Configurations spaces, algebraic topology and operads: Tutorials

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Lecture given at the CIMPA school "Crossroads of geometry, representation theory and higher structures"

This file contains the exercises for the tutorials. Thanks to Victor Roca i Lucio and Pedro Tamaroff for helping out!

Exercices

1. Let $I = (0,1)$. Prove that $\mathrm{Conf}_I(r)$ is homeomorphic to $\Sigma_r \times \mathring{\Delta}^r$, where $\mathring{\Delta}^r = \{t \in I^{r+1} \mid \sum_{i=0}^r t_i = 1\}$.
   (Hint: draw a picture!)
2. Prove that $\mathrm{Conf}_{\mathbb{R}^n}(2)$ is homeomorphic to $S^{n-1} \times \mathbb{R}_{>0} \times \mathbb{R}^n$.
   (Hint: think "angle, radius, center".)
3. (Fadell-Neuwirth fibrations.) Given a manifold M, consider the map $\pi_1 : \mathrm{Conf}_M(r) \to \mathrm{M}, \pi_1(x_1, \dots, x_r) = x_1$ that forgets everything but the first point.
   a) What is the fiber $F_\pi = \pi^{-1}(\{x_0\})$ (for some fixed x_0) of π?
   b) Let $(y_1, \dots, y_m) \in \mathrm{Conf}_M(m)$ be a fixed configuration with $m \geq 1$ and let $Y = M \setminus \{y_1, \dots, y_m\}$. Prove that $\mathrm{Conf}_Y(r) \to Y$ is split, i.e., it is isomorphic to the product bundle $Y \times F \to Y$.
   c) Suppose that G is a Lie group and let $x_0 \in G$. Show that there exists a group morphism $\theta : G \to \mathrm{Homeo}(G)$ such that $\theta(x)(x) = x_0$ and $\theta(x_0)(x) = x$ for all $x \in G$.
   d) Use is to show that $\pi : \mathrm{Conf}_G(r) \to G, (x_1, \dots, x_r) \mapsto x_1$ is split, i.e., it is isomorphic to the product bundle $G \times F_\pi \to G$.
   e) To what is $\mathrm{Conf}_{S^1}(r)$ diffeomorphic?
   f) Prove that $\mathrm{Conf}_{S^3}(\mathrm{r})$ is diffeomorphic to $S^3 \times \mathrm{Conf}_{\mathbb{R}^3}(r-1)$.
4. Let $A = S(V)$ be a free graded algebra on some graded vector space $V = \{V^n\}_{n \geq 0}$. Prove that as an algebra, there exists an isomorphism $A = \mathbb{Q}[V^{\mathrm{even}}] \otimes \Lambda(V^{\mathrm{odd}})$, where $V^{\mathrm{even}} = \bigoplus_k V^{2k}$ and $V^{"odd"} = \bigoplus_k V^{2k+1}$.
5. Compute the ranks of the rational homotopy groups of S^n.
   (You can use the fact that if $(S(V), d)$ is a minimal model of X, then the rank of $\pi_k(X)$ is equal to the dimension of V^k).
6. Recall that, as a ring, we have:
   $\mathrm{H}^*(\mathbb{CP}^r) = S(x)/x^{r+1} = \mathbb{Q}\langle 1, x, x^2, \dots x^r \rangle$,
   where $\deg x = 2$.
   a) Is $H^*(\mathbb{CP}^r)$ minimal? If not, find a minimal CDGA quasi-isomorphic to it.
   b) Use that minimal CDGA to prove that $\mathbb{CP}^r$ is a formal space.
7. Find the ranks of the rational homotopy groups $\pi_k(S^n)$ and $\pi_k(\mathbb{CP}^n)$
8. (Difference between real and rational models.) In Example 2.84, we defined (for $\alpha \in \mathbb{Q}$):
   $A_\alpha = (S(e_2, x_4, y_7, z_9), d_\alpha e = 0, d_\alpha x = 0, d_\alpha y = x^2 + \alpha e^4, d_\alpha z = e^5)$.
   Prove that A_α and A_β are quasi-isomorphic if and only if α/β is a square.
9. Prove that any element of $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ is a linear combination of terms of the form $\omega_{i_1 j_1} \dots \omega_{i_k j_k}$ with $i_1 < \dots < i_k$ and $i_1 < j_1, \dots, i_k < j_k$.
10. Using the description of $H^*(\mathrm{Conf}_{\mathbb{R}^n}(r))$ and the fact that this space is formal, find a minimal model of $\mathrm{Conf}_{\mathbb{R}^n}(3)$, try to do the same for every r.
11. Use it to compute the ranks of the homotopy groups of $\mathrm{Conf}_{\mathbb{R}^n}(r)$.

References for the solution

- [FN] Edward Fadell and Lee Neuwirth. "Configuration Spaces", *Math. Scand.* 10 (1962), pp. 111-118. DOI:10.7146/math.scand.a-10517

- [FOT] Yves Félix, John Oprea, Daniel Tanré. *Algebraic Models in Geometry*. Oxford University Press (2008). ISBN 978-0-19-920651-3.
- [I] Najib Idrissi. *Real Homotopy of Configuration Spaces*: Peccot Lecture, Collège de France, March and May 2020. Lecture Notes in Mathematics 2303. Springer (2023). ISBN: 978-3-031-04427-4. [DOI:10.1007/978-3-031-04428-1](https://doi.org/10.1007/978-3-031-04428-1).
  Note: a preprint of the book is available at https://hal.science/hal-03821309.
- [K] Ben Knudsen. *Configuration spaces in algebraic topology*. arXiv:1803.11165.

Reference for each exercise

1. [K] Example 2.1.2
2. [K] Example 2.1.3
3. Everything is found in the paper [FN].
4. [FOT] right after Definition 2.6.
5. [FOT] Example 2.43
6. [FOT] Example 2.44
7. Same examples as previous two. The proof that V^n has the rank of $\pi_n(X)$ is Theorem 2.50 there.
8. [FOT] Example 2.38.
9. [I] Lemma 2.89 (the result is not due to me, of course!)
10. [I] Theorem 2.103 (same comment)
11. Same reference (and same comment)