FM}_M(U \sqcup I) \to \mathrm{FM}_M(U)} \bigwedge_{u \in U \sqcup} \alpha_u \wedge \bigwedge_{e \in E} \phi_e$
- Where $\phi \in \Omega^{n-1}_{\mathrm{PA}}(\mathrm{FM}_M(2))$ is the "propagator": $d\phi = \Delta_M$
- Problem: could get extra terms in the differential from "partition function" $Z: \mathrm{GC}_R \to \mathbb{R}$, incomputable integrals
- Not the case thanks to degree counting! Several simplifications (unary internal, bivalent internal…)
- Get $\mathrm{Graphs}_R(r) \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}\left(\mathrm{Conf}_M(r)\right)$, quasi-iso thanks to the first result
- In dimension ≤ 3, different arguments: obvious in dim ≤ 2, need Poincaré conjecture for dim 3

Operads

Motivation: Factorization homology

- Goal: produce invariants of manifolds that are finer than homotopy invariants
  - For example, distinguish $L_{7,1}$ and $L_{7,2}$
  - Motivation from physics: "charged" particles in the manifold
  - More precisely, given space of decorations A, particles are decorated by elements of A that get multiplied when they collide



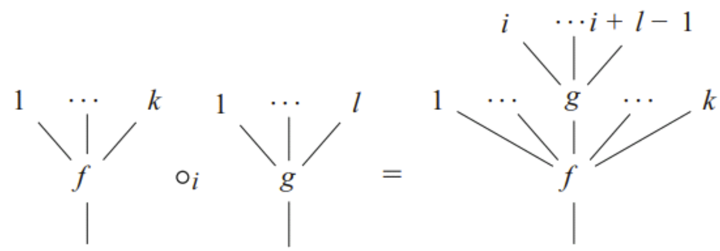
- What kind of multiplication do we want?
  - If we take A commutative+associative, everything works.
  - The result is known: higher Hochschild homology…
  - But this is a homotopy invariant!
  - We need to have different multiplications when points collide in different ways
  - Fulton-MacPherson compactification is the answer!

Introduction to operads

- Operads are designed to encode "categories of algebras"
  - Associative algebras
  - Commutative algebras
  - Lie algebras

- …

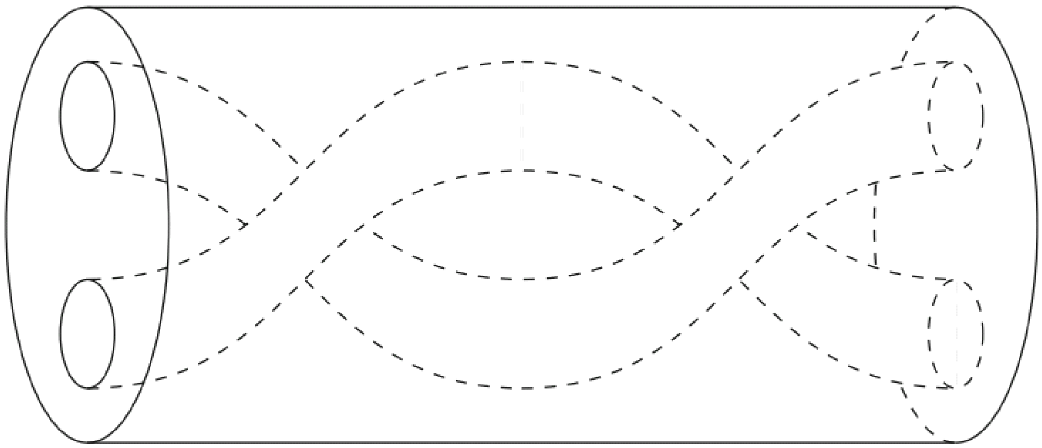
- (Analogy) group/monoid : representation :: operad : algebra
    - Operads are basically monoids, but operations are multivariable
    - Prototypical monoid: $\mathrm{End}(X)$ for some object X
    - Analogue: endomorphism operad $\mathrm{End}_X = \{\mathrm{Hom}(X^{\otimes r}, X)\}$
        - Can renumber inputs of operations (symmetric)
        - Can compose operations: $\circ_i : \mathrm{Hom}(X^{\otimes r}, X) \times \mathrm{Hom}(X^{\otimes s}, X) \to \mathrm{Hom}(X^{\otimes r+s-1}, X)$



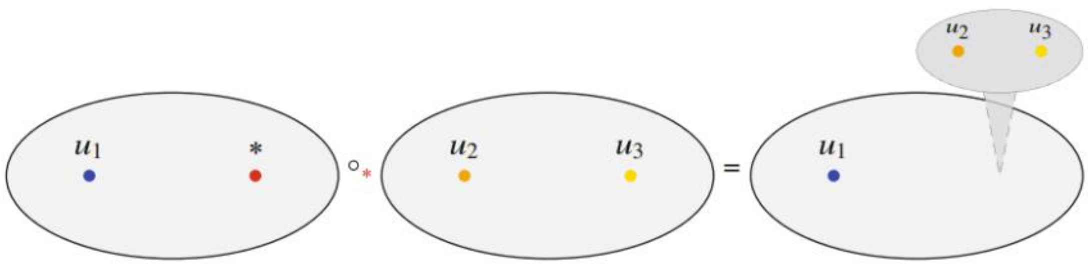
        - There is an identity: $\mathrm{id}_X \in \mathrm{Hom}(X, X)$
- An operad is a collection of abstract "operations" $\mathcal{P} = \{\mathcal{P}(r)\}_{r \geq 0}$ plus:
    - Renumbering
    - Composition
    - Identity for the composition
    - Associativity of composition
    - Equivariance of composition
    - Draw pictures of trees to understand what it's all about!
- Algebra = object A equipped with $\mathcal{P}(r) \otimes A^{\otimes r} \to A$ satisfying several conditions
    - Draw trees again!
- Examples of operads and their algebras
    - $\mathcal{A}s(r) = \mathfrak{S}_r$, algebras = monoids
    - $\mathcal{C}om(r) = \{*_r\}$, algebras = commutative monoids
- Why do all that…?
    - Operads in other categories: algebra, topology, geometry…
    - General framework to encode "algebraic structures"

Operadic structure of FM_n and FM_M

- Theorem: there is an operad structure on FM_n
  Structure maps: $\mathrm{FM}_n(r) \times \mathrm{FM}_n(s) \to \mathrm{FM}_n(r+s-1)$ "insert" infinitesimally small configurations of points
  Given by insertion of boundary facets $\partial_W \mathrm{FM}_n(A)$
- For $n = 1$: $\mathrm{FM}_1(r)$ is $r!$ copies of several polytopes known as associahedra
- For $n = 2$:
    - $r = 0$: $\mathrm{FM}_2(0) = \{*\}$ is a unit
    - $r = 1$: $\mathrm{FM}_2(1) = \{\mathrm{id}\}$ is the identity
    - $r = 2$: $\mathrm{FM}_2(2) = S^1$: all the ways of multiplying two elements in 2d
    - $r = 3$: contains homotopies between all the ways of multiplying three elements



- There is a right action of FM_n on FM_M if M is parallelized
  Insert of configurations into the tangent space
  Given by insert of boundary facets too



- Factorization homology: for a parallelized manifold M and an FM_n-algebra B,

$$\int_M B = \bigsqcup_{r\geq 0} \mathrm{FM}_M(r) \times A^r \,/\, \Big((x \circ_i y)(b_1, \dots, b_{r+s-1}) \sim x\big(b_1, \dots, b_{i-1}, y(b_i, \dots, b_{i+r-1}), b_{i+r}, \dots, a_{r+s-1}\big)\Big)$$

Connection with algebraic models

- The operad FM_n is formal as an operad
- The LS model G_A is compatible with the operad action
- Consequence: can compute factorization homology using G_A
- Example: $B = S(\mathfrak{g}[1-n])$ then $\int_M B = C_*^{\mathrm{CE}}\big(\Omega^{n-*}(M; \mathfrak{g})\big)$.

References

- Benoit Fresse. *Homotopy of Operads and Grothendieck–Teichmüller Groups*. Volume 1: “The Algebraic Theory and its Topological Background.” Mathematical Surveys and Monographs 217, AMS (2017). ISBN: 978-1-4704-3481-6. DOI:10.1090/surv/217.1
  Note: Parts Ia and Ib are the relevant parts.
- Fresse, Benoit. "Little discs operads, graph complexes and Grothendieck-Teichmüller groups". *Handbook of homotopy theory*, 405–441, CRC Press/Chapman Hall Handb. Math. Ser., CRC Press (2020).
- Najib Idrissi. *Real Homotopy of Configuration Spaces*: Peccot Lecture, Collège de France, March and May 2020. Lecture Notes in Mathematics 2303. Springer (2023). ISBN: 978-3-031-04427-4. DOI:10.1007/978-3-031-04428-1.
  Note: a preprint of the book is available at https://hal.science/hal-03821309.
- Ben Knudsen. *Configuration spaces in algebraic topology*. arXiv:1803.11165.
- Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. Grundlehren der mathematischen Wissenschaften 346. Springer (2012). ISBN: 978-3-642-30361-6. DOI:10.1007/978-3-642-30362-3.