Worksheet 1: **Homotopy theory of topological spaces I**

Definition (*Homotopy commutative diagram*)**.** A diagram like

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & Y \end{array}$$

is *homotopy commutative* if the two composites are homotopic, i.e. here $gf \sim kh$.

Exercise 1. ***Homeomorphism***
Show that the quotient space $I^n/\partial I^n$ is homeomorphic to the n-dimensional sphere S^n.

Exercise 2. ***Deformation retract***
Show that the circle S^1 is a strong deformation retract of the complement $S^3 \setminus S^1$.

Exercise 3. ***Compact-open topology and exponential law***
Let us recall that a space is *locally compact* if each neighbourhood of a point x contains a compact neighbourhood. (Notice that the product of two locally compact spaces is again locally compact.)

(1) Let X be a locally compact space. Show that the evaluation map

$$e : Y^X \times X \to Y$$

defined by the assignment $(f, x) \mapsto f(x)$ is continuous.

(2) Let $f : X \times Y \to Z$ be a continuous map, where Y is locally compact. Show that its set-theoretical adjoint map $\tilde{f} : X \to Z^Y$ is continuous.

(3) Give a sufficient condition (C) under which the set-theoretical adjunction

$$- \times Y : \mathsf{Set} \underset{}{\overset{\perp}{\rightleftarrows}} \mathsf{Set} : -^Y$$

given, for any $Y \in \mathsf{Top}$, by $\mathsf{Set}(X \times Y, Z) \cong \mathsf{Set}(X, Z^Y)$ induces the adjunction

$$- \times Y : \mathsf{Top} \underset{}{\overset{\perp}{\rightleftarrows}} \mathsf{Top} : -^Y$$

(4) Show that under Condition (C), the aforementioned adjunction induces an adjunction on the level of the homotopy category

$$- \times Y : \mathsf{hoTop} \underset{}{\overset{\perp}{\rightleftarrows}} \mathsf{hoTop} : -^Y$$

(5) Show that the exponential law, that is the natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, Z^Y)$$

of the adjunction of Question (3) is a homeomorphism when X and Y are locally compact.

(6) Let $\mathrm{cst}_a : Z \to A$ be the constant map with value $a \in A$. Show that the map

$$\psi : X^Z \times A \to (X \times A)^Z, \quad (\varphi, a) \mapsto (\varphi, \mathrm{cst}_a)$$

is continuous.

Recall that the *push-forward* of a map $f : X \to Y$ is the map $f_* : X^Z \to Y^Z$ defined by $f_*(g) \coloneqq fg$ and that its *pullback* is the map $f^* : Z^Y \to Z^X$ defined by $f^*(g) \coloneqq gf$.

(7) Let Z be a locally compact space. Let $H : X \times I \to Y$ be a homotopy from $f : X \to Y$ to $g : X \to Y$. We denote by $H_t : X \to Y$ the map defined by $H_t(x) \coloneqq H(x,t)$, for any $t \in I$. Show that the map $H^Z : X^Z \times I \to Y^Z$ defined by $H^Z(-,t) \coloneqq (H_t)_* : X^Z \to Y^Z$ is a homotopy from $f_* : X^Z \to Y^Z$ to $g_* : X^Z \to Y^Z$. Show that the map $Z^H : Z^Y \times I \to Z^X$ defined by $Z^H(-,t) \coloneqq (H_t)^* : Z^Y \to Z^X$ is a homotopy from $f^* : Z^Y \to Z^X$ to $g^* : Z^Y \to Z^X$.

(8) Let $f : X \xrightarrow{\sim} Y$ be a homotopy equivalence. Show that its pullback $f^* : \mathsf{Top}(Y,Z) \xrightarrow{\sim} \mathsf{Top}(X,Z)$ and its push-forward $f_* : \mathsf{Top}(Z,X) \xrightarrow{\sim} \mathsf{Top}(Z,Y)$ are homotopy equivalences for any locally compact space $Z \in \mathsf{Top}$.

Exercise 4. ***Cofibre sequence***

The goal of this exercise is to prove that the cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f_1} \Sigma\,\mathrm{Cone}(f) \xrightarrow{\Sigma p(f)} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 f_1} \Sigma^2\,\mathrm{Cone}(f) \xrightarrow{\Sigma^2 p(f)} \cdots$$

is h-coexact.

(1) Show that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{f_2} \mathrm{Cone}(f_1) \xrightarrow{f_3} \mathrm{Cone}(f_2)$$

is h-coexact.

(2) Show that all the bottom maps of the following commutative diagram

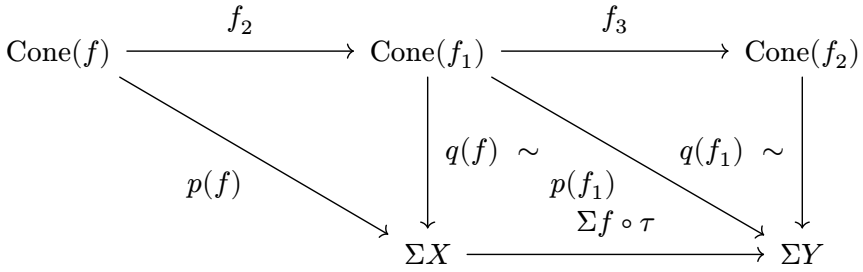
$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i_1} & \mathrm{Cone}(Y) \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle j_1} \\
\mathrm{Cone}(X) & \xrightarrow{j} & \mathrm{Cone}(f) & \xrightarrow{f_2} & \mathrm{Cone}(f_1) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p(f)} & & \downarrow{\scriptstyle q(f)}\ \sim \\
\mathrm{Cone}(X)/i_1(X) & \xrightarrow{\cong} & \mathrm{Cone}(f)/f_1(Y) & \xrightarrow{\cong} & \mathrm{Cone}(f_1)/j_1(\mathrm{Cone}(Y)) \\
\| & & \| & & \| \\
\Sigma X & & \Sigma X & & \Sigma X
\end{array}$$

are homeomorphisms with the suspension ΣX of the space X. (The above two squares are the pushouts defining the respective mapping cones.)

(3) Show that $q(f)$ is a homotopy equivalence.

We denote by $\tau : \Sigma X \to \Sigma X$ the orientation reversing homeomorphism defined by $(x, t) \mapsto (x, 1 - t)$.

(4) We consider the following diagram



Show that the triangles on the left-hand side and on the right-hand side are commutative and that the middle triangle is homotopy commutative, that is $\Sigma f \circ \tau \circ q(f) \sim p(f_1)$.

(5) Conclude that the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{Cone}(f) \xrightarrow{p(f)} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is h-coexact.

(6) Show that there exists a homeomorphism $\chi : \mathrm{Cone}(\Sigma f) \xrightarrow{\cong} \Sigma\,\mathrm{Cone}(f)$, which satisfies

$$\chi \circ (\Sigma f)_1 = \Sigma f_1.$$

(7) Conclude.

Exercise 5. ***Abelian group***

Prove directly that the product $+_1$ on $\left[\Sigma^2 X, Y\right]_*$ is abelian.

Exercise 6. ***Compatibility between the fibre sequence and the cofibre sequence***

(1) Describe the unit $\eta : X \to \Omega\Sigma X$ and the counit $\varepsilon : \Sigma\Omega X \to X$ of the Σ-Ω adjunction.

(2) Let $f : X \to Y$ be a pointed map. Show that the assignment

$$(x, \varphi) \mapsto \begin{cases} \varphi(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ (x, 2(1-t)) & \text{for } \frac{1}{2} \le t \le 1, \end{cases}$$

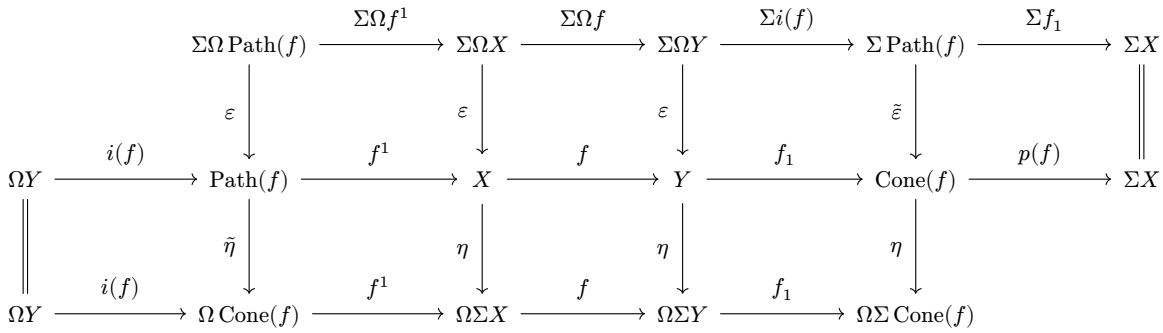
defines a pointed map

$$\tilde{\eta} : \mathrm{Path}(f) \to \Omega\,\mathrm{Cone}(f).$$

(3) Describe the adjoint map

$$\tilde{\varepsilon} : \Sigma\,\mathrm{Path}(f) \to \mathrm{Cone}(f).$$

(4) Show that the following diagram



is homotopy commutative.