Homotopy II

Lecture Notes

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Foreword

This document gives a quick summary of the courses *Introduction to Homotopy Theory* (2019–2020) and *Homotopy II* (2020–2021) given at Université de Paris as part of the M2 Mathématiques Fondamentales. They are not complete and the content may still change or be reorganized. In particular, the 2019–2020 course contained a lot of material on simplicial sets which were integrated in 2020–2021 into the *Homotopy I* course of Vallette [Val20]. The content covered by the *Homotopy I* course or by the *Homology* course is indicated by a symbol ↺.

These notes are strongly inspired by lecture notes of Grégory Ginot [Gin19] and follow almost the same outline. I would also like to thank Jenny Amanda, Carlo Buccisano, Pierre Elis, Jonah Frébault, Bruno Galvez Araneda, Clémentine Lemarié-Rieusset, Timothée Moreau, Maxime Ramzi, Tommaso Rossi and Gabriel Saadia for pointing out errors in the text.

Suggested reading :

- Reviews of algebraic topology and homological algebra : Bredon [Bre93], Hatcher [Hat02], Saint-Gervais [Sai17], Schapira [Sch15], Spanier [Spa95] et Weibel [Wei94].
- Model categories : Dwyer et Spalinski [DS95] et Hovey [Hov99].
- Simplicial sets : Goerss et Jardine [GJ99] et Friedman [Fri08].
- Rational homotopy theory : Félix, Halperin et Thomas [FHT01], Hess [Hes07], Félix, Oprea et Tanré [FOT08] et Griffiths et Morgan [GM13].
- Quasi-categories : Lurie [Lur09].

These notes were translated from the French original, in part using automatic translation software. It has been checked and corrected, but may still contain errors – if you find any, please let me know !

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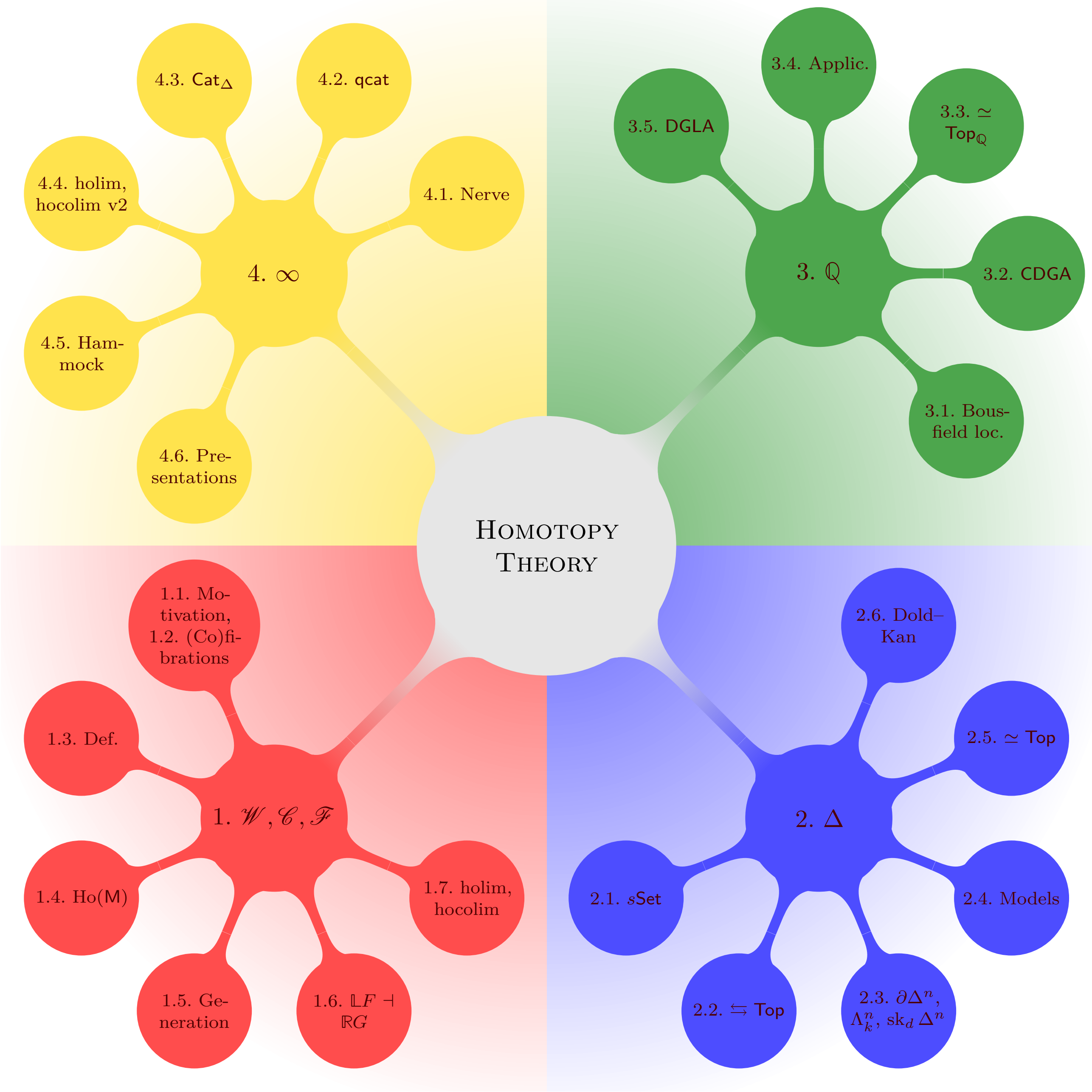


(Adapted from the TikZ & PGF manual [Tan19, Section 6].)

1 Model categories

1.1 Motivation

1.1.1 Topological spaces

Let Top denote the category of topological spaces. From now on and unless otherwise stated, we will only consider continuous functions.

Definition 1.1.1. Two functions $f, g : A \to X$ are *homotopic* if there exists a function $H : A \times [0,1] \to X$ such that $H(-,0) = f$ and $H(-,1) = g$. In this case we write $f \simeq g$. We also write $[A, X]$ for the set of homotopy classes of functions $A \to X$.

Definition 1.1.2. Two spaces are *homotopy equivalent* (also written $X \simeq Y$) if $\exists f : X \leftrightarrows Y : g$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. The maps f and g are then called *homotopy equivalences.*

This defines an “equivalence relation” on topological spaces. Homotopy theory is the study of spaces “up to homotopy”, meaning that we consider two spaces to be “the same” if they are homotopy equivalent.

The Question : given X, Y, how do we test whether $X \simeq Y$? In general, one uses *homotopy invariants.*

Definition 1.1.3. A *pointed space* is a pair (X, x_0) where X is a space and $x_0 \in X$. A *pointed map* is a map that preserves the base point. We write Top_* for the category of pointed topological spaces.

Definition 1.1.4. A *pointed homotopy* is a homotopy that remains constant on the base point. We write $[(A, a_0), (X, x_0)]$ for the set of pointed homotopy classes of pointed maps $(A, a_0) \to (X, x_0)$.

Definition 1.1.5. For a space X, we write $\pi_0(X)$ for the set of connected components of X. Let (X, x_0) be a pointed space. We also define the higher homotopy groups by :

$$\pi_n(X, x_0) \coloneqq [(S^n, *), (X, x_0)].$$

This defines functors $\pi_0 : \mathsf{Top} \to \mathsf{Set}$ and $\pi_n : \mathsf{Top}_* \to \mathsf{Set}$. Moreover, π_1 is a group, and π_n is an abelian group for $n \geq 2$.

Proposition 1.1.6. *Let $f : X \to Y$ be a homotopy equivalence. Then $\pi_0(f)$ is a bijection, and for all $x \in X$, $\pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism of groups.*

Definition 1.1.7. A *weak homotopy equivalence* is a map satisfying the conclusion of the preceding proposition. We write $\xrightarrow{\sim}$ for weak homotopy equivalences.

Example 1.1.8. A homotopy equivalence is a weak homotopy equivalence, but the converse is false : $\mathbb{N} \to \{0\} \cup \{1/n \mid n > 0\}$ is a weak homotopy equivalence but is not a homotopy equivalence.

Definition 1.1.9. Two spaces X, Y are said to be *weakly equivalent* if there exists a zigzag :

$$X \xleftarrow{\sim} X_1 \xrightarrow{\sim} \dots \xleftarrow{\sim} X_n \xrightarrow{\sim} Y.$$

Proposition 1.1.10. *Two weakly equivalent spaces have the same homotopy groups.*

The converse is false : for example $\mathbf{RP}^2 \times S^3$ and $\mathbf{RP}^3 \times S^2$ have the same homotopy groups, but they are not weakly equivalent (since otherwise they would have the same homology).

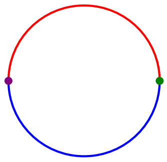
Recall : a CW-complex X is a topological space obtained as follows : one starts with a discrete space X_0, then obtains X_1 from X_0 by attaching cells of dimension 1, etc.

Theorem 1.1.11 (Whitehead [Whi49]). ↺ *If $f : X \to Y$ is a weak homotopy equivalence between two CW-complexes, then it is a homotopy equivalence.*

Example 1.1.12. This is not true if the spaces X and Y are not CW-complexes. Consider the pseudo-circle $P = \{g, d, h, b\}$ equipped with the topology whose open sets are :

$$\tau = \{\varnothing, \{h\}, \{b\}, \{h, b\}, \{g, h, b\}, \{d, h, b\}, P\}.$$

There exists a map $f : S^1 \to P$ that sends the left point to g, the right point to d, all points of the upper semicircle to h, and all points of the lower semicircle to b, as in the following picture :



$$\to \{h, b, d, g\}.$$

One can easily verify that f is a weak homotopy equivalence. However, there is no map $g : P \to S^1$ that is a homotopy inverse of f (all maps $P \to S^1$ are constant).

Sketch of proof. Suppose first that $f : X \to Y$ is the inclusion of a subcomplex. By induction, for all cells of Y that are not in X, one can find a homotopy that retracts that cell into X using the hypothesis. For the general case, one first shows that f is homotopic to a cellular map, then applies the previous result to the mapping cylinder of f (cf. Example 1.2.14) which retracts onto Y. ☐

Theorem 1.1.13. ↺ *Let X be an arbitrary space. Then there exists a CW-complex Z and a weak homotopy equivalence $Z \xrightarrow{\sim} X$.*

We thus have two different frameworks for homotopy theory :

- the “strong” homotopy category, where one formally inverts homotopy equivalences ;
- the “weak” homotopy category, where one formally inverts weak homotopy equivalences.

In the first case, one is closer to what one wishes to study, but it is harder to test algebraically ; in the second case, it is weaker, but simpler to test algebraically. Whitehead’s theorem tells us that if we restrict to CW-complexes, the two notions coincide.

In this course, the goal is to generalize this framework to an arbitrary category :

- we will want to define what it means for two objects to be “the same up to homotopy” ;
- what is a homotopy, and a strong homotopy equivalence ;
- which are the “nice” objects (models) on which it suffices to look at things only up to weak homotopy.

Another problem we will solve : limits and colimits in Top do not preserve weak equivalences. For example :

$$\begin{tikzcd}
* \arrow[d, "\sim"] & S^0 \arrow[l] \arrow[r] \arrow[d, "\sim"] & * \arrow[d, "\sim"] \\
D^1 & S^0 \arrow[l] \arrow[r] & D^1
\end{tikzcd}$$

The colimit of the top diagram is $* \cup_{S^0} * = *$, whereas the colimit of the bottom one is $D^1 \cup_{S^0} D^1 = S^1$. We will see that the bottom one is better : the maps composing it are inclusions of subspaces.

1.1.2 Chain complexes

Here is another example of a framework in which one can do homotopical algebra. Let R be a commutative ring.

Definition 1.1.14. A *chain complex* is a diagram of R-modules :

$$\dots \xrightarrow{d} C_i \xrightarrow{d} C_{i-1} \xrightarrow{d} \dots$$

satisfying $d \circ d = 0$. A morphism is a morphism of diagrams. We write $\mathsf{Ch}(R)$ for the category of chain complexes. We also write $\mathsf{Ch}_{\geq 0}(R)$ for the full subcategory of chain complexes C satisfying $C_i = 0$ for $i < 0$.

Definition 1.1.15. Two morphisms $f, g : C \to D$ are *homotopic* (written $f \simeq g$) if there exists a sequence of maps $h : C_n \to D_{n+1}$ such that $f - g = hd + dh$.

Definition 1.1.16. A *homotopy equivalence* is a pair of maps $f : C \leftrightarrows D : g$ such that $f \circ g \simeq \mathrm{id}_D$ and $g \circ f \simeq \mathrm{id}_C$.

Proposition 1.1.17. *A homotopy equivalence induces an isomorphism in homology.*

Definition 1.1.18. A *quasi-isomorphism* is a morphism of chain complexes that induces an isomorphism in homology. Two chain complexes C, D are said to be *quasi-isomorphic* if there exists a zigzag of quasi-isomorphisms :

$$C = X_0 \xleftarrow{\sim} X_1 \xrightarrow{\sim} X_2 \xleftarrow{\sim} \dots \xrightarrow{\sim} X_n = D.$$

Remark 1.1.19. A quasi-isomorphism is not necessarily a homotopy equivalence. The following example defines a quasi-isomorphism that is not a homotopy equivalence :

$$\begin{tikzcd}
C = \arrow[d, "f"] & \dots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, two heads] & 0 \arrow[r] & \dots \\
D = & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

We have $H_0(C) = H_0(D) = \mathbb{Z}/2\mathbb{Z}$ and $H_i(C) = H_i(D) = 0$ for all $i \neq 0$, and f does indeed induce an isomorphism. However, there is no nonzero morphism $D \to C$, and hence no homotopy inverse of f.

One can thus study chain complexes up to homotopy equivalence, or up to quasi-isomorphism. Which are the "nice" complexes for which these coincide ?

Definition 1.1.20. A R-module P is said to be *projective* if for every surjection $p : A \to B$ and every map $f : P \to B$, there exists a lifting $g : P \to A$ such that $p \circ g = f$.

Example 1.1.21. For $R = \mathbb{Z}$, the (finitely generated) projective R-modules are the free modules $\mathbb{Z}^n$.

Definition 1.1.22. A R-module I is said to be *injective* if for every injection $i : A \to B$ and every map $f : A \to I$, there exists an extension $g : B \to I$ such that $g \circ i = f$.

Example 1.1.23. For $R = \mathbb{Z}$, the injective R-modules are the divisible abelian groups.

Proposition 1.1.24. ↻ *Let $f : C \to D$ be a quasi-isomorphism of chain complexes concentrated in non-negative degrees.*

- *If D is projective in every degree, then f is a homotopy equivalence.*
- *If C is injective in every degree, then f is a homotopy equivalence.*

We thus have three possible frameworks for homotopy theory in chain complexes :

- The "strong" framework, where one works up to homotopy equivalence and all objects are "nice".
- The "projective" framework, where one works up to quasi-isomorphism, the "nice objects on the source side" are projective complexes and all modules are "nice on the target side". Every R-module M has a projective resolution $P_* \to M$. These projective resolutions play the role of CW-complexes in this framework.

- The “injective” framework, where one works up to quasi-isomorphism, all modules are “nice on the source side” and the “nice objects on the target side” are injective modules. Every module M has an injective resolution $M \to I_*$. These injective resolutions play a dual role to CW-complexes in this framework.

We will also see how to resolve the problems that arise when certain functors do not preserve quasi-isomorphisms. This is the case for the functor $\otimes$, which is only right exact. The correct version from a homotopical point of view is its left derived functor, the functor Tor (which involves a projective resolution, i.e. one works in the first framework). One can begin by showing by induction that every module has a projective resolution. Then one shows a certain lifting property, which allows one to conclude that any two projective resolutions are homotopic. One then defines $\mathrm{Tor}(M, N)$ as $H_i(P_* \otimes N)$ where P_* is a projective resolution of M. Since any two projective resolutions are homotopic and Tor preserves homotopies (but not quasi-isomorphisms), one deduces that this is well defined.

This is also the case for the functor Hom, which is only left exact. Its correct version from a homotopical point of view is its right derived functor, Ext (which involves either a projective resolution of the source, in which case one works in the first framework, or an injective resolution of the target, in which case one works in the second framework). In each of the two cases, to obtain a “correct” functor, one first resolves the object to which it is applied, then applies the original functor. Since module categories are abelian, resolutions go through chain complexes. In the setting of non-abelian categories (for example, algebras), this no longer necessarily works as such ; one of the goals of Quillen’s “homotopical algebra” is to provide a framework in which one can make sense of these resolutions.

1.2 (Co)fibrations

We have seen previously that in chain complexes, what mattered from a homotopical point of view was knowing whether maps were injective/surjective and knowing whether one could « lift » a map along an injection/surjection. We will briefly recall the analogue of injections and surjections in the category of topological spaces, their abstract properties, and use them as a basis for defining the notion of a model category.

1.2.1 Long exact sequences

Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of chain complexes, i.e. i is injective, p is surjective, and $\ker p = \operatorname{im} i$. Then we have an induced long exact sequence :

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \to H_0(C) \to 0.$$

The maps i_* and p_* are simply the induced maps. The map ∂ can be constructed in two ways :

- One defines $C(i)$, the cone of i, as the complex given in degree n by $B_n \oplus A_{n-1}$, with differential $d(b,a) = (d_B b + i(a), d_A a)$. One checks (exercise) that this cone is quasi-isomorphic to C, which follows from the fact that i is injective. Moreover, there is a projection $C(i) \to A[-1]$ which induces ∂ in homology.
- One defines $K(p)$, the homotopy kernel of p, as the complex given in degree n by $B_n \oplus C_{n+1}$, with differential $d(b,c) = (d_B b, p(b) + d_C c)$. One checks (exercise) that $K(p)$ is quasi-isomorphic to A, which follows from the fact that p is surjective. Moreover, there is an injective map $C[1] \to K(p)$, $c \mapsto (0,c)$ which induces ∂ in homology.

These are the notions we will generalize to topological spaces.

1.2.2 Fibrations

Definition 1.2.1. A *(Hurewicz) fibration* is a continuous map $p : E \to B$ such that for every space X and all maps $\tilde{H}_0$ and H in the following diagram, one can find a lift $\tilde{H}$:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook, "\sim"] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

Concretely, this means that if one has a homotopy $H : X \times I \to B$ between $f, g : X \to B$ and a map $\tilde{f} : X \to E$ that lifts f (i.e. $p \circ \tilde{f} = f$), then one can find a homotopy $\tilde{H}$ that lifts H (and thus $\tilde{g} = \tilde{H}(-,1)$ lifts g).

Definition 1.2.2. A *Serre fibration* is a map $p : E \to B$ that satisfies the above lifting property for $X = [0,1]^n$ (for all $n \geq 0$).

A (Hurewicz) fibration is a Serre fibration. The converse is false : e.g. the « universal covering » – constructed in the usual way – of the Hawaiian earring is not a Hurewicz fibration.

Example 1.2.3. A covering is a Serre fibration. A projection $p : F \times B \to B$ is a fibration. More generally, a fiber bundle is a Serre fibration ; if the base is paracompact (i.e. Hausdorff and every open cover admits a locally finite refinement), then it is a Hurewicz fibration.

Proposition 1.2.4. ↻ *The pullback of a fibration (Serre or Hurewicz) is a fibration (Serre or Hurewicz).*

Example 1.2.5. Let $f : X \to Y$ be a continuous map. One defines its *path space* :

$$P_f = Y^{[0,1]} \times_Y X = \{(\gamma, x) \in Y^{[0,1]} \times X \mid \gamma(0) = f(x)\}$$

Then $\mathrm{ev}_1 : P_f \to Y$, $\mathrm{ev}_1(\gamma, x) = \gamma(1)$ is a Hurewicz fibration : if one has a commutative diagram

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "{(\gamma,\varphi)}"] \arrow[d, hook] & P_f \arrow[d, "\mathrm{ev}_1"] \\
A \times [0,1] \arrow[r, "\psi"] & Y
\end{tikzcd}$$

then one can define a lift $\lambda : A \times [0,1] \to P_f$ by

$$\lambda(a,t) = \Big(\gamma(a) \cdot \psi(a,-)|_{[0,t]}, \varphi(a)\Big).$$

Note that f factors as $X \overset{\sim}{\hookrightarrow} P_f \xrightarrow{\mathrm{ev}_1} Y$: this allows one to « replace » any map by an equivalence followed by a fibration.

Proposition 1.2.6. $\circlearrowleft$ *Let $p : E \to B$ be a fibration. If B is path-connected, then all spaces $E_b = p^{-1}(b)$ are homotopy equivalent.*

One generally denotes by F the fiber of p. The following proposition says roughly that fibrations are the « short exact sequences » in topological spaces :

Proposition 1.2.7. $\circlearrowleft$ *Let $p : E \to B$ be a Serre fibration and suppose that B is path-connected. Let $b_0 \in B$ be a point, $F = p^{-1}(b_0)$ the fiber above b_0, and $f_0 \in F$ a point. Then we have a long exact sequence :*

$$\dots \to \pi_n(F, f_0) \to \pi_n(E, f_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, f_0) \to \dots$$

Remark 1.2.8. This generalizes the long exact sequence in homology associated to a short exact sequence of chain complexes. Indeed, for a surjective linear map $p : X \to Y$, the « fibers » $p^{-1}(Y)$ are all isomorphic to the kernel.

1.2.3 Cofibrations

Fibrations are interesting when one looks at what happens « on the right » in the functor $\mathrm{Hom}(-,-)$. Let us now look at the dual case, cofibrations, which are interesting « on the left ».

Definition 1.2.9. A *(Hurewicz) cofibration* is a continuous map $i : A \to X$ such that for every space Y and all maps f, h in the following diagram, the lift H exists :

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "f"] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}$$

Concretely, this means that if one has a map $f : X \to Y$ and a homotopy between the « restriction » $f \circ i = f|_A$ and another map $A \to Y$, then one can extend this homotopy to X.

Proposition 1.2.10 ([Hat02, Proposition 4H.1])**.** ↺ *Let $i : A \to X$ be a cofibration. Then it is a homeomorphism onto its image.*

Démonstration. Consider the retraction $r : X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ from the previous question.

Let C be the mapping cylinder of i (Example 1.2.14), given by the quotient $(A \times [0,1] \cup X)/\sim$ where $(a,0) \sim i(x)$. We define a diagram :

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"] & C^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "f"] & C
\end{tikzcd}$$

where $h(a,t) = [a,t]$ and $f(x) = [x]$. Since i is a cofibration, there exists $H : X \to C^{[0,1]}$ such that $H(i(a))(t) = h(a)(t) = [a,t]$ and $H(x)(0) = f(x) = [x]$. We have $a \neq a' \implies [a,1] \neq [a',1] \implies i(a) \neq i(a')$, so i is injective. Moreover, $g = h(-)(1)$ is a homeomorphism onto its image $A \times \{1\}$, so by commutativity of the diagram $g^{-1} \circ H(-)(1)$ is a continuous inverse of i. ☐

Proposition 1.2.11. ↺ *An inclusion $i : A \hookrightarrow X$ is a cofibration if and only if $X \times [0,1]$ retracts onto $X \times \{0\} \cup A \times [0,1]$.*

Démonstration. One implication is clear : if i is a cofibration, then one can find a lift (where $h(a)(t) = i(a)$) :

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "\mathrm{id}_X"'] \arrow[ur, dashed, "H"] & X
\end{tikzcd}$$

Then $r(x,t) = H(x)(t)$ defines a retraction of the inclusion $X \times \{0\} \cup A \times [0,1]$. Indeed, $r(x,0) = H(x)(0) = \mathrm{ev}_0(H(x)) = \mathrm{id}_X(x) = x$ and $r(i(a),t) = H(i(a))(t) = h(a)(t) = i(a)$.

Conversely, let $r : X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ be a retraction of the inclusion. Consider a commutative diagram as in Definition 1.2.9. One can find a lift $X \times [0,1] \to Y$ by considering the composite

$$X \times [0,1] \xrightarrow{r} X \times \{0\} \cup A \times [0,1] \xrightarrow{(f,h)} Y.$$ ☐

Proposition 1.2.12 ([Die08, Problem 1])**.** ↺ *Let $i : A \subset X$ be a cofibration. If X is Hausdorff then A is closed.*

Démonstration. Consider the retraction $r : X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ from the previous proposition. Then

$$A = \{x \in X \mid r(x,1) = (x,1)\} = \varphi^{-1}(\Delta)$$

where $\Delta \subset (X \times [0,1])^2$ is the diagonal and $\varphi(x) = (r(x,1),(x,1))$. Now $X \times [0,1]$ is Hausdorff so its diagonal is closed, which allows us to conclude. ☐

Example 1.2.13. The inclusion of a cellular subcomplex is a cofibration. One constructs by induction on the skeleton a retraction as in the previous lemma. (In fact all cofibrations are retracts of such inclusions.)

Example 1.2.14. ↺ Let $f : A \to X$ be a map. One defines its *mapping cylinder* :

$$\mathrm{Cyl}_f = \big(A \times [0,1] \sqcup X\big)/\sim.$$

The relation $\sim$ is generated by $(a, 0) \sim f(a)$. We denote by $[x]$, respectively $[a, t]$, the equivalence classes in Cyl_f.

The inclusion $A = A \times \{1\} \subset \mathrm{Cyl}_f$ is then a cofibration. Indeed, one can construct a retraction $r : \mathrm{Cyl}_f \times [0,1] \to \mathrm{Cyl}_f \times \{0\} \cup A \times \{1\} \times [0,1]$ as follows. Choose a retraction $\rho : [0,1]^2 \to U$ where

$$U = \{0\} \times [0,1] \cup [0,1] \times \{0\} \cup \{1\} \times [0,1] = \bigsqcup \subset [0,1]^2.$$

One then defines :

$$\begin{aligned} r : \mathrm{Cyl}_f \times [0,1] &\to \big(\mathrm{Cyl}_f \times \{0\}\big) \cup \big(A \times \{1\} \times [0,1]\big), \\ ([x], t) &\mapsto ([x], 0), \\ ([a, s], t) &\mapsto \begin{cases} ([f(a)], 0), & \text{if } \rho(s,t) = (0, \tau), \\ ([a, \sigma], 0), & \text{if } \rho(s,t) = (\sigma, 0), \\ ([a, 1], \tau), & \text{if } \rho(s,t) = (1, \tau). \end{cases} \end{aligned}$$

Moreover, $f : A \hookrightarrow \mathrm{Cyl}_f \underset{\sim}{\xrightarrow{p}} X$ factors as a cofibration followed by a weak equivalence ($p([x]) = x$, $p([a,t]) = f(a)$) passing through Cyl_f. This allows one to « replace » any map by a cofibration up to equivalence.

Remark 1.2.15. Let $\gamma : A \sqcup A \to A$ be the obvious map. Let $f, g : A \to X$ be two maps. A homotopy between f and g is exactly the data of a map $H : \mathrm{Cyl}_\gamma \to Y$ such that the composite $A \sqcup A \to \mathrm{Cyl}_\gamma \to X$ is given by (f, g).

Remark 1.2.16. This last statement dualizes but is slightly more involved. Let $f, g : A \to X$ be two maps. Let $\delta : X \to X \times X$ be the diagonal and consider the path space P_δ. Then a map $H : X \to P_\delta$ satisfying $\mathrm{ev}_0 \circ H = f$ and $\mathrm{ev}_1 \circ H = g$ is equivalent to the data of a third map $u : A \to X$ together with two homotopies $f \simeq u$ and $u \simeq g$.

Definition 1.2.17. Let (X, A) be a pair of spaces, let $a_0 \in A$ be a basepoint, and let $n \geq 1$. The nth relative homotopy group is the set of maps $\gamma : [0,1]^n \to X$ such that $\gamma(\partial[0,1]^n) \subset A$ and $\gamma\big(\partial[0,1]^n \setminus ([0,1]^{n-1} \times \{0\})\big) = \{a_0\}$, modulo the relation of homotopy rel A. It is a group for $n \geq 2$, and is abelian for $n \geq 3$.

Remark 1.2.18. One sometimes defines $\pi_0(X, A)$ as the set of path-connected components that do not meet A. One may also consider $\pi_0(X, A)$ as simply undefined.

Proposition 1.2.19. ↻ *Let (X, A) be a pair of topological spaces. Then we have a long exact sequence :*

$$\dots \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \dots$$

Proposition 1.2.20. ↻ *If $i : A \to X$ is a cofibration, A is r-connected and X is s-connected, then $\pi_n(X, A) \cong \pi_n(X/A)$ for $n \leq r + s$ and $\pi_{r+s+1}(X, A) \to \pi_{r+s+1}(X/A)$ is surjective.*

Remark 1.2.21. In general, $\pi_n(X, A) \neq \pi_n(X/A)$. For example, $\pi_n(D^2, S^1) = 0$ if $n \neq 2$ and $\pi_2(D^2, S^1) = \mathbb{Z}$, whereas $\pi_3(D^2/S^1) = \pi_3(S^2) = \mathbb{Z}/2\mathbb{Z}$.

1.3 Axiomatization

Definition 1.3.1 (Quillen [Qui67]). A *model category*[1] is a category M equipped with three classes of morphisms :

Class	Name of elements	Notation
$\mathscr{W}$	Weak equivalences	$\xrightarrow{\sim}$
$\mathscr{C}$	Cofibrations	$\hookrightarrow$
$\mathscr{F}$	Fibrations	$\twoheadrightarrow$

satisfying the following axioms :

(MC1) M is complete and cocomplete ;

(MC2) *2 out of 3* : Let f and g be two composable morphisms ; if two of the morphisms f, g and $g \circ f$ are in $\mathscr{W}$, then so is the third ;

(MC3) If f is a retract of g and $g \in \mathscr{W}$ (resp. $\mathscr{F}$, $\mathscr{C}$) then $f \in \mathscr{W}$ (resp. $\mathscr{F}$, $\mathscr{C}$) ;

(MC4) Consider the commutative diagram formed by the solid arrows :

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(i) If $i \in \mathscr{C}$ and $p \in \mathscr{F} \cap \mathscr{W}$ then a lift (dashed arrow) exists ;

(ii) If $i \in \mathscr{C} \cap \mathscr{W}$ and $p \in \mathscr{F}$ then a lift exists ;

(MC5) Every morphism $f : X \to Y$ admits two factorizations, functorial in f :

$$X \overset{\sim}{\hookrightarrow} P_f \twoheadrightarrow Y \qquad\qquad X \hookrightarrow C_f \overset{\sim}{\twoheadrightarrow} Y$$

Remark 1.3.2. In axiom (MC4), the lift is in general not unique.

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[1] Quillen called them *closed* model categories.

Definition 1.3.3. The elements of $\mathscr{C} \cap \mathscr{W}$ are called acyclic cofibrations, and those of $\mathscr{F} \cap \mathscr{W}$ are called acyclic fibrations.[2]

Definition 1.3.4. In the diagram of (MC4), if the dashed arrow always exists, we say that i has the *left lifting property* (LLP) with respect to p, and that p has the *right lifting property* (RLP) with respect to i. We sometimes write $i \perp p$.

Remark 1.3.5. By (MC1), the category M has an initial object $\varnothing$ (the empty colimit) and a terminal object $*$ (the empty limit).

Definition 1.3.6. An object $X \in \mathsf{M}$ is said to be *cofibrant* if the unique morphism $\varnothing \to X$ is a cofibration, and *fibrant* if the unique morphism $X \to *$ is a fibration.

Remark 1.3.7. Every object admits functorial fibrant and cofibrant replacements by (MC5) :

$$\begin{tikzcd} \varnothing \arrow[r, hook] & Q(X) \arrow[r, two heads, "\sim"] & X & Y \arrow[r, hook, "\sim"] & R(Y) \arrow[r, two heads] & * \end{tikzcd}$$

Example 1.3.8. Let M be an arbitrary category. Then for $\mathscr{W} = \{\text{isomorphisms}\}$ and $\mathscr{C} = \mathscr{F} = \mathsf{M}$ we obtain a model category. One can also choose $(\mathscr{W}, \mathscr{C}, \mathscr{F}) = (\text{isos}, \mathsf{M}, \text{isos})$ or the dual.

Example 1.3.9. Let $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ be a model category.

- The product of M with another model category admits an obvious model category structure.
- $(\mathsf{M}^{\mathrm{op}}, \mathscr{W}^{\mathrm{op}}, \mathscr{F}^{\mathrm{op}}, \mathscr{C}^{\mathrm{op}})$ is a model category.

Proposition 1.3.10. *Let* $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ *be a model category.*

- *i is a cofibration $\iff$ i satisfies the LLP with respect to all acyclic fibrations.*
- *i is an acyclic cofibration $\iff$ i satisfies the LLP with respect to all fibrations.*
- *p is a fibration $\iff$ p satisfies the RLP with respect to all acyclic cofibrations.*
- *p is an acyclic fibration $\iff$ p satisfies the RLP with respect to all cofibrations.*
- *f is a weak equivalence $\iff$ f factors as $p \circ i$ where p is an acyclic fibration and i is an acyclic cofibration.*

Corollary 1.3.11. *Let* $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ *be a model category.*

- *The data of two of the classes $\mathscr{W}$, $\mathscr{C}$ and $\mathscr{F}$ determines the third.*
- *The classes $\mathscr{W}$, $\mathscr{C}$ and $\mathscr{F}$ are closed under composition.*
- *The classes $\mathscr{C}$ and $\mathscr{C} \cap \mathscr{W}$ are closed under pushout, and the classes $\mathscr{F}$ and $\mathscr{F} \cap \mathscr{W}$ are closed under pullback.*
- *Isomorphisms belong to all three classes $\mathscr{W}$, $\mathscr{C}$ and $\mathscr{F}$.*

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[2] One sometimes finds "trivial cofibrations" and "trivial fibrations", but it is better to avoid this terminology as it is reminiscent of trivial fiber bundles, i.e. products.

Proof of Proposition 1.3.10. The first four points are proved in nearly identical fashion. Let us prove the first one. The $\Longrightarrow$ direction follows from (MC4). Now let $i : A \to B$ be a map with the LLP with respect to all acyclic fibrations. By (MC5), we can factor i as $A \hookrightarrow X \overset{\sim}{\twoheadrightarrow} B$. Using the LLP of i, we can find h making the following diagram commute :

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "i"] & X \arrow[d, two heads, "\sim"] \\
B \arrow[r, equal] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

We then see that i is a retract of $A \hookrightarrow X$ and conclude by (MC3) :

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "i"] & A \arrow[r, equal] \arrow[d, hook] & A \arrow[d, "i"] \\
B \arrow[r, "h"] \arrow[rr, bend right, "\mathrm{id}_B"'] & X \arrow[r, two heads, "\sim"] & B
\end{tikzcd}$$

For the fifth point, we use (MC2) in both cases. ☐

Proof of Corollary 1.3.11.

- Clear by the proposition.
- The fact that $\mathscr{W}$ is closed under composition follows from (MC2). Let us show for example that $\mathscr{C}$ is closed under composition. Let $A \overset{i}{\hookrightarrow} B \overset{j}{\hookrightarrow} C$ be two cofibrations. We show that $j \circ i$ has the LLP with respect to acyclic fibrations. We find lifts in the following diagram in two steps (first l then l') :

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "i"] & X \arrow[dd, two heads, "\sim"] \\
B \arrow[d, hook, "j"] \arrow[ur, dashed, "l"] & \\
C \arrow[r] \arrow[uur, dashed, "l'"'] & Y
\end{tikzcd}$$

- The proof of stability under pushout and pullback is similar.
- It is clear that an isomorphism has the LLP and the RLP with respect to any map. ☐

We will not prove the following examples right away.

Example 1.3.12 (Quillen [Qui67])**.** Top has a model category structure where the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations, and the cofibrations are the retracts of generalized cellular inclusions.[3] All objects are fibrant, and the cofibrant objects are the retracts of generalized CW-complexes.

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[3] That is, maps $i : A \to B$ where $B = \operatorname{colim} B_n$, $B_0 = A$, and B_{n+1} is obtained from B_n by attaching cells.

Example 1.3.13 (Strøm [Str72])**.** Top has another model category structure where the weak equivalences are the homotopy equivalences, the cofibrations are the retracts of closed Hurewicz cofibrations, and the fibrations are the Hurewicz fibrations. Every object is fibrant and cofibrant.

Example 1.3.14 (Cole [Col06])**.** There also exists a "mixed" model category structure on Top whose weak equivalences are the weak homotopy equivalences (Quillen structure) and whose fibrations are the Hurewicz fibrations (Strøm structure). The cofibrations are the cofibrations of the Strøm structure that factor as $f \circ i$ where i is a cofibration of the Quillen structure and f is a (strong) homotopy equivalence.

Example 1.3.15. The category $\mathsf{Ch}(R)$ has several model category structures :

- the *projective* structure : $\mathscr{W} = \{\text{quasi-iso}\}$, $\mathscr{C} = \{\text{injections with projective cokernel}\}$, $\mathscr{F} = \{\text{surjections}\}$;
- the *injective* structure : $\mathscr{W} = \{\text{quasi-iso}\}$, $\mathscr{C} = \{\text{injections}\}$, $\mathscr{F} = \{\text{surjections with injective kernel}\}$;
- a third structure "in the style of Strøm" : $\mathscr{W} = \{\text{homotopy equivalences}\}$, $\mathscr{C} = \mathrm{LLP}(B^I \xrightarrow{ev_0} B)$, $\mathscr{F} = \mathrm{RLP}(A \xrightarrow{i_0} A \otimes I)$ where $I = N_*(\Delta^1)$.

The projective structure generalizes to $\mathsf{Ch}_{\geq 0}(R)$, by replacing surjections with surjections in degree ≥ 1.

1.4 Homotopy category

1.4.1 Localization

General case

Let M be a category and $\mathscr{W}$ a class of morphisms.

Definition 1.4.1 ([GZ67])**.** A *(Gabriel–Zisman) localization* of M with respect to $\mathscr{W}$ is a category $\mathsf{M}[\mathscr{W}^{-1}]$ equipped with a functor $\lambda : \mathsf{M} \to \mathsf{M}[\mathscr{W}^{-1}]$ satisfying the following universal property : for every functor $F : \mathsf{M} \to \mathsf{D}$, if F sends the morphisms of $\mathscr{W}$ to isomorphisms, then there exists a unique (up to natural isomorphism) functor $G : \mathsf{M}[\mathscr{W}^{-1}] \to \mathsf{D}$ such that $G \circ \lambda \cong F$.

Remark 1.4.2. In particular, the morphisms of $\mathscr{W}$ are sent to isomorphisms in $\mathsf{M}[\mathscr{W}^{-1}]$. The converse is false in general, but we will see that it holds for model categories.

Proposition 1.4.3. $\circlearrowleft$ *The localization* $\mathsf{M}[\mathscr{W}^{-1}]$ *exists and is unique up to equivalence. We also denote it* $\mathrm{Ho}(\mathsf{M})$ *when* $\mathscr{W}^{-1}$ *is implicit.*

Remark 1.4.4. This proposition is only approximately true. Indeed, in the proof that follows, the morphisms between two objects do not necessarily form a set ! We will see later that in a model category, the morphisms in the homotopy category always form a set.

Proof of Proposition 1.4.3. Let us explicitly describe the category $\mathrm{Ho}(\mathsf{M})$. Its objects are the same as those of M. The morphisms of $\mathrm{Ho}(\mathsf{M})$ are given by the quotient

$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(X,Y) = \mathrm{Path}_{\mathscr{W}}(X,Y)/\!\sim,$$

where :

- $\mathrm{Path}_{\mathscr{W}}(X,Y)$ is the class of “paths” between X and Y formed by morphisms of M and morphisms of $\mathscr{W}$ in the reverse direction ;
- the relation $\sim$ is generated by $X \xrightarrow{f} Y \xleftarrow[\sim]{f} X \sim X$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \sim X \xrightarrow{g\circ f} Y$.

Composition is given by concatenation of paths. One verifies that this forms a category and that $\mathsf{M} \to \mathsf{M}[\mathscr{W}^{-1}]$ (which sends morphisms to paths of length 1) is a functor satisfying the universal property. ☐

The problem with this construction (aside from the set-theoretic issue) is that it is very difficult to perform computations. For example, determining whether two morphisms are equal in $\mathrm{Ho}(\mathsf{M})$ is a non-trivial problem.

Example 1.4.5. A category M is said to be *concrete* if there exists a faithful functor $\mathsf{M} \to \mathsf{Set}$. This is the case for many categories we are accustomed to working with : topological spaces, groups, etc. Let $\mathsf{M} = \mathsf{Top}$ and $\mathscr{W}$ the class of weak equivalences. A theorem of Freyd [Fre70] states that $\mathrm{Ho}(\mathsf{Top})$ is *not* concrete.

Remark 1.4.6 (Exercise)*.* There is another construction, slightly less ad hoc (but it is equally difficult to perform computations with it). Let M be a category and $\mathscr{W}$ a class of morphisms. Denote by Arr the category with two objects x, y and a unique non-trivial morphism $x \to y$. Denote also by Iso the category with two objects x, y and two non-trivial morphisms $x \to y$ and $y \to x$ inverse to each other. There is an obvious inclusion $\mathsf{Arr} \subset \mathsf{Iso}$. The data of a functor $\mathsf{Arr} \to \mathsf{M}$ is equivalent to the data of a morphism of M, while a functor $\mathsf{Iso} \to \mathsf{M}$ is equivalent to the data of an isomorphism of M. One then verifies that the localization $\mathsf{M}[\mathscr{W}^{-1}]$ is equivalent to the pushout :

$$\begin{tikzcd}
\bigsqcup_{f\in\mathscr{W}} \mathsf{Arr} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & \mathsf{M} \arrow[d, dashed] \\
\bigsqcup_{f\in\mathscr{W}} \mathsf{Iso} \arrow[r, dashed] & \mathsf{M}[\mathscr{W}^{-1}]
\end{tikzcd}$$

Example 1.4.7 (Exercise)**.** Let M be a category having a unique object $*$ and let $M = \mathrm{Hom}_{\mathsf{M}}(*,*)$ be its endomorphism monoid. Set $\mathscr{W} = M$. Then $\mathsf{M}[\mathscr{W}^{-1}]$ is the category with one object whose endomorphism monoid is M^{+}, the group completion of M.

Example 1.4.8 (Exercise)**.** Let $F : \mathsf{M} \to \mathsf{D}$ be a functor having a right adjoint $G : \mathsf{D} \to \mathsf{M}$. Let $\mathscr{W} = \{f : X \to Y \mid F(f) \text{ is an isomorphism}\} \subset \mathsf{M}$. The following statements are equivalent :

- the functor G is fully faithful ;
- the counit $F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ is an isomorphism ;
- the natural functor $\mathsf{M}[\mathscr{W}^{-1}] \to \mathsf{D}$ is an equivalence of categories.

Example 1.4.9 (Exercise)**.** Let $\mathsf{M} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$ be the category of abelian groups, viewed as $\mathbb{Z}$-modules. Let p be a prime number. Define the class $\mathscr{W}$ as the morphisms $f : X \to Y$ such that $\ker f$ and $\operatorname{coker} f$ are p-torsion. Then the localization $\mathsf{Mod}_{\mathbb{Z}}[\mathscr{W}^{-1}] \simeq \mathsf{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is equivalent to the category of $\mathbb{Z}[\frac{1}{p}]$-modules.

Remark 1.4.10 (Exercise : right fractions)*.* Suppose that the class $\mathscr{W}$ satisfies the following properties :

- $\mathscr{W}$ is closed under composition ;
- for every diagram $A \xleftarrow{w} C \xrightarrow{f} B$ where $w \in \mathscr{W}$, one can complete the following square, where $w' \in \mathscr{W}$:

$$\begin{tikzcd}
 & C \arrow[ld, "w"', "\sim"] \arrow[rd, "f"] & \\
A \arrow[rd, "f'"'] & & B \arrow[ld, "w'", "\sim"'] \\
 & C' &
\end{tikzcd}$$

- for all $f, g : A \to X$, if there exists $w \in \mathscr{W}$ such that $f \circ w = g \circ w$, then there exists $w' \in \mathscr{W}$ such that $w' \circ f = w' \circ g$.

Then the localization $\mathsf{M}[\mathscr{W}^{-1}]$ can be described as follows. Its objects are the same as those of M. The morphisms $A \to X$ are zigzags of length 2 of the form $A \to X' \leftarrow X$, where the arrow going in the wrong direction is in $\mathscr{W}$, modulo the following equivalence relation. Two zigzags $A \to X' \leftarrow X$ and $A \to X'' \leftarrow X$ are equivalent if one can complete the following diagram :

$$\begin{tikzcd}
 & X' \arrow[d] & \\
A \arrow[ru] \arrow[rd] & \bar{X} & X \arrow[lu] \arrow[ld] \\
 & CX'' \arrow[u] &
\end{tikzcd}$$

where the arrow $X \to \bar{X}$ is in $\mathscr{W}$. Composition is defined using the second property above.

In a model category

Definition 1.4.11. Let $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ be a model category. We define its *homotopy category* by $\mathrm{Ho}(\mathsf{M}) = \mathsf{M}[\mathscr{W}^{-1}]$.

Remark 1.4.12. The homotopy category obviously depends only on the class $\mathscr{W}$, not on $\mathscr{C}$ and $\mathscr{F}$. This is where we see the idea that fibrations and cofibrations are merely auxiliary data serving to study the object we are truly interested in, namely the homotopy category of M.

Definition 1.4.13. We define the full subcategories $\mathsf{M}_c, \mathsf{M}_f, \mathsf{M}_{cf} \subset \mathsf{M}$ generated respectively by the cofibrant objects, by the fibrant objects, and by the cofibrant and fibrant objects. We also define their homotopy categories as their localizations with respect to $\mathscr{W}$.

Lemma 1.4.14. *The inclusions induce equivalences of categories :*

$$\begin{tikzcd}
 & \mathrm{Ho}(\mathsf{M}_c) \arrow[dr, "\sim"] & \\
\mathrm{Ho}(\mathsf{M}_{cf}) \arrow[ur, "\sim"] \arrow[dr, "\sim"] & & \mathrm{Ho}(\mathsf{M}) \\
 & \mathrm{Ho}(\mathsf{M}_f) \arrow[ur, "\sim"] &
\end{tikzcd}$$

Démonstration. We treat the case $\mathsf{M}_f \subset \mathsf{M}$; the others are similar. The inclusion $\mathsf{M}_f \subset \mathsf{M}$ sends weak equivalences to weak equivalences by definition. It therefore induces a functor $\mathrm{Ho}(\mathsf{M}_f) \to \mathrm{Ho}(\mathsf{M})$. The "fibrant replacement" functor $R : \mathsf{M} \to \mathsf{M}_f$ similarly induces a functor $\mathrm{Ho}(\mathsf{M}) \to \mathrm{Ho}(\mathsf{M}_f)$. One then verifies using the axioms of model categories that these functors are inverse (up to natural equivalence) to each other. □

1.4.2 Homotopies

We will now show that $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(A, X)$ forms a set that can be described as a quotient of a set of morphisms in M by a homotopy relation. Note that there are two dual notions of homotopy, one on the left (for the source) and one on the right (for the target).

Left homotopy

Definition 1.4.15. A *cylinder* of $A \in \mathsf{M}$ is a factorization

$$A \sqcup A \hookrightarrow C \xrightarrow{\sim} A$$

of the canonical map $A \sqcup A \to A$. We denote by $i_0, i_1 : A \to C$ the two components of the cofibration.

Definition 1.4.16. Let $f, g : A \to X$ be two morphisms. A *left homotopy* between f and g is the data of a cylinder $A \sqcup A \xhookrightarrow{(i_0,i_1)} C \xrightarrow{\sim} A$ and a map $H : C \to X$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$, which can be summarized by the following diagram :

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook] \arrow[dr, "(id_A, id_A)"'] \arrow[rr, bend left, "(f,g)"] & C \arrow[r, "H"'] \arrow[d, "\sim"] & X \\
 & A &
\end{tikzcd}$$

If such an H exists, we write $f \simeq_l g$.

Example 1.4.17. ↺ In Top with the Quillen model structure, one can for instance choose $C = A \times [0,1]$ when A is cofibrant. A left homotopy is exactly what is usually called a homotopy.

Example 1.4.18. ↺ In $\mathsf{Ch}(R)$ with the injective model structure, let A_* be a chain complex. One can choose as cylinder $C_n = A_n \oplus A_n \oplus A_{n-1}$ with $d(x,y,z) = (dx + (-1)^n z, dy - (-1)^n z, dz)$. A homotopy $f \simeq g : A \to X$ is equivalent to the data of $h : A_{n-1} \to X_n$ such that $dh \pm hd = f - g$ (exactly the usual definition).

Remark 1.4.19. In general it is not possible to choose a cylinder once and for all ; two maps may be left homotopic with respect to one cylinder but not with respect to another that has been fixed in advance.

Proposition 1.4.20. *If* $f \simeq_l g : A \to X$ *then* $h \circ f \simeq_l h \circ g : A \to Y$ *for every morphism* $h : X \to Y$*.*

Démonstration. Let $H : C \to X$ be a homotopy between f and g. Then $h \circ H$ is a homotopy between $h \circ f$ and $h \circ g$. ☐

Lemma 1.4.21. *If* $A \sqcup A \hookrightarrow C \xrightarrow{\sim} A$ *is a cylinder and* A *is cofibrant, then* $i_0, i_1 : A \to C$ *are acyclic cofibrations.*

Démonstration. We first note that $A \to A \sqcup A$ is a cofibration, since it is the pushout of :

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d, hook] & A \arrow[d] \\
A \arrow[r] & A \sqcup A
\end{tikzcd}$$

Since i_0 is the composite of two cofibrations, it is a cofibration. Axiom (MC2) implies that it is acyclic :

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[rrr, bend left, equal] \arrow[rr, bend right, "i_0"'] & A \sqcup A \arrow[r, hook] & C \arrow[r, "\sim"] & A
\end{tikzcd}$$

☐

Proposition 1.4.22. *If* A *is cofibrant, then* $\simeq_l$ *defines an equivalence relation on* $\mathrm{Hom}_{\mathsf{M}}(A, X)$*.*

Démonstration. We verify the properties one by one.

- Reflexivity : Let $f : A \to X$ and let $A \sqcup A \to C \xrightarrow{\sim} A$ be any cylinder, obtained for instance by (MC5). Then $H : C \xrightarrow{\sim} A \xrightarrow{f} X$ is a homotopy between f and f.
- Symmetry : If $f \simeq_l g$ via a homotopy $H : C \to X$, then we define a new cylinder $A \sqcup A \cong A \sqcup A \hookrightarrow C \to A$ where the first morphism exchanges the two factors. The morphism H' obtained by composing H with this exchange then defines a homotopy between g and f : $H' \circ i_0 = H \circ i_1 = g$ and $H' \circ i_1 = H \circ i_0 = f$.

- Transitivity : This is where we need the hypothesis “A is cofibrant”. Suppose $f \simeq_l g$ and $g \simeq_l h$ via homotopies $H : C \to X$ and $H' : C' \to X$. We construct a new cylinder C'' as a pushout (draw a picture : we are gluing two cylinders) :

$$\begin{tikzcd}
A \arrow[r, hook, "i'_0", "\sim"'] \arrow[d, hook, "i_1", "\sim"'] & C' \arrow[d] \\
C \arrow[r] & C''
\end{tikzcd}$$

  The morphisms $C \to C''$ and $C' \to C''$ are acyclic cofibrations (as pushouts of acyclic cofibrations). The universal property induces a morphism $C'' \to A$, which is a weak equivalence by (MC2) ($C \xrightarrow{\sim} C'' \to X$ is a weak equivalence). One verifies that this gives a cylinder by factoring (i_0, i'_1) as a cofibration followed by an acyclic fibration. Moreover, again by the universal property, H and H' induce $H'' : C'' \to X$ which restricts to f and h, i.e., it is a homotopy $f \simeq_l h$. ☐

Proposition 1.4.23. *Let A be a cofibrant object and $h : X \xrightarrow{\sim} Y$ a weak equivalence. If h is an (acyclic) fibration or if X and Y are fibrant, then h_* is a bijection :*

$$h_* : \mathrm{Hom}_{\mathsf{M}}(A, X)/{\simeq_l} \to \mathrm{Hom}_{\mathsf{M}}(A, Y)/{\simeq_l}.$$

Démonstration. By the previous proposition, $h_*(f) = h \circ f$ passes to the quotient.

- Suppose that $h : X \xrightarrow{\sim} Y$ is an acyclic fibration. Surjectivity of h_* follows immediately from (MC4). For injectivity, suppose that $f, g : A \to X$ are two morphisms such that $h \circ f \simeq_l h \circ g$. Let $A \sqcup A \hookrightarrow C \xrightarrow{\sim} A$ be a cylinder and $H : C \to Y$ a homotopy between $h \circ f$ and $h \circ g$. We can find a lift :

$$\begin{tikzcd}
A \sqcup A \arrow[r, "{(f,g)}"] \arrow[d, hook] & X \arrow[d, two heads, "h", "\sim"'] \\
C \arrow[r, "H"] \arrow[ur, dashed, "K"] & Y
\end{tikzcd}$$

  and K is a homotopy between f and g.

- Now suppose that X and Y are fibrant. We will deduce bijectivity from the previous case using Brown’s Lemma (see Lemma 1.6.6 for the general statement). We have just shown that $F = \mathrm{Hom}_{\mathsf{M}}(A, -)/{\simeq_l}$ sends acyclic fibrations to bijections. Let us show that it then sends weak equivalences between fibrant objects to bijections as well. We can factor the morphism $(\mathrm{id}_X, h) : X \to X \times Y$ as $X \overset{\sim}{\hookrightarrow} W \to X \times Y$. Since X and Y are fibrant, the morphisms $X \leftarrow X \times Y \to Y$ are fibrations (as pullbacks of fibrations). We obtain the following diagram :

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "h"', "\sim"] \arrow[d, hook, "i"', "\sim"] \arrow[dr, bend left, "\mathrm{id}_X", "\sim"'] & \\
 & W \arrow[d, two heads, "\pi"'] & \\
Y & X \times Y \arrow[l, two heads, "p_Y"] \arrow[r, two heads, "p_X"] & X
\end{tikzcd}$$

- $p_Y \circ \pi$ is an acyclic fibration by (MC2), so $F(p_Y \circ \pi) : F(W) \to F(Y)$ is a bijection ;
- similarly $F(p_X \circ \pi) : F(W) \to F(X)$ is a bijection ;
- $p_X \circ \pi \circ i$ is the identity, so $F(p_X \circ \pi \circ i)$ is the identity, hence in particular a bijection ; since $F(p_X \circ \pi)$ is a bijection, we deduce that $F(i)$ is a bijection ;
- finally, $F(h) = F(p_X \circ \pi) \circ F(i)$ is a bijection as a composite of two bijections.

□

Proposition 1.4.24. *If X is fibrant, $f \simeq_l g : A \to X$ and $h : B \to A$ is a morphism, then $f \circ h \simeq_l g \circ h$.*

Démonstration. Let $A \sqcup A \hookrightarrow C \xrightarrow{\sim} A$ be a cylinder and $H : C \to X$ a homotopy between f and g. Factor $C \xrightarrow{\sim} A$ as $C \overset{\sim}{\hookrightarrow} C' \xrightarrow{\sim} A$ by (MC5) (and by (MC2) both morphisms are weak equivalences). Since X is fibrant, we can find H' such that :

$$\begin{tikzcd}
C \arrow[r, "H"] \arrow[d, hook, "\sim"] & X \arrow[d, two heads] \\
C' \arrow[r] \arrow[ur, dashed, "H'"'] & *
\end{tikzcd}$$

Then $H' : C' \to X$ is still a homotopy between f and g. The advantage being that we have a cylinder of the form $A \sqcup A \hookrightarrow C' \xrightarrow{\sim} A$.

Now, let $B \sqcup B \hookrightarrow D \xrightarrow{\sim} B$ be a cylinder for B. We can find a lift G :

$$\begin{tikzcd}
B \sqcup B \arrow[r, "h \sqcup h"] \arrow[d, hook] & A \sqcup A \arrow[r, hook] & C' \arrow[d, two heads, "\sim"] \\
D \arrow[r, "\sim"] \arrow[urr, dashed, "G"] & B \arrow[r, "h"] & A
\end{tikzcd}$$

One then verifies that $H \circ G$ is a homotopy between $f \circ h$ and $g \circ h$. □

Right homotopy

Everything can also be dualized. All the statements and proofs that follow are formally dual to those of the previous section.

Definition 1.4.25. A *path object* of $X \in \mathsf{M}$ is a factorization

$$X \xrightarrow{\sim} P \to X \times X$$

of the canonical map $X \to X \times X$. We denote by $p_0, p_1 : P \to X$ the two components of the fibration.

Example 1.4.26. ↻ In Top with the Quillen model structure, a path object of a CW-complex X is given by $X^{[0,1]}$.

Example 1.4.27. In $\mathsf{Ch}_{\geq 0}(R)$ with the projective model structure, let A_* be a chain complex. A path object of a chain complex can be given by $C_n = A_n \oplus A_n \oplus A_{n+1}$ with $d(x, y, z) = (dx, dy, dz + y - x)$.

Definition 1.4.28. Let $f, g : A \to X$ be two morphisms. A *right homotopy* between f and g is the data of a path object $X \xrightarrow{\sim} P \xrightarrow{(p_0,p_1)} X \times X$ and a map $H : A \to P$ such that $p_0 \circ H = f$ and $p_1 \circ H = g$, which can be summarized by the following diagram :

$$\begin{tikzcd}
A \arrow[r, "H"'] \arrow[rr, bend left, "{(f,g)}"] & P \arrow[r, two heads] & X \times X \\
 & X \arrow[u, "\sim"] \arrow[ur, "{(\mathrm{id}_X,\mathrm{id}_X)}"'] &
\end{tikzcd}$$

If such an H exists, we write $f \simeq_r g$.

Proposition 1.4.29. *If* $f \simeq_r g : A \to X$ *then* $f \circ h \simeq_r g \circ h : B \to X$ *for every morphism* $h : B \to A$.

Lemma 1.4.30. *If* $X \xrightarrow{\sim} P \to X \times X$ *is a path object and* X *is cofibrant, then* $p_0, p_1 : P \to X$ *are acyclic fibrations.*

Proposition 1.4.31. *If* X *is fibrant, then* $\simeq_r$ *defines an equivalence relation on* $\mathrm{Hom}_{\mathsf{M}}(A, X)$.

Proposition 1.4.32. *Let* X *be a fibrant object and* $h : A \xrightarrow{\sim} B$ *a weak equivalence. If* h *is an (acyclic) cofibration or if* A *and* B *are cofibrant, then* h^* *is a bijection :*

$$h^* : \mathrm{Hom}_{\mathsf{M}}(B, X)/{\simeq_r} \to \mathrm{Hom}_{\mathsf{M}}(A, X)/{\simeq_r}.$$

Proposition 1.4.33. *If* A *is cofibrant,* $f \simeq_r g : A \to X$ *and* $h : X \to Y$ *is a morphism, then* $h \circ f \simeq_r h \circ g$.

1.4.3 Explicit description

Proposition 1.4.34. *Let* $f, g : A \to X$ *be two morphisms.*

- *If* A *is cofibrant then* $f \simeq_l g \implies f \simeq_r g$.
- *If* X *is fibrant then* $f \simeq_r g \implies f \simeq_l g$.

Démonstration. We prove the first point ; the other is dual. Suppose that $f \simeq_l g$. Let $A \sqcup A \hookrightarrow C \xrightarrow[\sim]{j} A$ be a cylinder for A and $H : C \to X$ a homotopy. We have seen that i_0 is an acyclic cofibration. Choose any path object $X \xrightarrow{\sim} P \twoheadrightarrow X \times X$ for X. We can find a lift :

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i_0", "\sim"'] & X \arrow[r, "\sim"] & P \arrow[d, two heads] \\
C \arrow[rru, dashed, "K"] \arrow[rr, "{(fj,H)}"'] & & X \times X
\end{tikzcd}$$

Then $Ki_1 : A \to P$ is a right homotopy between f and g. $\square$

Definition 1.4.35. Let A be a cofibrant object and X a fibrant object. We denote by $[A, X]$ the quotient of $\mathrm{Hom}_{\mathsf{M}}(A, X)$ by the equivalence relation of the previous proposition.

Definition 1.4.36. The category $\pi\mathsf{M}_{cf}$ has as objects the fibrant and cofibrant objects of M, and as morphisms $\mathrm{Hom}_{\pi\mathsf{M}_{cf}}(A, X) = [A, X]$.

Theorem 1.4.37 (Analogue of Whitehead's theorem)**.** *Let $f : A \to X$ be a morphism between two fibrant and cofibrant objects. Then f is a weak equivalence if and only if there exists $g : X \to A$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities of X and A.*

Démonstration. First, suppose that f is a weak equivalence. We can factor it as $A \xhookrightarrow[\sim]{i} W \xrightarrow[\sim]{p} X$. Since i is an acyclic cofibration and A is fibrant, we deduce that there exists a right inverse $r : W \to A$ (i.e., $ri = \mathrm{id}_A$). Since W is fibrant and i is an acyclic cofibration, we deduce that $i^* : \mathrm{Hom}_{\mathsf{M}}(W, W)/{\simeq_r} \to \mathrm{Hom}_{\mathsf{M}}(A, W)/{\simeq_r}$ is a bijection. Now, $i^*([ir]) = [iri] = [i] = i^*([\mathrm{id}_W])$, hence $ir \simeq_r \mathrm{id}_W$. We deduce that r is an inverse of i up to (right) homotopy. A dual argument gives $s : X \to W$ which is an inverse of p up to (left) homotopy.

Conversely, suppose that f is a homotopy equivalence. We factor f as $A \xhookrightarrow[\sim]{i} W \xrightarrow{p} X$; it suffices to show that p is a weak equivalence. Note that W is bifibrant. Let $g : X \to A$ be a homotopy inverse of f and $H : C \to X$ a homotopy between fg and id_X (where C is a cylinder of X). Then we can find a lift H' in the following diagram :

$$\begin{tikzcd}
X \arrow[r, "ig"] \arrow[d, hook, "i_0", "\sim"'] & W \arrow[d, two heads, "p"] \\
C \arrow[r, "H"] \arrow[ru, dashed, "H'"] & X
\end{tikzcd}$$

Let $s = H' \circ i_1$, which is homotopic to ig via H'. Since i is a weak equivalence, it is a homotopy equivalence by what we have just shown ; let r be a homotopy inverse of i. Since $pi = f$, we have $p \sim pir = fr$. Moreover $s \sim ig$ so $sp \sim igp \sim igfr \sim ir \sim \mathrm{id}_C$. We deduce that sp is a weak equivalence, and p is a retract of sp :

$$\begin{tikzcd}
W \arrow[r, "\mathrm{id}_W"] \arrow[d, "p"] & W \arrow[r, "\mathrm{id}_W"] \arrow[d, "sp"] & W \arrow[d, "p"] \\
X \arrow[r, "s"] & W \arrow[r, "p"] & X
\end{tikzcd}$$

□

For reference, we denote by $Q(X) \overset{\sim}{\twoheadrightarrow} X$ (resp. $X \overset{\sim}{\hookrightarrow} R(X)$) the functorial cofibrant (resp. fibrant) replacement.

Theorem 1.4.38 (Description of the homotopy category)**.** *The category* $\mathrm{Ho}(\mathsf{M})$ *is equivalent to $\pi\mathsf{M}_{cf}$. For all objects $A, X \in \mathsf{M}$,*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(A, X) \cong \mathrm{Hom}_{\mathsf{M}}(Q(A), R(Y))/{\sim}.$$

If $f : A \to X$ is a morphism that becomes an isomorphism in $\mathrm{Ho}(\mathsf{M})$*, then it is a weak equivalence.*

Démonstration. Since we already know that $\mathrm{Ho}(\mathsf{M}) \simeq \mathrm{Ho}(\mathsf{M}_{cf})$ (Lemma 1.4.14), it suffices to show that the quotient $\pi : \mathsf{M}_{cf} \to \pi\mathsf{M}_{cf}$ satisfies the universal property of localization with respect to $\mathscr{W}$. Let $F : \mathsf{M}_{cf} \to \mathsf{D}$ be a functor that sends weak equivalences to isomorphisms. We want to show that there exists a unique functor $\bar{F} : \pi\mathsf{M}_{cf} \to \mathsf{D}$ such that $\bar{F} \circ \pi = F$. On objects, we must set $\bar{F}(X) = F(X)$. Let us show that F passes to the quotient, i.e., that if $f, g : A \to X$ are two homotopic morphisms between fibrant and cofibrant objects, then $F(f) = F(g)$.

Let $A \sqcup A \hookrightarrow C \underset{\sim}{\xrightarrow{j}} A$ be a cylinder for A and $H : C \to X$ a homotopy between f and g, i.e., $H \circ i_0 = f$ and $H \circ i_1 = g$. (C is cofibrant ; we saw in the proof of Proposition 1.4.24 that we could also choose it so that j is a fibration, hence C is also fibrant since A is.) We will show that $F(i_0) = F(i_1)$. Since $j : C \to A$ is a weak equivalence, we deduce that $F(j)$ is an isomorphism. Now, $j \circ i_0 = j \circ i_1 = \mathrm{id}_A$, so $F(j) \circ F(i_0) = F(j) \circ F(i_1)$, hence $F(i_0) = F(i_1)$. We then deduce that

$$F(f) = F(H \circ i_0) = F(H \circ i_1) = F(g). \tag{1.4.39}$$

This allows us to conclude that π indeed satisfies the universal property.

Let us now show that $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(A, X) \cong \mathrm{Hom}_{\mathsf{M}}(QA, RX)$. By what we have just shown,

$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(A, X) \cong [QRA, QRX] = \mathrm{Hom}_{\mathsf{M}}(QRA, QRX)/\simeq. \tag{1.4.40}$$

Since QRA is cofibrant and $QRX \xrightarrow{\sim} RX$ is a weak equivalence between fibrant objects, we have $\mathrm{Hom}_{\mathsf{M}}(QRA, QRX)/\sim \, \simeq \mathrm{Hom}_{\mathsf{M}}(QRA, RX)/\sim$. Similarly, the latter is in bijection with $\mathrm{Hom}_{\mathsf{M}}(QA, RX)/\sim$.

Finally, suppose that $[f : A \to X]$ is an isomorphism in $\mathrm{Ho}(\mathsf{M})$. Thanks to the diagram

$$\begin{tikzcd}
QRA \arrow[r, "QRf"] \arrow[d, "\sim"] & QRX \arrow[d, "\sim"] \\
RA \arrow[r, "Rf"] & RX \\
A \arrow[r, "f"] \arrow[u, "\sim"] & X \arrow[u, "\sim"]
\end{tikzcd}$$

we see that it suffices to show that $QR(f)$ is a weak equivalence. By point 1 of the theorem, there exists $g : QRX \to QRA$ such that $[QRf \circ g] = [\mathrm{id}_{QRX}] \in [QRX, QRX]$ and $[g \circ QRf] = [\mathrm{id}_{QRA}] \in [QRA, QRA]$. In other words, $QRf \circ g \simeq \mathrm{id}_{QRX}$ and $g \circ QRf \simeq \mathrm{id}_{QRA}$. By Whitehead's theorem, we deduce that QRf is a weak equivalence. □

1.5 Cofibrantly Generated

We will begin by proving that the projective structure defines a model category on $\mathsf{Ch}_{\geq 0}(R)$. The proof will involve ideas that we will formalize under the name of « cofibrantly generated model categories » (or cofibrantly generated).

1.5.1 Example : chain complexes

As a reminder, $\mathsf{Ch}_{\geq 0}(R)$ denotes the category of chain complexes concentrated in nonnegative degrees. For a chain complex $M = (M_n, d_n)_{n\geq 0}$ we define :

- the cycles $Z_k(M)$ by $Z_0(M) = M_0$ and $Z_k(M) = \ker(d : M_k \to M_{k-1})$;
- the boundaries $B_k(M)$ by $B_k(M) = \operatorname{im}(d : M_{k+1} \to M_k)$;
- the homology $H_k(M) = Z_k(M)/B_k(M)$.

A chain complex is said to be acyclic if $H_k(M) = 0$ for all $k \in \mathbb{N}$.

Recall also that an R-module P is said to be projective if for every surjection $A \to B$, the induced map $\operatorname{Hom}(P, A) \to \operatorname{Hom}(P, B)$ is surjective. This is equivalent to requiring that there exists a free module R^n that decomposes as $P \oplus Q$ (for some Q), or alternatively that every surjective morphism $A \to P$ has a section.

Definition 1.5.1. We say that a morphism $f : M \to N$ of chain complexes is :

- a weak equivalence if it is a quasi-isomorphism ;
- a cofibration if for all $k \geq 0$, the map $f_k : M_k \to N_k$ is injective and its cokernel is projective ;
- a fibration if for all $k > 0$, the map $f_k : M_k \to N_k$ is surjective.

Theorem 1.5.2 (Projective structure on $\mathsf{Ch}_{\geq 0}(R)$)**.** *These three classes of morphisms define a model category structure on the category* $\mathsf{Ch}_{\geq 0}(R)$*, called the « projective structure ».*

Example 1.5.3. Let A and B be two R-modules. Consider A as a complex concentrated in degree m, denoted $A[-m]$; and B as a complex concentrated in degree n, denoted $B[-n]$. Then in the homotopy category, $[\Sigma^m A, \Sigma^n B] \cong \operatorname{Ext}_R^{n-m}(A, B)$.

Lemma 1.5.4. $\mathsf{Ch}_{\geq 0}(R)$ *satisfies axioms (MC1), (MC2), (MC3).*

Démonstration. Limits and colimits are computed degree by degree in $\mathsf{Ch}_{\geq 0}(R)$. Axiom (MC2) is clear. For axiom (MC3), we note that the retract of an isomorphism (resp. epimorphism, monomorphism) is an isomorphism (resp. epimorphism, monomorphism), and moreover that a retract of a projective module is projective. ☐

Lemma 1.5.5. $\mathsf{Ch}_{\geq 0}(R)$ *satisfies axiom (MC4(i)).*

Démonstration. Consider a commutative diagram of chain complexes :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"] & X \arrow[d, two heads, "p", "\sim"'] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

By definition, p_k is surjective for all $k > 0$. Moreover, since $H_0(X) \to H_0(Y)$ is an isomorphism, a small diagram chase shows that p_0 is surjective as well. Let $K = \ker(p)$, then we have a short exact sequence $0 \to K \to X \to Y \to 0$, so K is acyclic.

We seek a lift $l : B \to X$ which we will construct by induction on k.

- ($k = 0$) Let $P_0 = B_0/A_0$. We have a short exact sequence $0 \to A_0 \to B_0 \to P_0 \to 0$. Since P_0 is projective, this sequence splits : we can find a section $\sigma_0 : P_0 \to B_0$ of the projection $B_0 \to P_0$. We therefore deduce that B_0 is isomorphic to $A_0 \oplus P_0$ as R-modules. We then define $l_0 : A_0 \oplus P_0 \to X_0$ by taking f_0 on A_0 and any lift $P_0 \to X_0$ of the map $P_0 \subset B_0 \to Y_0$. This argument can be summarized by the following diagram :

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, hook, "i_0"] & X_0 \arrow[d, two heads, "p_0"] \\
B_0 \cong A_0 \oplus P_0 \arrow[r, "g_0"] \arrow[ur, dotted, "{(f_0, g_0\sigma)}"] \arrow[d, two heads] & Y_0 \\
P_0 \arrow[u, dashed, bend left, "\exists\sigma"] \arrow[ur, dashed, "g_0\sigma"'] &
\end{tikzcd}$$

- (induction) Let $k > 0$. Suppose that we have constructed maps $l_j : B_j \to X_j$ (for all $0 \leq j < k$) satisfying :
  (i) $dl_j = l_{j-1}d$ for $1 \leq j < k$,
  (ii) $p_j l_j = g_j$ and $l_j i_j = f_j$ for $0 \leq j < k$,

  We want to find l_k. As in the case $k = 0$, we have a splitting $B_k \cong A_k \oplus P_k$ and we can therefore define $\tilde{l}_k : B_k \to X_k$ satisfying the second point, but it is not necessarily compatible with the differential. We therefore define $\xi : B_k \to X_{k-1}$ by $\xi(b) = d(\tilde{l}_k(b)) - l_{k-1}(db)$. Then one verifies that :
  - $d\xi = 0$, since $dl_{k-1} = l_{k-2}d$,
  - $p_{k-1}\xi = 0$ since $p_k\tilde{l}_k = g_k$ which commutes with d,
  - $\xi i_k = 0$ since $\tilde{l}_k i_k = f_k$ which commutes with d.

  In other words, ξ induces $\xi' : B_k/i_k(A_k) \cong P_k \to Z_{k-1}(K)$. Now, K is acyclic, so $d : K_k \to Z_{k-1}(K)$ is surjective. By projectivity of P_k, we can lift ξ' to $\xi'' : P_k \to K_k$, which we can compose with the inclusion to obtain $\xi''' : P_k \to X_k$. One then readily verifies that $l_k := \tilde{l}_k - \xi'''$ satisfies the required equations. ☐

Lemma 1.5.6. $\mathsf{Ch}_{\geq 0}(R)$ *satisfies (MC4(ii)).*

The idea is that the cokernel of an acyclic cofibration has a very particular form.

Definition 1.5.7. Let A be an R-module. For $n > 0$, we define $D_n(A) \in \mathsf{Ch}_{\geq 0}(R)$:

$$\cdots \to 0 \to \underbrace{A}_{n} \xrightarrow{\mathrm{id}_A} \underbrace{A}_{n-1} \to 0 \to \dots$$

Lemma 1.5.8. *For any $M \in \mathsf{Ch}_{\geq 0}(R)$, there is a natural bijection :*

$$\mathrm{Hom}_{\mathsf{Ch}_{\geq 0}(R)}(D_n(A), M) \cong \mathrm{Hom}_{R-\mathsf{Mod}}(A, M_n).$$

In particular, if $M \to N$ is surjective in degree n and A is projective, then any map $D_n(A) \to N$ lifts to M. ☐

Lemma 1.5.9. *Let $P \in \mathsf{Ch}_{\geq 0}(R)$ be an acyclic complex which is projective in each degree. Then all the $Z_k(P)$ are projective, and P is isomorphic in $\mathsf{Ch}_{\geq 0}(R)$ to $\bigoplus_{k \geq 1} D_k(Z_{k-1}(P))$.*

Démonstration. We define a subcomplex (for $k \geq 1$) :

$$P_n^{(k)} = \begin{cases} P_n & n \geq k \\ B_{k-1}(P) & n = k-1 \\ 0 & n \leq k-2 \end{cases}$$

Since P is acyclic, one verifies that $P^{(k)}/P^{(k+1)} \cong D_k(Z_{k-1}(P))$. Since $Z_0(P) = P_0$ is projective, using the previous lemma we deduce that $P = P^{(1)} \cong P^{(2)} \oplus D_1(Z_0(P))$. We now apply the same argument to $P^{(2)}$ to find $P^{(2)} \cong P^{(3)} \oplus D_2(Z_1(P))$, etc. ☐

Proof of Lemma 1.5.6. We are given a diagram :

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i", "\sim"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}$$

The morphism i is injective and its cokernel $P = B/A$ satisfies the hypotheses of the previous lemma. We therefore deduce that the exact sequence $0 \to A \to B \to P \to 0$ splits, so $B \cong A \oplus P$ as chain complexes. We can therefore define $l : B \to X$ by using f for the factor A and using any lift of $P \subset B \xrightarrow{g} Y$ for the factor P. ☐

Axiom (MC5)

It remains to prove (MC5). Rather than doing this directly, we will illustrate what is called « the small object argument », which allows one to produce factorizations with good lifting properties in many model categories. This argument will be formalized later.

We consider $\mathbb{N}$ as a category whose objects are the natural numbers and

$$\mathrm{Hom}_{\mathbb{N}}(m, n) = \begin{cases} * & \text{if } m \leq n, \\ \varnothing & \text{otherwise.} \end{cases}$$

A functor (diagram) $X : \mathbb{N} \to \mathsf{M}$ is nothing other than a sequence of objects and morphisms (note this is not a chain complex!) :

$$X_0 \to X_1 \to X_2 \to \dots$$

The natural maps $X_n \to \operatorname{colim} X$ induce, for any object $A \in \mathsf{M}$, a map :

$$\operatorname{colim}_{n \in \mathbb{N}} \mathrm{Hom}_{\mathsf{M}}(A, X_n) \to \mathrm{Hom}_{\mathsf{M}}(A, \operatorname{colim}_{n \in \mathbb{N}} X_n).$$

Definition 1.5.10. An object $A \in \mathsf{M}$ is $\mathbb{N}$-*small* (or sequentially small) if for every diagram $X : \mathbb{N} \to \mathsf{M}$, the above map is a bijection.

Example 1.5.11. A set is $\mathbb{N}$-small if and only if it is finite. An R-module is $\mathbb{N}$-small if and only if it is finitely presented. A chain complex $X \in \mathsf{Ch}_{\geq 0}(R)$ is $\mathbb{N}$-small if and only if $X_n = 0$ except for finitely many values, and each X_n is finitely presented.

Let $\mathcal{M} = \{f_i : A_i \to B_i\}_{i\in I}$ be a *set* of morphisms of M. Fix a map $p : X \to Y$. We seek to factor p as $X \to X' \to Y$ such that $X' \to Y$ has the RLP with respect to all morphisms of $\mathcal{M}$. Of course, one could take $X' = Y$, but we want X' to resemble X as closely as possible.

We define $S(\mathcal{M}, p)$ to be *the set* of all possible commutative diagrams :

$$S(\mathcal{M},p) = \left\{ (i \in I, g : A_i \to X, h : B_i \to Y) \;\middle|\; \begin{tikzcd} A_i \arrow[r, "g"] \arrow[d, "f_i"] & X \arrow[d, "p"] \\ B_i \arrow[r, "h"] & Y \end{tikzcd} \text{ commutes} \right\}$$

The gluing construction $G^1(\mathcal{M}, p)$ is then given as the pushout :

$$\begin{tikzcd} \bigsqcup_{(i,g,h)\in S(\mathcal{M},p)} A_i \arrow[r, "\bigsqcup g"] \arrow[d, "\bigsqcup f_i"] \arrow[dr, phantom, "\ulcorner", very near end] & X \arrow[d, "i_1"] \arrow[dr, "f"] & \\ \bigsqcup_{(i,g,h)\in S(\mathcal{M},p)} B_i \arrow[r] \arrow[rr, bend right, "\bigsqcup h"'] & G^1(\mathcal{M},p) \arrow[r, dashed, "p_1"] & Y \end{tikzcd}$$

The fact that all diagrams in $S(\mathcal{M}, p)$ commute shows that we have an induced map $p_1 : G^1(\mathcal{M}, p) \to Y$. We can then define by induction $G^k(\mathcal{M}, p) = G^1(\mathcal{M}, p_{k-1})$. We obtain :

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "p"] & G^1(\mathcal{M},p) \arrow[r, "i_2"] \arrow[d, "p_1"] & G^2(\mathcal{M},p) \arrow[r, "i_2"] \arrow[d, "p_2"] & \dots \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & \dots \end{tikzcd}$$

We set $G^\infty(\mathcal{M}, p)$ to be the colimit of the top row. It is equipped with maps $i_\infty : X \to G^\infty(\mathcal{M}, p)$ and $p_\infty : G^\infty(\mathcal{M}, p) \to Y$ such that $p_\infty i_\infty = p$.

Proposition 1.5.12. *Suppose that all the A are $\mathbb{N}$-small in $\mathcal{M}$. Then p_∞ has the right lifting property with respect to all morphisms of $\mathcal{M}$.*

Démonstration. We seek to solve the following lifting problem :

$$\begin{tikzcd} A_i \arrow[r, "g"] \arrow[d, "f_i"] & G^\infty(\mathcal{M},p) \arrow[d, "p_\infty"] \\ B_i \arrow[r, "h"] & Y \end{tikzcd}$$

Since A_i is $\mathbb{N}$-small, we can factor g through a finite-stage construction G^k, that is :

$$A_i \xrightarrow{g'} G^k(\mathcal{M}, p) \to G^\infty(\mathcal{M}, p). \tag{1.5.13}$$

We thus obtain a diagram :

$$\begin{tikzcd}
A_i \arrow[r, "g'"] \arrow[d, "f_i"] & G^k(\mathcal{M},p) \arrow[r, "i_{k+1}"] \arrow[d, "p_k"] & G^{k+1}(\mathcal{M},p) \arrow[r] \arrow[d, "p_{k+1}"] & G^\infty(\mathcal{M},p) \arrow[d, "p_\infty"] \\
B_i \arrow[r, "h"] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

The triple (i, g', h) indexes one of the morphisms in the colimit that defines $G^{k+1}(\mathcal{M},p)$ from $G^k(\mathcal{M},p)$. By construction, we therefore find a map $B_i \to G^{k+1}(\mathcal{M},p)$ making the diagram commute, which we can then post-compose with the morphism to $G^\infty(\mathcal{M},p)$. We thus obtain the desired lift. □

Everything above (generalized to an arbitrary cardinal) can be summarized as follows :

Theorem 1.5.14 (Small object argument)**.** *Let* M *be a cocomplete category and* $\mathcal{M} = \{f_i : A_i \to B_i\}$ *a set of morphisms of* M*. Suppose that all the* A_i *are* κ*-small for a fixed cardinal* κ*. Every morphism* $f : X \to Y$ *factors as* $X \xrightarrow{i_\infty} G^\infty(f,\mathcal{M}) \xrightarrow{p_\infty} Y$ *where* i_∞ *is a relative* $\mathcal{M}$*-cellular complex and* p_∞ *has the right lifting property with respect to all morphisms of* $\mathcal{M}$*.* □

We will now show that the cofibrations and acyclic cofibrations are « generated » by a small number of (acyclic) cofibrations which we will describe explicitly. We will produce the factorization of an arbitrary map by using the construction G^∞, either with respect to the generating cofibrations or with respect to the generating acyclic cofibrations.

Definition 1.5.15. For $n \geq 1$, we define the « n-disk » $D_n(R)$ and for $n \geq 0$ the « n-sphere » by :

$$D_n(R) : \dots \to 0 \to \underbrace{R}_{n} \xrightarrow{\mathrm{id}_R} \underbrace{R}_{n-1} \to 0 \to \dots$$

$$S_n(R) : \dots \to 0 \to \underbrace{R}_{n} \to 0 \to \dots$$

We also define $D_0(R) = R$ concentrated in degree 0, and $S_{-1}(R) = 0$. There is an obvious inclusion $i_n : S_{n-1}(R) \to D_n(R)$ (which is a cofibration). We also write $j_n : 0 \to D_n(R)$ for the inclusion (which is an acyclic cofibration).

Lemma 1.5.16 (Exercise)**.** *A morphism* $f : X \to Y$ *in* $\mathsf{Ch}_{\geq 0}(R)$ *is :*

- *an acyclic fibration* $\iff$ *it has the RLP with respect to all the* i_n *;*
- *a fibration* $\iff$ *it has the RLP with respect to all the* j_n*.*

Lemma 1.5.17. *The category* $\mathsf{Ch}_{\geq 0}(R)$ *satisfies (MC5).*

Démonstration. Let us first verify (MC5(i)), i.e. we want to factor a morphism $f : X \to Y$ as an acyclic cofibration followed by a fibration. Consider the class $\mathcal{J} = \{j_n : 0 \to D_n(R)\}$. The small object argument gives a factorization of f as $X \xrightarrow{i_\infty} G^\infty(\mathcal{J}, f) \xrightarrow{p_\infty} Y$. By the previous lemma, p_∞ is a fibration. At each step, the map $G^k(\mathcal{J}, f) \to G^{k+1}(\mathcal{J}, f)$ is obtained by gluing acyclic cofibrations (the j_n), hence it is an acyclic cofibration, and therefore so is the map to the colimit.

Let us now verify (MC5(ii)). We do the same thing, except that we use the class $\mathcal{I} = \{i_n : S_{n-1}(R) \to D_n(R)\}$ instead. $\square$

We have finished proving the theorem !

1.5.2 Definition and existence theorem

Cofibrantly generated model categories are very important in homotopy theory. (We have seen the chain complexes ; one can also identify the fact that Serre fibrations are defined in a similar way...) In general, proving axiom MC5 is difficult, but if one manages to find a set of generating (acyclic) cofibrations, then it is simpler, thanks to the small object argument. This will also allow us to compute homotopy (co)limits more easily later on, and many theorems apply only to these categories.

Let $\mathcal{M} = \{A_i \xrightarrow{f_i} B_i\}$ be a set of morphisms of M. (One should think of this as the class of generating cofibrations or the class of generating acyclic cofibrations.)

Definition 1.5.18. A *relative $\mathcal{M}$-cellular complex* is a morphism $X \to Y$ obtained as a colimit of the form $\operatorname{colim}_{\alpha<\kappa} X_\alpha$ where κ is an ordinal[4], $X_0 = X$, $X_{\alpha+1}$ is obtained from X_α as a pushout of the form $X_{\alpha+1} = X_\alpha \cup_{A_i} B_i$ for a morphism $g : A_i \to X_\alpha$, and if $\lambda \in \kappa$ is a limit ordinal then $X_\lambda = \operatorname{colim}_{\alpha<\lambda} X_\alpha$. We denote by $\mathcal{M}$-cell the class of such morphisms. An *$\mathcal{M}$-cellular complex* is an object Y such that $\varnothing \to Y \in \mathcal{M}$-cell.

Example 1.5.19. In Top, relative cellular complexes are obtained for $\mathcal{M} = \{\partial I^n \to I^n\}$.

Definition 1.5.20. A morphism $p : X \to Y$ is *$\mathcal{M}$-injective* if it has the RLP with respect to all morphisms of $\mathcal{M}$. We denote by $\mathcal{M}^\perp$ the class of $\mathcal{M}$-injective morphisms.

Definition 1.5.21. A morphism $i : A \to B$ is *$\mathcal{M}$-cofibrant* if it has the LLP with respect to all morphisms of $\mathcal{M}^\perp$. We denote by ${}^\perp(\mathcal{M}^\perp)$ the class of $\mathcal{M}$-cofibrant morphisms.

Definition 1.5.22. A model category $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ is *cofibrantly generated* if there exist two sets of morphisms $\mathcal{I}$ (« generating cofibrations ») and $\mathcal{J}$ (« generating acyclic cofibrations ») such that :

- the acyclic fibrations are the $\mathcal{I}$-injective morphisms ($\mathscr{F} \cap \mathscr{W} = \mathcal{I}^\perp$) ;
- the fibrations are the $\mathcal{J}$-injective morphisms ($\mathscr{F} = \mathcal{J}^\perp$) ;
- the sources of the morphisms of $\mathcal{I}$ are small with respect to the class $\mathcal{I}$-cell ;
- the sources of the morphisms of $\mathcal{J}$ are small with respect to the class $\mathcal{J}$-cell.

Remark 1.5.23. The first two points imply that $\mathscr{C} = {}^\perp(\mathcal{I}^\perp)$ and $\mathscr{C} \cap \mathscr{W} = {}^\perp(\mathcal{J}^\perp)$. The last two points are often verified by choosing $\mathcal{I}$ and $\mathcal{J}$ such that their sources are compact (hence small with respect to everything).

---

[4] in general one can restrict to $\kappa = \mathbb{N}$

Example 1.5.24. We have seen that the projective structure on $\mathsf{Ch}_{\geq 0}(R)$ is cofibrantly generated by $\mathcal{I} = \{S_n(R) \to D_n(R)\}$ and $\mathcal{J} = \{0 \to D_n(R)\}$.

Example 1.5.25. The Quillen structure on Top is cofibrantly generated by $\mathcal{I} = \{\partial I^n \hookrightarrow I^n\}$ and $\mathcal{J} = \{I^n \hookrightarrow I^{n+1}\}$. The Strøm structure is not.

Proposition 1.5.26. *Let* M *be a model category cofibrantly generated by* $(\mathcal{I}, \mathcal{J})$*. Then the cofibrations (resp. acyclic cofibrations) are the retracts of relative* $\mathcal{I}$*-cellular complexes (resp. relative* $\mathcal{J}$*-cellular complexes).*

Démonstration. Let us prove for example that $\mathscr{C} = \mathrm{Retract}(\mathcal{I}\text{-cell})$ (the other proof is similar). Since the morphisms of $\mathcal{I}$ are cofibrations, so are the retracts of relative $\mathcal{I}$-cellular complexes. Conversely, let $i : A \to B$ be a cofibration. Reusing the small object argument, i factors as $A \xrightarrow{i_\infty} G^\infty(\mathcal{I}, i) \xrightarrow{p_\infty} B$. By construction, i_∞ is a relative $\mathcal{I}$-cellular complex. We have also shown that p_∞ has the RLP with respect to $\mathcal{I}$, so it is an acyclic fibration, and therefore it has the RLP with respect to i. We can therefore find a lift in the following diagram :

$$\begin{tikzcd}
A \arrow[r, "i_\infty"] \arrow[d, "i"] & G^\infty(\mathcal{I}, i) \arrow[d, "p_\infty"] \\
B \arrow[r, equal] \arrow[ur, dashed, "f"] & B
\end{tikzcd}$$

We then find that i is a retract of i_∞ :

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "i"] & A \arrow[r, equal] \arrow[d, "i_\infty"] & A \arrow[d, "i"] \\
B \arrow[r, "f"] & B \arrow[r, "p_\infty"] & B
\end{tikzcd}$$

□

In theory, given a category M, one can recover the entire model category structure : the (acyclic) cofibrations are the retracts of relative $\mathcal{I}$-cellular ($\mathcal{J}$-cellular) complexes, the (acyclic) fibrations are the $\mathcal{J}$-injective ($\mathcal{I}$-injective) morphisms, and the weak equivalences are the morphisms obtained by composing an acyclic cofibration with an acyclic fibration. But the whole question is whether this actually yields a model category !

Theorem 1.5.27 (Existence of a cofibrantly generated structure)**.** *Let* M *be a complete and cocomplete category,* $\mathscr{W}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *three classes of morphisms. Then* M *is a model category with* $\mathscr{W}$ *as weak equivalences and* $\mathcal{I}$*,* $\mathcal{J}$ *as generating (acyclic) cofibrations if and only if :*

1. *$\mathscr{W}$ satisfies 2-out-of-3 and is closed under retracts ;*
2. *the sources of the morphisms of $\mathcal{I}$ are small with respect to $\mathcal{I}$-cell ;*
3. *the sources of the morphisms of $\mathcal{J}$ are small with respect to $\mathcal{J}$-cell ;*
4. *$\mathcal{J}$-cell $\subset \mathscr{W} \cap {}^\perp(\mathcal{I}^\perp)$;*
5. *$\mathcal{I}^\perp \subset \mathscr{W} \cap \mathcal{J}^\perp$;*
6. *either ${}^\perp(\mathcal{I}^\perp) \cap \mathscr{W} \subset {}^\perp(\mathcal{J}^\perp)$, or $\mathcal{J}^\perp \cap \mathscr{W} \subset \mathcal{I}^\perp$.*

Démonstration. The conditions are clearly necessary. Let us show that they are sufficient. Recall that we take $\mathscr{F} = \mathcal{J}^{\perp}$ and $\mathscr{C} = {}^{\perp}(\mathcal{I}^{\perp})$. (MC1) is satisfied by hypothesis. (MC2) and the stability of $\mathscr{W}$ under retracts follow from (1). The stability under retracts of $\mathscr{C}$ and $\mathscr{F}$ comes from the fact that they are defined by lifting properties, so we have (MC3).

Conditions (4) and (5) show that the morphisms of $\mathcal{I}^{\perp}$ are acyclic fibrations, and those of $\mathcal{J}$-cell are acyclic cofibrations. The small object argument, which works thanks to (3), provides a factorization of every morphism $f : A \to X$ as $A \xrightarrow{i_\infty} G^\infty(\mathcal{M}, f) \xrightarrow{p_\infty} X$, where $\mathcal{M} = \mathcal{I}$ (resp. $\mathcal{J}$). The morphism i_∞ is a $\mathcal{I}$-cellular (resp. $\mathcal{J}$-cellular) complex. The same argument as in the previous proposition shows that it is indeed a cofibration (resp. acyclic cofibration). The morphism p_∞ is $\mathcal{I}$-injective (resp. $\mathcal{J}$-injective), so it is an acyclic fibration by (5) (resp. a fibration by definition of $\mathscr{F}$). We therefore have (MC5).

It remains to verify (MC4), the lifting axiom. We will use (6) ; suppose that $\mathscr{C} \cap \mathscr{W} := {}^{\perp}(\mathcal{I}^{\perp}) \cap \mathscr{W} \subset {}^{\perp}(\mathcal{J}^{\perp})$ (the other case is dual). Then the acyclic cofibrations indeed have the LLP with respect to the fibrations, which is half of the axiom. Now suppose that $f : X \to Y \in \mathcal{J}^{\perp} \cap \mathscr{W}$ is an acyclic fibration ; let us show that it has the RLP with respect to cofibrations. By definition of $\mathscr{C} = {}^{\perp}(\mathcal{I}^{\perp})$, it suffices to verify that f has the RLP with respect to elements of $\mathcal{I}$. By the small object argument, we factor $f = p_\infty i_\infty$ where $i_\infty \in \mathcal{J}\text{-cell} \subset \mathscr{W} \cap {}^{\perp}(\mathcal{I}^{\perp}) = \mathscr{W} \cap \mathscr{C}$ (4), and $p_\infty \in \mathcal{I}^{\perp} \subset \mathscr{W}$ (5). We have just shown that f has the RLP with respect to $i_\infty \in {}^{\perp}(\mathcal{J}^{\perp})$. We therefore have a lift :

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "i_\infty"] & X \arrow[d, "f"] \\
G^\infty(\mathcal{J}, f) \arrow[r, "p_\infty"] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

and therefore f is a retract of p_∞ :

$$\begin{tikzcd}
X \arrow[r, "i_\infty"] \arrow[d, "f"] & G^\infty(\mathcal{J}, f) \arrow[r, "h"] \arrow[d, "p_\infty"] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Since $p_\infty \in \mathcal{I}^{\perp} \subset \mathscr{F}$ by (4), we deduce by (MC3) that $f \in \mathscr{F}$. □

Example 1.5.28 (Exercise)**.** Let $\mathsf{M} = \mathsf{Top}$ and consider the Quillen structure (see Example 1.3.12). Take as generating cofibrations the class of inclusions $\{S^{n-1} \to D^n\}_{n \geq 0}$ and as generating acyclic cofibrations the inclusions $\{D^n \to D^n \times [0,1]\}_{n \geq 0}$. Show that the hypotheses of the theorem are satisfied and that the classes of fibrations, cofibrations and weak equivalences are those of Example 1.3.12. (One may refer to [Hov99, Sections 2.1 and 2.4] as well as a similar proof for simplicial sets in Chapter 2).

1.6 Quillen Adjunctions

We now know how to define the homotopy category of a model category and perform computations in it, especially when the category is cofibrantly generated. Now, if we have a functor between two model categories, under what conditions do we get an induced functor between their homotopy categories ?

Proposition 1.6.1. *Let $F : \mathsf{M} \leftrightarrows \mathsf{D} : G$ be an adjunction between two model categories. The following properties are equivalent :*

- *F preserves cofibrations and acyclic cofibrations ;*
- *G preserves fibrations and acyclic fibrations ;*
- *F preserves cofibrations and G preserves fibrations ;*
- *F preserves acyclic cofibrations and G preserves acyclic fibrations.*

Definition 1.6.2. A *Quillen adjunction* is an adjunction satisfying these properties.

Démonstration. It suffices to play a little with the lifting properties. Let us show for example that if F preserves cofibrations then G preserves acyclic fibrations. (All other statements are obtained similarly.) Let $p : X \to {\sim}Y$ be an acyclic fibration. Let us show that $G(p) : X \to Y$ is an acyclic fibration. It suffices to show that $G(p)$ has the RLP with respect to cofibrations. We are given a diagram and we seek a lift :

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "i"] & G(X) \arrow[d, "G(p)"] \\
B \arrow[r] \arrow[ur, dashed] & G(Y)
\end{tikzcd}$$

By adjunction, the lift exists if and only if a lift exists in the following diagram :

$$\begin{tikzcd}
F(A) \arrow[r] \arrow[d, "F(i)"] & X \arrow[d, two heads, "p", "\sim"'] \\
F(B) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Now F preserves cofibrations, so $F(i)$ is a cofibration in D. Since p is an acyclic fibration, we deduce that the lift exists. ☐

Example 1.6.3. Let $f : R \to S$ be a morphism of commutative rings. Then we have an adjunction $f_! : \mathsf{Ch}_{\geq 0}(R) \leftrightarrows \mathsf{Ch}_{\geq 0}(S) : f^*$. One readily verifies that this is a Quillen adjunction for the projective structures.

We recall that we denote by $\lambda : \mathsf{M} \to \mathrm{Ho}(\mathsf{M})$ the localization functor.

Definition 1.6.4. Let M be a model category and let H be an arbitrary category. [5] Let $F : \mathsf{M} \to \mathsf{H}$ be a functor.

---

[5] We will think of it as $\mathrm{Ho}(\mathsf{D})$.

1. A *left derived functor* of F is a functor $\mathbb{L}F : \mathrm{Ho}(\mathsf{M}) \to \mathsf{H}$ and a natural transformation $\alpha : \mathbb{L}F \circ \lambda \Rightarrow F$ satisfying the following universal property : for every pair $(G : \mathrm{Ho}(\mathsf{M}) \to \mathsf{H}, \beta : G \circ \lambda \Rightarrow F)$, there exists a unique natural transformation $\theta : G \to \mathbb{L}F$ such that β is the composite

$$\begin{tikzcd}
G \circ \lambda \arrow[rr, Rightarrow, "\forall\beta"] \arrow[dr, Rightarrow, dashed, "(\exists!\theta)\circ\lambda"'] & & F \\
& \mathbb{L}F \circ \lambda \arrow[ur, Rightarrow, "\alpha"'] &
\end{tikzcd}$$

2. A *right derived functor* of F is a functor $\mathbb{R}F : \mathrm{Ho}(\mathsf{M}) \to \mathsf{H}$ and a natural transformation $\varepsilon : F \Rightarrow \mathbb{R}F \circ \lambda$ satisfying the following universal property : for every pair $(G : \mathrm{Ho}(\mathsf{M}) \to \mathsf{H}, \varphi : F \Rightarrow G \circ \lambda)$, there exists a unique natural transformation $\theta : \mathbb{R}F \to G$ such that φ is the composite

$$\begin{tikzcd}
F \arrow[rr, Rightarrow, "\forall\varphi"] \arrow[dr, Rightarrow, "\varepsilon"'] & & G \circ \lambda \\
& \mathbb{R}F \circ \lambda \arrow[ur, Rightarrow, dashed, "(\exists!\theta)\circ\lambda"'] &
\end{tikzcd}$$

If a derived functor (right or left) exists, it is unique up to isomorphism, as usual with universal properties. A left derived functor is, in a certain sense, the best possible left approximation (on the source side) of F as a functor that factors through λ. In technical terms, it is a right Kan extension (and not a left one !) along λ.

Proposition 1.6.5. *Let M be a model category, H an arbitrary category, and $F : \mathsf{M} \to \mathsf{H}$ a functor.*

1. *If F sends acyclic cofibrations between cofibrant objects to isomorphisms, then $\mathbb{L}F$ exists.*
2. *If F sends acyclic fibrations between fibrant objects to isomorphisms, then $\mathbb{R}F$ exists.*

The proof relies on Brown's Lemma :

Lemma 1.6.6 (Brown's Lemma [Bro73])**.** *Let $F : \mathsf{M} \to \mathsf{D}$ be a functor between two model categories.*

1. *If F sends acyclic cofibrations between cofibrant objects to weak equivalences, then F sends weak equivalences between cofibrant objects to weak equivalences.*
2. *If F sends acyclic fibrations between fibrant objects to weak equivalences, then F sends weak equivalences between fibrant objects to weak equivalences.*

Démonstration. Let $f : A \xrightarrow{\sim} B$ be a weak equivalence between cofibrant objects. We have a pushout :

$$\begin{tikzcd}
\varnothing \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\
B \arrow[r] & A \sqcup B
\end{tikzcd}$$

Therefore the inclusions $i_A : A \to A \sqcup B$ and $i_B : B \to A \sqcup B$ are cofibrations.

The morphism f induces a map $(\mathrm{id}, f) : A \sqcup B \to B$. Let us factor it as a cofibration followed by an acyclic fibration. We obtain a commutative diagram :

$$\begin{tikzcd}
 & B & \\
 & X \arrow[u, two heads, "\sim"', "p"] & \\
A \arrow[uur, bend left, "f", "\sim"'] \arrow[r, hook, "i_A"] & A \sqcup B \arrow[u, hook, "j"] & B \arrow[l, hook', "i_B"'] \arrow[uul, equal, bend right]
\end{tikzcd}$$

- Since $p \circ (j \circ i_B) = \mathrm{id}_B \in \mathscr{W}$ and $p \in \mathscr{W}$, we deduce that $j \circ i_B \in \mathscr{W}$. Moreover $j \in \mathscr{C}$ and $i_B \in \mathscr{C}$ so $j \circ i_B \in \mathscr{C}$. Furthermore, its source (B) and target (X) are cofibrant. By the hypothesis on F, we have that $F(j \circ i_B)$ is a weak equivalence.
- In the same way, we deduce that $F(j \circ i_A)$ is a weak equivalence.
- Since $F(p) \circ F(j \circ i_B) = F(\mathrm{id}_B)$ is an isomorphism and $F(j \circ i_B)$ is an isomorphism, we have that $F(p)$ is a weak equivalence.
- Finally, $F(f) = F(p) \circ F(j \circ i_A)$ is the composite of two weak equivalences, hence it is a weak equivalence. ☐

Corollary 1.6.7. *A left (resp. right) Quillen adjoint sends weak equivalences between cofibrant (resp. fibrant) objects to weak equivalences.*

Proof of Proposition 1.6.5. Suppose that F sends acyclic cofibrations between cofibrant objects to isomorphisms (the other case is dual). By Brown's Lemma, we deduce that F sends weak equivalences between cofibrant objects to isomorphisms.

We recall that $Q(X) \xrightarrow[\sim]{p_X} X$ is the functorial cofibrant resolution. We set $\mathbb{L}F(X) \coloneqq F(Q(X))$. The natural arrow $Q(X) \to X$ defines a natural transformation $\alpha : \mathbb{L}F \Rightarrow F$. Since F sends weak equivalences between cofibrant objects to isomorphisms, $\mathbb{L}F$ as defined factors through $\mathrm{Ho}(\mathsf{M})$.

Let us show that the pair $(\mathbb{L}F, \alpha)$ satisfies the universal property. Let $(G : \mathrm{Ho}(\mathsf{M}) \to \mathsf{H}, \beta : G \circ \lambda \Rightarrow F)$ be as in the definition. We want to construct $\theta_X : G(X) \to \mathbb{L}F(X) = F(Q(X))$ that is natural in X and such that $\beta_X = \alpha_X \circ \theta_{\lambda(X)} : G(\lambda(X)) \to \mathbb{L}F(\lambda(X)) \to F(X)$. We have that $Q(X) \xrightarrow{\sim} X$ becomes an isomorphism in $\mathrm{Ho}(\mathsf{M})$, so its image under G is an isomorphism in H. We can therefore find a lift :

$$\begin{tikzcd}
G(Q(X)) \arrow[r, "\beta_{Q(X)}"] \arrow[d, "\cong"', "G(p_X)"] & \mathbb{L}F(X) = F(Q(X)) \arrow[d, "\alpha_X"] \\
G(X) \arrow[r, "\beta_X"] \arrow[ur, dashed, "\theta_X"] & F(X)
\end{tikzcd} \tag{1.6.8}$$

We can therefore define a natural transformation θ by $\theta_X \coloneqq \beta_{Q(X)} \circ G(p_X)^{-1}$, and it indeed satisfies $\beta_X = \alpha_X \circ \theta_{\lambda(X)}$.

Let us now show that θ is unique. Let θ' be another natural transformation satisfying the hypothesis. If we draw diagram (1.6.8) replacing X by QX, the right vertical arrow (α_{QX}) is an isomorphism by Corollary 1.6.7. We deduce that $\theta'_{QX} = \alpha^{-1}_{QX}\beta_{QX}$. Now by naturality of θ', we must have $F(Q(p_X))\theta'_{QX} = \theta'_X G(p_X)$. But since F sends weak equivalences between cofibrant objects to isomorphisms, one verifies that $F(Q(p_X)) = F(p_{QX}) =: \alpha_{QX}$ by considering the cube formed by all ways of passing from $QQQX$ to X. Therefore necessarily :

$$\theta'_X = \alpha_{QX}\theta_{QX}G(p_X)^{-1} = \alpha_{QX}\alpha^{-1}_{QX}\beta_{QX}G(p_X)^{-1} = \beta_{QX}G(p_X)^{-1} = \theta_X. \tag{1.6.9}$$

□

Definition 1.6.10. Let $F : \mathsf{M} \to \mathsf{D}$ be a functor between two model categories.

1. A *total left derived functor* of F is a left derived functor of $\lambda F : \mathsf{M} \to \mathsf{D} \to \mathrm{Ho}(\mathsf{D})$.
2. A *total right derived functor* of F is a right derived functor of $\lambda F : \mathsf{M} \to \mathsf{D} \to \mathrm{Ho}(\mathsf{D})$.

We will still write (by abuse of notation...) $\mathbb{L}F$ and $\mathbb{R}F$ for the total derived functors.

Proposition 1.6.11 (Exercise)**.** *Suppose that $F : \mathsf{M} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ are two functors such that F, G, and $G \circ F$ admit left derived functors. Then there exists a canonical natural transformation $(\mathbb{L}G) \circ (\mathbb{L}F) \Rightarrow \mathbb{L}(G \circ F)$.*

Theorem 1.6.12 (Adjunction between homotopy categories)**.** *Let $F : \mathsf{M} \leftrightarrows \mathsf{D} : G$ be a Quillen adjunction between two model categories. Then the total derived functors $\mathbb{L}F$, $\mathbb{R}G$ exist and form an adjunction :*

$$\mathbb{L}F : \mathrm{Ho}(\mathsf{M}) \leftrightarrows \mathrm{Ho}(\mathsf{D}) : \mathbb{R}G.$$

Démonstration. The fact that they exist follows from the proposition we just proved and from the definition of Quillen adjunctions. In the proof, we saw that

$$\mathbb{L}F(A) = F(Q(A)), \qquad \mathbb{R}G(X) = G(R(X)),$$

where $Q(-)$ is the functorial cofibrant resolution and $R(-)$ the functorial fibrant resolution. We seek a natural bijection :

$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{D})}(\mathbb{L}F(A), X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(A, \mathbb{R}G(X)).$$

By the description of morphisms in the homotopy category, this amounts to seeking a natural bijection :

$$\mathrm{Hom}_{\mathsf{D}}(F(Q(A)), R(X))/\!\sim \; \cong \mathrm{Hom}_{\mathsf{M}}(Q(A), G(R(X)))/\!\sim.$$

Since $F \dashv G$, we already have a bijection before passing to the quotient. We need to show that it passes to the quotient. Suppose that $f, g : F(Q(A)) \to R(X)$ are homotopic. Let $R(Y) \xrightarrow{\sim} P \to R(Y) \times R(Y)$ be a path object and $H : F(Q(A)) \to P$ a homotopy. Since G preserves fibrations and limits, we have that $G(R(Y)) \to G(P) \to R(Y) \times R(Y)$. Moreover, $R(Y)$ and P are fibrant, so G sends the weak equivalence between them to a weak equivalence, hence $G(P)$ is a path object for $G(R(Y))$. The morphism H is adjoint to $H' : Q(A) \to R(P)$ and one verifies that this is a (right) homotopy between the adjoints of f and g. The converse is dual. □

Proposition 1.6.13 (Exercise)**.** *Let* $F : \mathsf{M} \to \mathsf{D}$ *and* $G : \mathsf{D} \to \mathsf{E}$ *be two left Quillen adjoints between model categories. Then the natural transformation* $\mathbb{L}G \circ \mathbb{L}F \Rightarrow \mathbb{L}(G \circ F)$ *is an isomorphism.*

Theorem 1.6.14 (Equivalences between homotopy categories)**.** *Let* $F : \mathsf{M} \leftrightarrows \mathsf{D} : G$ *be a Quillen adjunction. The following statements are equivalent :*

1. *The induced adjunction* $\mathbb{L}F : \mathrm{Ho}(\mathsf{M}) \leftrightarrows \mathrm{Ho}(\mathsf{D}) : \mathbb{R}G$ *is an equivalence of categories.*
2. *For every cofibrant object* $A \in \mathsf{M}$ *and every fibrant object* $X \in \mathsf{D}$*, a morphism* $F(A) \to Y$ *is a weak equivalence if and only if its adjoint* $A \to G(Y)$ *is a weak equivalence.*
3. *For every cofibrant object* $A \in \mathsf{M}$ *and every fibrant object* $X \in \mathsf{D}$*, the two arrows :*

$$A \xrightarrow{\eta} G(F(A)) \to G(R(F(A)))$$

$$F(Q(G(Y))) \to F(G(Y)) \xrightarrow{\varepsilon} Y$$

are weak equivalences.

Démonstration. The proof proceeds in several steps.

2. $\implies$ 3. The morphism $A \to GRF(A)$ is adjoint to $F(A) \to RF(A)$, which is a fibrant resolution, hence a weak equivalence. By 2, we deduce that $A \to GRF(A)$ is a weak equivalence. The second part is dual.

3. $\implies$ 2. Let A be cofibrant, X fibrant, and $f : F(A) \to Y$ a morphism. We denote by $\bar{f} : A \xrightarrow{\eta} GF(A) \xrightarrow{G(f)} G(Y)$ its adjoint. We have a commutative diagram :

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[d, equal] & GF(A) \arrow[r, "G(f)"] \arrow[d] & G(Y) \arrow[d, two heads, "\sim"] \\
A \arrow[r, "\sim"] & GRF(A) \arrow[r, "\sim"] & GR(Y)
\end{tikzcd}$$

where $A \to GRF(A)$ is a weak equivalence by hypothesis, $G(Y) \to GR(Y)$ is a weak equivalence since G preserves weak equivalences between fibrant objects, and $GRF(A) \to GR(Y)$ is a weak equivalence for the same reason. We deduce by 2-out-of-3 that $\bar{f}$ (the top row) is a weak equivalence. The other case is dual.

2. $\implies$ 1. The unit of the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ is

$$\tilde{\eta} \in \mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(A, \mathbb{R}G(\mathbb{L}F(A))) = \mathrm{Hom}_{\mathsf{M}}(QA, GRFQ(A)). \tag{1.6.15}$$

It is adjoint to $FQ(A) \to RFQ(A)$, which is a weak equivalence whose source is cofibrant, so its adjoint is indeed a weak equivalence (i.e. an isomorphism in $\mathrm{Ho}(\mathsf{M})$). Dually, the counit of the adjunction is an isomorphism ; we deduce that the adjunction between homotopy categories is an equivalence of categories.

1. $\implies$ 3. Suppose that A is cofibrant. We have a commutative diagram :

$$\begin{tikzcd}
QA \arrow[r, "\sim"] \arrow[d, two heads, "\sim"] & GRFQ(A) \arrow[d] \\
A \arrow[r] & GRF(A)
\end{tikzcd}$$

Since the unit of the adjunction is an isomorphism, $QA \to GRFQ(A)$ is a weak equivalence (cf. above). Since $QA \to A$ is a weak equivalence between cofibrant objects, $F(QA) \to F(A)$ is a weak equivalence, so $RFQ(A) \to RF(A)$ is a weak equivalence between fibrant objects, so $GRFQ(A) \to GRF(A)$ is a weak equivalence. We deduce that the bottom arrow is a weak equivalence, which is what we wanted to prove. The other case is dual. □

Definition 1.6.16. A Quillen adjunction satisfying the hypotheses of the previous theorem is called a *Quillen equivalence.*

We have a very useful criterion for verifying that a Quillen adjunction is a Quillen equivalence :

Proposition 1.6.17. *Let $F : \mathsf{M} \leftrightarrows \mathsf{D} : G$ be a Quillen adjunction. The following statements are equivalent :*

1. *The adjunction is a Quillen equivalence.*
2. *The functor F reflects weak equivalences between cofibrant objects*[6] *and for every fibrant object $X \in \mathsf{D}$, the morphism $FQG(X) \to X$ is a weak equivalence.*
3. *The functor G reflects weak equivalences between fibrant objects and for every cofibrant object $A \in \mathsf{M}$, the morphism $A \to GRF(A)$ is a weak equivalence.*

Démonstration. Exercise. □

Example 1.6.18. There is a Quillen equivalence between the projective structure on $\mathsf{Ch}(R)$ and the injective structure on $\mathsf{Ch}(R)$. This equivalence is simply given by the identity (cf. Hovey [Hov99, Section 2.3])

Example 1.6.19. The identity $\mathrm{id} : \mathsf{Ch}_{\geq 0}(R) \leftrightarrows \mathsf{Ch}_{\geq 0}(R) : \mathrm{id}$ is a Quillen adjunction between the projective structure and the Strøm structure. Is it a Quillen equivalence ?

1.7 Homotopy limits and colimits

Let I be a small category (i.e. one has a set of objects and a set of morphisms). A "diagram indexed by I" is simply a functor $X : I \to \mathsf{M}$, which we will write slightly differently to emphasize that it is a diagram : the value at $i \in I$ is denoted X_i and the image of $\alpha : i \to j$ is denoted $\alpha_* : X_i \to X_j$. A morphism between diagrams is a natural transformation. We denote by M^I the category of diagrams.

We have a functor $\mathrm{cst} : \mathsf{M} \to \mathsf{M}^I$ such that $\mathrm{cst}(A)_i = A$ and $\alpha_* = \mathrm{id}_A$ for all i, α.

---

[6] This means that if $f : A \to B$ is a morphism between cofibrant objects and $F(f)$ is a weak equivalence, then f is a weak equivalence.

Proposition 1.7.1 (Exercise)**.** *If* M *is complete then* cst *has a right adjoint* $\lim_I : \mathsf{M}^I \to \mathsf{M}$*. If* M *is cocomplete then* cst *has a left adjoint* $\operatorname{colim}_I : \mathsf{M}^I \to \mathsf{M}$*.*

If M has a notion of weak equivalences $\mathscr{W}$, then so does M^I by considering $\mathscr{W}^I$ (i.e. weak equivalences are defined objectwise).

Definition 1.7.2. Let $(\mathsf{M}, \mathscr{W})$ be a category with weak equivalences and I a small category. A *homotopy colimit* is a total left derived functor :

$$\operatorname{hocolim}_I \coloneqq \mathbb{L}\operatorname{colim}_I : \operatorname{Ho}(\mathsf{M}^I) \to \operatorname{Ho}(\mathsf{M}).$$

A *homotopy limit* is a total right derived functor :

$$\operatorname{holim}_I \coloneqq \mathbb{R}\lim_I : \operatorname{Ho}(\mathsf{M}^I) \to \operatorname{Ho}(\mathsf{M}).$$

Remark 1.7.3. Warning : $\operatorname{hocolim}_I$ is *not* a colimit in $\operatorname{Ho}(\mathsf{M})$! The difference essentially comes from the fact that $\operatorname{Ho}(\mathsf{M})^I \neq \operatorname{Ho}(\mathsf{M}^I)$. For example, if $I = \{a \leftarrow c \to b\}$, $\mathsf{M} = \mathsf{Top}$, and $X \in \mathsf{Top}^I$ is defined by $X_a = X_b = *$ and $X_c = \{0, 1\}$ (with the only possible maps), then $\operatorname{hocolim}_I X = S^1$ whereas the colimit in $\operatorname{Ho}(\mathsf{Top})^I$ is contractible. One can also find examples where the (co)limit in $\operatorname{Ho}(\mathsf{M})$ does not even exist, e.g. the pushout of $* \leftarrow S^1 \xrightarrow{z \mapsto z^2} S^1$ in $\operatorname{Ho}(\mathsf{Top}_*)$ (exercise).

We will now give conditions for holim and hocolim to exist. What follows is a direct consequence of what we know about derived functors of Quillen adjunctions.

Lemma 1.7.4. *Let* M *be a model category and* I *a small category.*

1. *If* M^I *admits a model structure whose weak equivalences contain* $\mathscr{W}^I$ *and such that* $\mathrm{cst} : \mathsf{M} \to \mathsf{M}^I$ *is a right Quillen adjoint, then* $\operatorname{hocolim}_I$ *exists.*
2. *If* M^I *admits a model structure whose weak equivalences contain* $\mathscr{W}^I$ *and such that* $\mathrm{cst} : \mathsf{M} \to \mathsf{M}^I$ *is a left Quillen adjoint, then* holim_I *exists.*

Definition 1.7.5. Let M be a model category and I a small category.

1. The *projective structure* on M^I has $\mathscr{W}^I$ as weak equivalences and $\mathscr{F}^I$ as fibrations. The cofibrations are defined by the lifting property.
2. The *injective structure* on M^I has $\mathscr{W}^I$ as weak equivalences and $\mathscr{C}^I$ as cofibrations. The fibrations are defined by the lifting property.

Lemma 1.7.6. *If the projective (resp. injective) structure defines a model category structure on* M^I*, then* $\operatorname{hocolim}_I$ *(resp.* holim_I*) exists.*

Remark 1.7.7. In this case, to compute $\operatorname{hocolim}_I X$, it suffices to replace X by a cofibrant object and compute its limit. Similarly for holim.

Definition 1.7.8. A category I is *very small* if it has a finite number of objects and if there exists $N > 0$ such that every sequence of morphisms $A_0 \xrightarrow{f_0} A_1 \dots A_n$ has at most N arrows that are not identities.

Example 1.7.9. A finite partially ordered set defines a very small category.

Theorem 1.7.10 (Structure on diagram categories)**.** *If I is very small then the projective and injective structures define model category structures on M^I.*

Démonstration. The complete proof is left as an exercise and is a special case of a much more general theory, that of Reedy categories (of which very small categories are examples). The proof essentially proceeds by induction. Indeed, a very small category induces a preorder on the objects of I, by setting $i \leq j \iff \mathrm{Hom}_I(i,j) \neq \varnothing$.

To show that the projective structure is a model category, we proceed as follows, by induction on the size of I. The axioms (MC1), (MC2) and (MC3) are clear. Since the poset $ob(I)$ is finite, it has at least one minimal element i_0. We let I' be the full subcategory on all other objects and we have a restriction functor $U : \mathsf{M}^I \to \mathsf{M}^{I'}$. By induction, $\mathsf{M}^{I'}$ is a model category. Then $f \in \mathrm{Hom}_{\mathsf{M}^I}(A,X)$ is a cofibration if and only if the following two conditions hold :

- The morphism $f_{i_0} : A_{i_0} \to X_{i_0}$ is a cofibration in M ;
- Let $\{j_1,\dots,j_k\}$ be the immediate successors of i_0 in the poset I. For $1 \leq l \leq k$, we define $\partial_l(f)$ as the pushout :

$$\begin{tikzcd}
A_{i_0} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner", very near end] & A_{j_l} \arrow[d, dashed] \\
X_{i_0} \arrow[r, dashed] & \partial_l(f)
\end{tikzcd}$$

  One can then define a diagram $A' \in \mathsf{M}^{I'}$ by $A'_j = A_j$ if $j \notin \{i_0, j_1, \dots, j_k\}$ and $A'_{j_l} = \partial_l(f)$. We have a morphism $A' \to U(X)$ in $\mathsf{M}^{I'}$. The second condition is that this morphism be a cofibration (defined inductively : the cardinality of I' is strictly less than the cardinality of I).

To prove the lifting or factorization axioms, one first handles i_0, and then finds liftings/factorizations for the morphism $X' \to U(Y)$ compatible with what happens at i_0. □

This allows us, for example, to define homotopy pushouts and homotopy pullbacks. However, not all index categories are very small ; for example $\mathbb{N}$ is not. We will introduce a condition on M which allows one to put a model category structure on M^I for every small category I.

Definition 1.7.11. A category I is *filtered* if every finite diagram has a cocone[7] : for every functor $F : \mathsf{D} \to I$ where D is a finite category, there exists an object $i \in I$ and a natural transformation $F \Rightarrow \mathrm{cst}_i$ between F and the constant functor equal to i. A colimit is *filtered* if it is indexed by a filtered category.

Example 1.7.12. A partially ordered set is a filtered category if and only if it is directed (i.e. every finite subset has an upper bound). Any category admitting a terminal object is filtered. A discrete category with at least two objects is not filtered.

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[7] One can think of a cocone as an "upper bound".

Definition 1.7.13. An object $A \in \mathsf{M}$ is (κ-)*compact* if $\mathrm{Hom}_C(A, -)$ commutes with (κ-)filtered colimits.

Example 1.7.14. The compact objects of Set are the finite sets.

Definition 1.7.15. A cocomplete category M is *presentable* if there exists a set S of compact objects such that every object of M is a filtered colimit of objects of S.

Remark 1.7.16. Some authors use the terminology "locally presentable" for this notion. Indeed, there exists a general notion of a presentable object in a category, and one might be tempted to think that a "presentable category" is a "presentable object in Cat", which is not the case : it is the objects of the category that are presentable, not the category itself.

Example 1.7.17. The category Set of sets is presentable, generated by finite sets (every set is indeed the filtered colimit of its finite subsets). The category of R-modules is presentable, generated by free modules of finite rank, etc.

Definition 1.7.18. A model category M is *combinatorial* if it is cofibrantly generated and presentable.

Example 1.7.19. The category $\mathsf{Ch}_{\geq 0}(R)$ is combinatorial, generated by the $D_n(R)$ and the $S_n(R)$. The category Top is not combinatorial, but it is Quillen equivalent to the category of simplicial sets (see Chapter 2) which is.

Theorem 1.7.20 (Projective structure on a diagram category)**.** *Let* M *be a cofibrantly generated model category and* I *a small category. Then the projective structure exists on* M^I *(hence homotopy colimits exist). If moreover* M *is combinatorial, then the injective structure on* M^I *exists (hence homotopy limits exist).*

Démonstration. Left as an exercise, the idea being to show that M^I is cofibrantly generated using the theorems we have already seen. The injective case is quite involved. □

Let us now give some examples of homotopy colimits. We introduce a technical notion :

Definition 1.7.21. A model category M is said to be *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. It is said to be *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. We say it is *proper* if it is both left and right proper.

Example 1.7.22 ([Hir03, Corollary 13.1.3])**.** If every object of M is cofibrant (resp. fibrant), then M is left proper (resp. right proper).

Example 1.7.23 (Exercise)**.** The category Top with the Quillen structure and the category $\mathsf{Ch}_{\geq 0}(R)$ with the projective structure are proper.

Example 1.7.24. Let $I = \{j \leftarrow i \rightarrow j'\}$, which is very small. A diagram $X \in \mathsf{M}^I$ is the data of three objects and two morphisms $B \xleftarrow{f} A \xrightarrow{g} C$. Then $\operatorname{colim}_I X$ is the pushout of f and g, $B \cup_A C$. This notion is clearly not invariant under homotopy : for example in Top, $* \cup_{\{0,1\}} * = *$ whereas $[0,1] \cup_{\{0,1\}} [0,1] = S^1$. To compute the homotopy pushout, $\operatorname{hocolim}_I X = B \cup_A^h C$, we must replace our diagram by a cofibrant diagram. One can show that a cofibrant replacement of the diagram is given by $Q(B) \hookleftarrow Q(A) \hookrightarrow Q(C)$ where the $Q(-)$ are cofibrant resolutions. One then simply computes the pushout of this new diagram to obtain the homotopy pushout.

Given a diagram $B \leftarrow A \rightarrow C$, if the category M is left proper one can replace $A \rightarrow C$ by a cofibration $A \hookrightarrow C' \xrightarrow{\sim} C$ and compute $B \cup_A^h C = B \cup_A C'$.

In Top, a standard way to replace an arrow by a cofibration is to consider its mapping cylinder, $\operatorname{Cyl}(g) = A \times [0,1] \sqcup C/((a,1) \sim g(c))$. One then finds $B \cup_A^h C = B \cup_A^h \operatorname{Cyl}(C)$.

In $\mathsf{Ch}_{\geq 0}(R)$ with the projective structure, to replace $A \xrightarrow{g} C$ by a cofibration, one can consider a formally similar construction :

$$\operatorname{Cyl}(g) = (A_n \oplus C_n \oplus A_{n-1}, d(a,c,a') = (da + a', dc - g(a'), da')).$$

Then the homotopy pushout is given by :

$$B \oplus_A^h C = (B_n \oplus C_n \oplus A_{n-1}, d(b,c,a) = (db + f(a), dc - g(a), da)).$$

The homotopy pullback is the dual notion. Exercise : compute the homotopy pullback in Top and $\mathsf{Ch}_{\geq 0}(R)$.

2 Simplicial sets

2.1 Definition and properties

A large part of this chapter was covered in the course *Homotopy I* ⟲.

Definition 2.1.1. The *simplex category* Δ has as objects the totally ordered finite sets $[n] = \{0 < 1 < \cdots < n\}$ and as morphisms the order-preserving maps.

We will denote a morphism $f \in \mathrm{Hom}_\Delta([m],[n])$ as a sequence :

$$f = f(0) \to f(1) \to \cdots \to f(m).$$

For example, the identity of $[n]$ is $0 \to 1 \to \cdots \to n$.

Every morphism of Δ factors uniquely as an order-preserving surjection followed by an order-preserving injection. Furthermore,

- the order-preserving injections are generated by the $\partial^i : [n-1] \to [n]$ (for $0 \leq i \leq n$) defined by
$$\partial^i = 0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n$$
- the order-preserving surjections are generated by the $\sigma^j : [n+1] \to [n]$ (for $0 \leq j \leq n$) defined by
$$\sigma^j = 0 \to \cdots \to j \to j \to \cdots \to n$$

We also have the “cosimplicial relations” (which need not be memorized... !)

$$\begin{aligned}
\partial^j \partial^i &= \partial^i \partial^{j-1} && \text{if } i < j, \\
\sigma^j \partial^i &= \partial^i \sigma^{j-1} && \text{if } i < j, \\
\sigma^j \partial^i &= \mathrm{id} && \text{if } i = j \text{ or } j+1, \\
\sigma^j \partial^i &= \partial^{i-1} \sigma^j && \text{if } i > j+1, \\
\sigma^j \sigma^i &= \sigma^{i-1} \sigma^j && \text{if } i \geq j.
\end{aligned}$$

Definition 2.1.2. A *simplicial set* is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Set}$. We denote by $s\mathsf{Set}$ the category of simplicial sets.

Concretely, a simplicial set is thus the data of :

- a sequence of sets, X_0, X_1, X_2, ... whose elements are called n-simplices (the elements of X_0 are sometimes called “vertices”, those of X_1 “edges”, and those of X_2 “faces”) ;

- "face" maps $d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$;
- "degeneracy" maps $s_j : X_n \to X_{n+1}$ for $0 \leq j \leq n$;
- satisfying the "simplicial relations", which are the same as the cosimplicial relations but with the order of compositions reversed (for example, $d_i d_j = d_{j-1} d_i$ for $i < j$, etc.).

A *simplicial map* $X_\bullet \to Y_\bullet$ is the data of maps $f_n : X_n \to Y_n$ that commute with the face and degeneracy maps.

Remark 2.1.3 (See, e.g., [Fri08, Section 3]). One should be careful with terminology when reading older texts. In the past, simplicial sets were called "complete semi-simplicial complexes". The motivation was that these objects resembled simplicial complexes, but a sequence of vertices did not determine a face uniquely. "Semi-simplicial complexes" were simplicial sets with only face maps, no degeneracies; they are called "semi-simplicial sets" today. Over time, "complex" became "set". Since "complete semi-simplicial complexes" were by far the most interesting, their name was first abbreviated to "semi-simplicial complexes" (the old semi-simplicial complexes fading into the background), then to "simplicial complex", and finally to "simplicial set". Semi-simplicial sets later came back into prominence and took their current name. Some authors also call semi-simplicial sets "Delta-sets" (or even "Δ-sets"), which can conflict with the fact that the category Δ defined above is the basic building block of simplicial sets. For simplicity, in these course notes, we only speak of simplicial sets (modern terminology) as defined above.

Definition 2.1.4. More generally, a *simplicial object* in a category C is a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. We denote by $s\mathsf{C}$ the category of simplicial objects in C.

Definition 2.1.5 (Dual)**.** A *cosimplicial set* is a covariant functor $\Delta \to \mathsf{Set}$; the corresponding category is denoted $c\mathsf{Set}$. We also define the category $c\mathsf{C}$ of cosimplicial objects in a given category.

Example 2.1.6. By the Yoneda Lemma, for fixed $n \geq 0$, we have a "canonical" simplicial set, the standard n-simplex $\Delta^n_\bullet$, defined by :

$$\Delta^n_k = \mathrm{Hom}_\Delta([k],[n]).$$

Note moreover that $\mathrm{Hom}_{s\mathsf{Set}}(\Delta^n_\bullet, X_\bullet) \cong X_n$ (again by the Yoneda Lemma). We then deduce that

$$X_\bullet \cong \mathrm{colim}_{f:\Delta^n_\bullet \to X_\bullet} \Delta^n_\bullet.$$

Example 2.1.7. We note that $\Delta^\bullet_\bullet$ defines a cosimplicial object in the category of simplicial sets. Let us give an example of a cosimplicial space that resembles it. We define :

$$\mathbb{\Delta}^n := \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^n \mid \forall i, t_i \geq 0, \ \sum t_i = 1\}.$$

The coface and codegeneracy maps are given by :

$$\begin{aligned} \partial^i(t_0, \dots, t_n) &= (t_0, \dots, t_i, 0, t_{i+1}, \dots, t_n), \\ \sigma^j(t_0, \dots, t_n) &= (t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_n) \end{aligned}$$

We see for example that $\mathbb{\Delta}^0$ is a point, $\mathbb{\Delta}^1 = [0,1]$, $\mathbb{\Delta}^2$ is a triangle, etc. The coface maps are the inclusions of the codimension 1 faces, while the codegeneracy maps "collapse" a dimension.

Definition 2.1.8. Let X be a simplicial set. A simplex $x \in X_n$ is called *degenerate* if $x = s_i(y)$ for some $y \in X_{n-1}$. Otherwise, it is called *non-degenerate.*

Lemma 2.1.9. *Let $w \in X_n$ be a simplex. Then there exists a unique pair (y, s) where $y \in X_m$ is a non-degenerate simplex and $s = \sigma^*$ is a degeneracy induced by a surjection $\sigma : \Delta^n \to \Delta^m$.*

Démonstration. Let $w = s_{i_0}s_{i_1}\dots s_{i_k}(x) = s_{j_0}s_{j_1}\dots s_{j_l}(y)$ be a degenerate simplex with $x \in X_a, y \in X_b$ non-degenerate. Let us show that $x = y$. Using the simplicial identities,

$$x = d_{i_k}\dots d_{i_0}(w) = d_{i_k}\dots d_{i_0}s_{j_0}\dots s_{j_l}(y).$$

Using the simplicial identities, we move all the s's to the left of the d's and obtain $x = s_{u_0}\dots s_{u_m}d_{v_1}\dots d_{v_n}(y)$. Since x is non-degenerate, we have $m = 0$ and $x = d_{v_1}\dots d_{v_n}(y)$. In particular, $a = b - n \leq b$. By symmetry, we also find $b \leq a$, hence $a = b$. Returning to the equality above, we deduce $x = y$ (there can be no $d_{\dots}$). ☐

2.2 Adjunction with topological spaces

We recall the standard cosimplicial space :

$$\mathbb{\Delta}^n = \{(t_0, \dots, t_n) \in (\mathbb{R}_+)^{n+1} \mid t_0 + \dots + t_n = 1\}.$$

Definition 2.2.1. Let $X_\bullet$ be a simplicial set. Its *geometric realization* is the quotient topological space :

$$|X_\bullet| \coloneqq \left(\bigsqcup_{n\geq 0} X_n \times \mathbb{\Delta}^n \right) /\!\sim,$$

where the equivalence relation is given by

$$(d_i(x), t) \sim (x, \partial^i(t)), \qquad (s_j(x), t) \sim (x, \sigma^j(t)).$$

Definition 2.2.2. Let Y be a topological space. Its *singular (simplicial) set* is given by :

$$S_\bullet(Y) \coloneqq \mathrm{Hom}_{\mathsf{Top}}(\mathbb{\Delta}^\bullet, Y),$$

with the face and degeneracy maps induced by $\mathbb{\Delta}^\bullet$.

Proposition 2.2.3. ↻ *The geometric realization and the singular set define an adjunction :*

$$|-| : s\mathsf{Set} \leftrightarrows \mathsf{Top} : S_\bullet.$$

Démonstration. It is clear that $|-|$ and $S_\bullet$ define functors. Let us show that they are adjoint by defining natural bijections

$$\varphi : \mathrm{Hom}_{\mathsf{Top}}(|X_\bullet|, Y) \leftrightarrows \mathrm{Hom}_{s\mathsf{Set}}(X, S_\bullet(Y)) : \psi.$$

– Let $f : X_\bullet \to S_\bullet(Y)$ be a simplicial map. We thus have maps $f_n : X_n \to S_n(Y) = \mathrm{Hom}_{\mathsf{Top}}(\mathbb{\Delta}^n, Y)$ that commute with the face and degeneracy maps. We will define a continuous map $\psi(f) : |X_\bullet| \to Y$. We begin by defining a continuous map

$$\bigsqcup_{n \geq 0} X_n \times \mathbb{\Delta}^n \to Y$$

and we will show that it is compatible with the equivalence relation. Concretely, if $(x, t) \in X_n \times \mathbb{\Delta}^n$, we associate to it $f_n(x)(t)$. This map is indeed continuous, and compatibility with the equivalence relation is easily verified (since f commutes with d_i and s_j).

– Let $g : |X_\bullet| \to Y$ be a continuous map. We define $\varphi(h) : X_\bullet \to S_\bullet(Y)$ as follows. For $x \in X_n$, we define $\varphi(h)_n(x) : \mathbb{\Delta}^n \to Y$ by $\varphi(h)_n(x) : t \mapsto h([x, t])$. One readily checks that $\varphi(h)_n(x)$ is continuous and that $\varphi(h)$ is simplicial.

One also readily checks that φ and ψ are natural and inverse to each other. ☐

Example 2.2.4. Let E be a set. We define the constant simplicial set $E_\bullet$ by $E_n = E$, with all face and degeneracy maps being identities. Then $|E_\bullet|$ is simply E viewed as a discrete space.

Example 2.2.5. By the Yoneda lemma, $|\Delta^n| = \mathbb{\Delta}^n$.

Example 2.2.6 (Exercise)**.** Describe the unique simplicial set that has exactly two non-degenerate simplices in respective dimensions 0 and 1. Show that its geometric realization is a circle.

Remark 2.2.7. One could replace $\mathbb{\Delta}^\bullet$ by any cosimplicial space and obtain an adjunction. More generally, given a cocomplete category C and a cosimplicial object $A^\bullet \in c\mathsf{C}$, one obtains an adjunction $s\mathsf{Set} \leftrightarrows \mathsf{C}$.

2.3 Boundaries, horns, skeletons

In this section, we describe some simplicial subsets of Δ^n that will prove useful in what follows.

Definition 2.3.1. Let $\mathrm{id}_{[n]} = v_n \in \Delta^n_n$ be the unique non-degenerate simplex. The *boundary* of the standard simplex $\partial\Delta^n_\bullet \subset \Delta^n_\bullet$ is the smallest simplicial subset containing all faces $d_i v_n$ $(0 \leq i \leq n)$. Concretely,

$$\partial\Delta^n_i = \{f : [i] \to [n] \mid f \text{ is not surjective}\} \subset \Delta^n_i.$$

We also define (by convention) $\partial\Delta^0 = \varnothing$.

Definition 2.3.2. Let $n \geq 1$ and $0 \leq k \leq n$. The *kth horn* $(\Lambda^n_k)_\bullet \subset \Delta^n_\bullet$ is the smallest simplicial subset containing the faces $d_i v_n$ for $i \neq k$. Concretely, $(\Lambda^n_k)_i$ consists of the order-preserving maps $[i] \to [n]$ whose image does not contain $[n] - \{r\}$.

Lemma 2.3.3. ↺ *We have identifications :*

$$\mathrm{Hom}_{s\mathsf{Set}}(\Lambda^n_k, X) = \{(y_0, \ldots, \hat{y}_k, \ldots, y_n) \in (X_{n-1})^{\times n} \mid d_i y_j = d_{j-1} y_i,\ \forall i < j\},$$

$$\mathrm{Hom}_{s\mathsf{Set}}(\partial\Delta^n, X) = \{(y_0, \ldots, y_n) \in (X_{n-1})^{\times n+1} \mid d_i y_j = d_{j-1} y_i,\ \forall i < j\}.$$

Definition 2.3.4. Let $X_\bullet \in s\mathsf{Set}$ be a simplicial set and $n \geq 0$ an integer. Its *n-skeleton* $\mathrm{sk}_n X_\bullet$ is the smallest simplicial subset of $X_\bullet$ containing all non-degenerate simplices of dimension $\leq n$.

Definition 2.3.5. We denote by $i_n : \Delta_{\leq n} \hookrightarrow \Delta$ the full subcategory whose objects are the $[k]$ with $k \leq n$. The category of n-truncated simplicial sets $s_{\leq n}\mathsf{Set}$ is the category $\mathrm{Hom}_{\mathsf{Cat}}(\Delta^{\mathrm{op}}_{\leq n}, \mathsf{Set})$.

Lemma 2.3.6. ↺ *The construction* sk_n *defines a functor* $\mathrm{sk}_n : s\mathsf{Set} \to s_{\leq n}\mathsf{Set}$.

Proposition 2.3.7. ↺ *The restriction functor* $i_{n*} : s\mathsf{Set} \to s_{\leq n}\mathsf{Set}$ *has a right adjoint* i_n^*. *There is a natural isomorphism :*

$$\mathrm{sk}_n X \cong i_n^* i_{n*} X.$$

Remark 2.3.8. This functor also has a left adjoint $i_n^!$, and $i_n^! i_{n*}$ is isomorphic to the « coskeleton » functor.

Lemma 2.3.9. ↺ *Let* $X_\bullet$ *be a simplicial set. The* k*-simplices of* $\mathrm{sk}_n X_\bullet$ *are the* $x \in X_k$ *such that there exist a surjection* $\sigma : \Delta^k \to \Delta^l$ *(where* $l \leq n$*) and a non-degenerate simplex* $y \in X_l$ *with* $x = \sigma^* y$.

Corollary 2.3.10. *The canonical map* $\mathrm{colim}_{n \geq 0} \mathrm{sk}_n X \to X$ *is an isomorphism of simplicial sets.*

Corollary 2.3.11. *Let* $n \geq 0$. *The boundary* $\partial\Delta^n$ *is the* $(n-1)$*-skeleton of* Δ^n *:* $\partial\Delta^n = \mathrm{sk}_{n-1} \Delta^n$.

Proposition 2.3.12. ↺ *The boundary* $\partial\Delta^n$ *is given as the coequalizer :*

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{0 \leq i \leq n} \Delta^{n-1} \to \partial\Delta^n$$

corresponding to the relation $d^j d^i = d^i d^{j-1}$.

Remark 2.3.13. This result can be used to define the skeleton by induction : there is a cocartesian diagram

$$\begin{tikzcd}
\coprod_{x \in X_n \text{ non-degen.}} \partial\Delta^n \arrow[r, hook] \arrow[d, hook] & \mathrm{sk}_{n-1} X \arrow[d, hook] \\
\coprod_{x \in X_n \text{ non-degen.}} \Delta^n \arrow[r, hook] & \mathrm{sk}_n X.
\end{tikzcd}$$

Horns can also be defined in this way :

Proposition 2.3.14. ↺ *The kth horn Λ^n_k is given as the coequalizer :*

$$\coprod_{\substack{0\leq i<j\leq n\\ i,j\neq k}} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0\leq i\leq n\\ i\neq k}} \Delta^{n-1} \to \Lambda^n_k.$$

2.4 Model Structure

We will define a model structure on $s\mathsf{Set}$ such that the preceding adjunction is a Quillen equivalence. The definitions of weak equivalences and fibrations resemble those of weak homotopy equivalences and Serre fibrations.

Definition 2.4.1. A *Kan fibration* is a simplicial map $p : X \to Y$ that has the RLP with respect to all inclusions $\Lambda^n_k \subset \Delta^n$:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook, "i"] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

Definition 2.4.2. A *Kan complex* is a simplicial set X such that the unique map to the terminal simplicial set $X \to *$ is a Kan fibration.

Concretely, a morphism is a Kan fibration if for every simplex $y \in Y_n$ and every n-tuple $z_0, \ldots, \hat{z}_k, \ldots, z_n \in X_{n-1}$ satisfying $p(z_i) = d_i(y)$ and $d_i z_j = d_{j-1} z_i \ \forall i < j$, there exists $x \in X_n$ such that $p(x) = y$ and $d_i(x) = z_i$. Geometrically, one can “fill” the horn.

Lemma 2.4.3 (Exercise)**.** *A continuous map $p : X \to Y$ is a Serre fibration if and only if $S_\bullet(p)$ is a Kan fibration.*

Remark 2.4.4. Not all simplicial sets are Kan complexes. For example, Δ^n is not one for $n \geq 2$. Consider the horn $(-, y_1, y_2) : \Lambda^2_0 \to \Delta^2$ where $y_1 = 0 \to 2 \in \Delta^2_1$ and $y_2 = 0 \to 1 \in \Delta^2_1$. Then there is no $x \in \Delta^2_2$ such that $d_1 x = y_1$ and $d_2 x = y_2$. We would necessarily have $x = 0 \to 2 \to 1$, which is not increasing.

Theorem 2.4.5 (Quillen [Qui67])**.** *There exists a cofibrantly generated and combinatorial model category structure on $s\mathsf{Set}$, called the* Quillen structure *where :*

- *the weak equivalences are the simplicial maps $f : X_\bullet \to Y_\bullet$ such that $|f| : |X_\bullet| \to |Y_\bullet|$ is a weak homotopy equivalence ;*
- *the fibrations are the Kan fibrations ;*
- *the cofibrations are the inclusions.*

To prove this theorem, we will apply Theorem 1.5.27 on cofibrantly generated model categories. Let us begin by noting that limits and colimits are simply defined degree by degree. We choose as generating cofibrations and generating acyclic cofibrations :

$$\mathcal{I} = \{\partial\Delta^n \subset \Delta^n\}_{n\geq 0}, \quad \mathcal{J} = \{\Lambda^n_k \subset \Delta^n\}_{n\geq 1}.$$

By definition we indeed have $\mathscr{F} = \mathcal{J}^{\perp}$. Moreover, the sources of the morphisms of $\mathcal{I}$ and $\mathcal{J}$ are small with respect to every simplicial set (thanks to the descriptions of $\mathrm{Hom}_{s\mathsf{Set}}(\partial\Delta^n, -)$ and $\mathrm{Hom}_{s\mathsf{Set}}(\Lambda^n_k, -)$).

Let us verify that the cofibrations are the correct ones.

Lemma 2.4.6. *A simplicial map is injective if and only if it is in* ${}^{\perp}(\mathcal{I}^{\perp})$*. Moreover, every cofibration is in* $\mathcal{I}$*-cell.*

Démonstration. Let us first show that maps in ${}^{\perp}(\mathcal{I}^{\perp})$ are injective. By the small object argument, $f : X \to Y \in {}^{\perp}(\mathcal{I}^{\perp})$ factors as $p_\infty i_\infty$ where $i_\infty \in \mathcal{I}$-cell and $p_\infty = \mathcal{I}^{\perp}$ (so it is a Kan fibration). In particular f has the LLP with respect to p_∞ and we have seen previously that as a consequence, f is a retract of i_∞. The map i_∞ is obtained as a transfinite composite of pushouts of injections. Now, in Set injections are always split (every injection admits a right inverse) and split monomorphisms are stable under pushouts and transfinite composites. Therefore i_∞ is an injection.

Conversely, suppose that $i : A \to X$ is injective and let us show that it is a countable composite of pushouts of coproducts of maps in $\mathcal{I}$, and therefore $i \in \mathcal{I}$-cell $\subset {}^{\perp}(\mathcal{I}^{\perp})$. Set $A_{(0)} = A$. Suppose we have defined an injection $i_k : A_{(k)} \to X$ that is an isomorphism on simplices of dimension $< k$, and let us extend it to an injection $A_{(k+1)} \to X$ of the same type. Let S_k be the set of k-simplices of X that are not in the image of i_k (they cannot be degenerate), which we put in correspondence with maps $\Delta^k \to X$. For $s \in S_k$, the restriction of s to $\partial\Delta^k$ factors through $A_{(k)}$. We then define $A_{(k+1)}$ as the pushout :

$$\begin{tikzcd}
\coprod_{s\in S_k} \partial\Delta^k \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & A_{(k)} \arrow[d, dashed] \\
\coprod_{s\in S_k} \Delta^k \arrow[r, dashed] & A_{(k+1)}.
\end{tikzcd}$$

The induced map $A_{(k+1)} \to X$ is surjective in dimension $\leq k$ by construction. It is moreover injective, since we only add non-degenerate simplices. □

Lemma 2.4.7. *A relative* $\mathcal{J}$*-cell complex is an acyclic cofibration :* $\mathcal{J}$*-cell* $\subset \mathscr{W} \cap {}^{\perp}(\mathcal{I}^{\perp})$*.*

Démonstration. The morphisms of $\mathcal{J}$ are injections, i.e. they are in ${}^{\perp}(\mathcal{I}^{\perp})$. A class of the form ${}^{\perp}(\dots)$ is stable under pushouts, so $\mathcal{J}$-cell $\subset {}^{\perp}(\mathcal{I}^{\perp})$.

The geometric realization of $\Lambda^n_k \to \Delta^n$ is isomorphic to $[0,1]^{n-1} \subset [0,1]^n$, so $|\mathcal{J}|$ is the set of acyclic cofibrations of Top. Now geometric realization is a left adjoint so $|{}^{\perp}(\mathcal{J}^{\perp})| \subset {}^{\perp}(|\mathcal{J}|^{\perp})$ (exercise), and ${}^{\perp}(|\mathcal{J}|^{\perp})$ consists exactly of the acyclic cofibrations of Top. In particular these are weak homotopy equivalences, so by definition ${}^{\perp}(\mathcal{J}^{\perp}) \subset \mathscr{W}$. We therefore deduce that $\mathcal{J}$-cell $\subset {}^{\perp}(\mathcal{J}^{\perp}) \subset \mathscr{W}$. □

Lemma 2.4.8. *The elements of $\mathcal{I}^\perp$ are acyclic Kan fibrations.*

Démonstration. Let $p : X \to Y \in \mathcal{I}^\perp$. We have shown that p has the RLP with respect to all inclusions. In particular it has the RLP with respect to $\mathcal{J}$, i.e. $p \in \mathcal{J}^\perp$, so by definition it is a Kan fibration. It remains to show that $|p|$ is a weak homotopy equivalence. Since p has the RLP with respect to all inclusions, we can find a lifting :

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, hook, "(\mathrm{id}_X, p)"'] & X \arrow[d, "p"] \\
X \times Y \arrow[r, "p_Y"] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}$$

And therefore p is a retract of p_Y :

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "p"] & X \times Y \arrow[r, "l"] \arrow[d, "p_Y"] & X \arrow[d, "p"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

In particular, $|p|$ is a retract of $|p_Y|$, which is a Serre fibration, so $|p|$ is also one. Let $F = \Delta^0 \times_Y X$ denote the fiber of p ; we want to show that $|F|$ is contractible. Since p has the RLP with respect to $\mathcal{I}$ and $F \to \Delta^0$ is a pullback of p, then $F \to \Delta^0$ also has the RLP with respect to $\mathcal{I}$. Therefore $F \to \Delta^0$ has the RLP with respect to all inclusions. In particular, F is non-empty ; let $f \in F_0$ be a 0-simplex, and let $f : F \to F$ be the constant map equal to f. Then we have a commutative diagram :

$$\begin{tikzcd}
F \times \partial\Delta^1 \arrow[r, "(\mathrm{id}, f)"] \arrow[d, hook] & F \arrow[d] \\
F \times \Delta^1 \arrow[r] \arrow[ur, dashed, "H"] & \Delta^0
\end{tikzcd}$$

where we can find a lifting, which is a homotopy between $\mathrm{id}_{|F|}$ and a constant map. We deduce that F is contractible, so by the long exact sequence in homotopy, $|p|$ is a weak homotopy equivalence, and therefore p is a weak equivalence. ☐

Lemma 2.4.9. *Acyclic fibrations are in $\mathcal{I}^\perp$.*

Démonstration. This is the key point of the proof, and the most difficult one. The proof relies on the theory of minimal fibrations and anodyne extensions. One may refer to [GJ99, Theorem I.7.10] for the case of an acyclic fibration between fibrant objects, and to [GJ99, Theorem I.11.2] for the case of an arbitrary acyclic fibration. ☐

Proof of Theorem 2.4.5. We have just verified all the hypotheses of Theorem 1.5.27, so $s\mathsf{Set}$ admits a cofibrantly generated model category structure. Moreover, $s\mathsf{Set}$ is combinatorial since $\{\Delta^n, \partial\Delta^n\}$ is a set of small generators. ☐

2.5 Equivalence with Top

Theorem 2.5.1 (Quillen equivalence between $s\mathsf{Set}$ and Top)**.** *The adjunction* $|-| : s\mathsf{Set} \leftrightarrows \mathsf{Top} : S_\bullet$ *of Proposition 2.2.3 is a Quillen equivalence.*

To prove this theorem, we will take a short detour through the enrichment of $s\mathsf{Set}$ over itself and through simplicial homotopy groups.

2.5.1 Enrichment

We will also need the fact that the category $s\mathsf{Set}$ is *enriched* over itself. This means that one can extend $\mathrm{Hom}_{sSet}(A, X)$ to a simplicial set whose vertices are exactly the simplicial maps $A \to X$. (The category $s\mathsf{Set}$ is thus an example of a simplicial category, see Section 4.3). The edges will correspond to homotopies between maps.

Definition 2.5.2. Let A, X be two simplicial sets. We define *the space of simplicial maps* $\mathrm{Map}_\bullet(A, X)$ by :

$$\mathrm{Map}_n(A, X) = \mathrm{Hom}_{s\mathsf{Set}}(A \times \Delta^n, X).$$

Its simplicial structure is induced by the cosimplicial structure of $\Delta^\bullet$.

Lemma 2.5.3. *There is a natural isomorphism* $\mathrm{Map}_\bullet(\Delta^0, X) \cong X_\bullet$.

Démonstration. This is essentially the Yoneda Lemma. Indeed, $\Delta^0 \times \Delta^n \cong \Delta^n$. Thus :

$$\mathrm{Map}_n(\Delta^0, X) \cong \mathrm{Hom}_{s\mathsf{Set}}(\Delta^n, X) \cong X_n.$$

One easily checks that this isomorphism is compatible with faces and degeneracies. ☐

Lemma 2.5.4. *There is a natural isomorphism in* (A, B, X) *:*

$$\mathrm{Hom}_{s\mathsf{Set}}(A, \mathrm{Map}(B, X)) \cong \mathrm{Hom}_{s\mathsf{Set}}(A \times B, X).$$

In more sophisticated terms, one says that $s\mathsf{Set}$ is a cartesian closed $s\mathsf{Set}$-enriched category.

Démonstration. Let us define a map

$$\psi : \mathrm{Hom}_{s\mathsf{Set}}(A, \mathrm{Map}(B, X)) \to \mathrm{Hom}_{s\mathsf{Set}}(A \times B, X). \tag{2.5.5}$$

For $f : A \to \mathrm{Map}(B, X)$, we set :

$$\begin{aligned} \psi(f) : A_n \times B_n &\to X_n, \\ (a, b) &\mapsto \underbrace{f(a)}_{\in \mathrm{Map}_n(B,X) = \mathrm{Hom}_{s\mathsf{Set}}(B \times \Delta^n, X)}(b, \mathrm{id}_{[n]}). \end{aligned}$$

Let us verify that $\psi(f)$ is a simplicial map. We check it for faces ; the proof for degeneracies is identical.

- On the one hand, we have $d_i(\psi(f)(a,b)) = d_i(f(a)(b,\mathrm{id}_{[n]}))$. Now $f(a) : B \times \Delta^n \to X$ is simplicial, so $d_i(f(a)(b,\mathrm{id}_{[n]})) = f(a)(d_i(b),\partial^i)$ where $\partial^i \in \Delta^n_{n-1}$.
- On the other hand, $\psi(f)(d_ia,d_ib) = f(d_ia)(d_ib,\mathrm{id}_{[n-1]})$. Now f is a simplicial map, so $f(d_ia)(d_ib,\mathrm{id}_{[n-1]}) = f(a)(d_ib,\partial^i \circ \mathrm{id}_{[n-1]})$. We indeed have equality.

Conversely, we define $\varphi : \mathrm{Hom}_{s\mathsf{Set}}(A \times B, X) \to \mathrm{Hom}_{s\mathsf{Set}}(A, \mathrm{Map}(B,X))$ as follows. For $g : A \times B \to X$, we set :

$$\varphi(g)_n : A_n \to \mathrm{Map}_n(B,X),$$
$$a \mapsto \begin{cases} B_k \times \Delta^n_k & \to X_k, \\ (b,u) & \mapsto g_k(u^*(a),b). \end{cases}$$

It remains to verify (exercise) that $\varphi(g)(a)$ is a simplicial map $B \times \Delta^n \to X$, that $\varphi(g)$ is a simplicial map $A \to \mathrm{Map}(B,X)$, and finally that φ and ψ are inverses of each other. □

We will now show that Map behaves well with respect to the model structure. We will use the following lemma when we study simplicial homotopy groups.

Let $i : A \to B$ and $p : X \to Y$ be two simplicial maps ; then we have a commutative diagram :

$$\begin{tikzcd}
\mathrm{Map}(B,X) \arrow[r, "p_*"] \arrow[d, "i^*"] & \mathrm{Map}(B,Y) \arrow[d, "i^*"] \\
\mathrm{Map}(A,X) \arrow[r, "p_*"] & \mathrm{Map}(A,Y)
\end{tikzcd}$$

which induces a canonical map

$$(i^*,p_*) : \mathrm{Map}(B,X) \to \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y). \tag{2.5.6}$$

Proposition 2.5.7. *Let $i : A \hookrightarrow B$ be a cofibration and $p : X \twoheadrightarrow Y$ a fibration. Then the canonical morphism (i^*,p_*) of Equation (2.5.6) is a fibration. If moreover i or p is acyclic, then (i^*,p_*) is acyclic.*

Lemma 2.5.8. *Let $i : A \overset{\sim}{\hookrightarrow} B$ be an acyclic cofibration and $j : K \hookrightarrow L$ a cofibration. Then*

$$i \mathbin{\square} j : (K \times B) \cup_{K \times A} (L \times A) \to L \times B$$

is an acyclic cofibration.

[REVIEWER NOTE : The proof of this lemma is incomplete as written and appears to require additional justification.]

We can already prove Proposition 2.5.7 and a corollary using this lemma.

Proof of Proposition 2.5.7. We need to show that (i^*,p_*) has the RLP with respect to the inclusions $\Lambda^n_k \overset{\sim}{\hookrightarrow} \Delta^n$. Now, a diagram of the type on the left is equivalent to a diagram of the type on the right :

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook, "\sim"] & \mathrm{Map}(B,X) \arrow[d, "(i^*,p_*)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\Lambda^n_k \times B) \cup_{\Lambda^n_k \times A} (\Delta^n \times A) \arrow[r] \arrow[d, "j"] & X \arrow[d, two heads, "p"] \\
\Delta^n \times B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Now $\Lambda^n_k \overset{\sim}{\hookrightarrow} \Delta^n$ is an acyclic cofibration and $A \subset B$ is a cofibration, so j is an acyclic cofibration by Lemma 2.5.8. We can therefore find a lifting in the diagram on the right, which corresponds to a lifting in the diagram on the left.

To show that (i^*, p_*) is acyclic when i or p is, we replace $\Lambda^n_k \subset \Delta^n$ by $\partial\Delta^n \subset \Delta^n$ and adapt. □

Corollary 2.5.9. *If $i : A \overset{\sim}{\hookrightarrow} B$ is an acyclic cofibration and X is a fibrant simplicial set, then $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is an acyclic fibration. Dually, if $p : X \overset{\sim}{\twoheadrightarrow} Y$ is an acyclic fibration and A is cofibrant, then $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ is an acyclic fibration.*

Démonstration. For the first case, it suffices to apply the previous proposition to $i : A \to B$ and to $p : X \to *$. We then have $\mathrm{Map}(A, X) \times_{\mathrm{Map}(A,*)} \mathrm{Map}(B, *) \cong \mathrm{Map}(A, X)$. The second case is dual. □

Proof of Lemma 2.5.8. We fix $j : K \hookrightarrow L$. Since i is an acyclic cofibration, it is a retract of a relative $\mathcal{J}$-cellular complex (where $\mathcal{J} = \{\Lambda^n_k \hookrightarrow \Delta^n\}$).

Let us show that the class of morphisms of the type $(-) \,\square\, j$ is stable under pushouts. Suppose that $D = B \cup_A C$. We want to show that we have a pushout :

$$\begin{tikzcd}
(K \times B) \cup_{K\times A} (L \times A) \arrow[r] \arrow[d] & L \times B \arrow[d] \\
(K \times D) \cup_{K\times C} (L \times C) \arrow[r] & L \times D
\end{tikzcd}$$

We note that $L \times D = L \times B \cup_{L\times A} L \times D$. Moreover, $K \times D = K \times C \cup_{K\times A} K \times B$, so

$$\begin{aligned} K \times D \cup_{K\times C} L \times C &= (K \times B \cup_{K\times A} K \times C) \cup_{K\times C} L \times C \\ &= K \times B \cup_{K\times A} L \times C, \end{aligned}$$

from which we deduce that

$$\begin{aligned} (K \times D \cup_{K\times C} L \times C) \cup_{K\times B\times_{K\times A} L\times A} L \times B &= (K \times B \cup_{K\times A} L \times C) \cup_{K\times B\times_{K\times A} L\times A} L \times B \\ &= L \times C \cup_{L\times A} L \times B = L \times D \end{aligned}$$

(where we used the fact that $K \subset L$ is an inclusion in the passage to the last line).

Now let $i : \Lambda^n_k \overset{\sim}{\hookrightarrow} \Delta^n$ be a generating acyclic cofibration. Let us show that

$$i \,\square\, j : K \times \Delta^n \cup_{K\times\Lambda^n_k} L \times \Lambda^n_k \to L \times \Delta^n$$

is an acyclic cofibration.

Lemma 2.5.10. *The generating acyclic cofibrations $i : \Lambda^n_k \overset{\sim}{\hookrightarrow} \Delta^n$ are retracts of maps of the type $\mathcal{J} \,\square\, f$ where $f : \Lambda^1_\varepsilon \overset{\sim}{\hookrightarrow} \Delta^1$ is an inclusion for $\varepsilon \in \{0, 1\}$.*

Démonstration. See Figure 2.1. Let first $k < n$. We construct a diagram :

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook, "\sim"] & \left(\Lambda^n_k \times \Delta^1\right) \cup_{\Lambda^n_k \times \{0\}} \left(\Delta^n \times \{0\}\right) \arrow[r] \arrow[d, hook] & \Lambda^n_k \arrow[d, hook, "\sim"] \\
\Delta^n \arrow[r, "\sigma"] & \Delta^n \times \Delta^1 \arrow[r, "r_k"] & \Delta^n
\end{tikzcd}$$

The map σ is induced by the map $[n] \to [n] \times [1]$, $j \mapsto (j, 1)$. The map r_k is induced by the map $[n] \times [1] \to [n]$ such that $r_k(j, 1) = j$, $r_k(j, 0) = j$ if $j \leq k$ and $r_k(j, 1) = k$ if $j > k$. It is clear that $r_k \circ \sigma = \mathrm{id}$ and that the maps restrict correctly as shown in the diagram.

If $k = n$, we instead set $\sigma'(j) = (j, 0)$ and $r'(j, 0) = j$, $r'(j, 1) = n$. $\square$

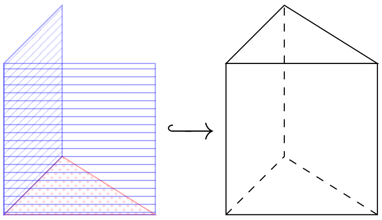


FIGURE 2.1 : Inclusion $\left(\Lambda^n_k \times \Delta^1\right) \cup_{\Lambda^n_k \times \{0\}} \left(\Delta^n \times \{0\}\right) \subset \Delta^n \times \Delta^1$ for $n = 2$

We deduce from this lemma that a map is an acyclic cofibration if and only if it is a $(\mathcal{J} \square \{f_0, f_1\})$-cellular complex where $f_\varepsilon = \Lambda^1_\varepsilon \overset{\sim}{\hookrightarrow} \Delta^1$. Using the stability under retracts and pushouts of the classes $(-) \square (-)$, to conclude it therefore suffices to show that $i \square j \square f_0$ and $i \square j \square f_1$ are acyclic cofibrations for any inclusion i and any generating acyclic cofibration $j \in \mathcal{J}$.

One easily sees that $i \square j$ is an inclusion. It is therefore a retract of a relative $\mathcal{I}$-cellular complex, and it suffices to show :

Lemma 2.5.11. *If $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ is a generating inclusion, then*

$$i_n \square f_\varepsilon : \left(\partial\Delta^n \times \Delta^1\right) \cup_{\partial\Delta^n \times \Lambda^1_\varepsilon} \Delta^n \times \Lambda^1_\varepsilon \to \Delta^n \times \Delta^1$$

is an acyclic cofibration for $\varepsilon \in \{0, 1\}$.

Démonstration. A k-simplex of $\Delta^n \times \Delta^1$ is given by an increasing map $[k] \to [n] \times [1]$. There are therefore $n + 1$ non-degenerate simplices $x_0, \ldots, x_n \in (\Delta^n \times \Delta^1)_{n+1}$ given by

$$x_j = \big((0, 0), \ldots, (j, 0), (j + 1, 1), \ldots, (n, 1)\big).$$

We note that $\partial\Delta^n \times \Delta^1$ is generated by the $d_i x_j$ for $i \neq j, j + 1$ and that $\Delta^n \times \Lambda^1_\varepsilon$ is generated by $d_0 x_0$ and $d_{n+1} x_n$. The map $i_n \square f_\varepsilon$ is thus obtained by gluing x_0 along a Λ^{n+1}_1, then x_1 along a Λ^{n+1}_2, etc. In the end, we find that $i_n \square f_\varepsilon$ is a relative $\mathcal{J}$-cellular complex and is therefore an acyclic cofibration. $\square$

To summarize :

- the maps $i_n \square j \square f_\varepsilon$ are acyclic cofibrations for $i_n \in \mathcal{I}$, $j \in \mathcal{J}$, $f_\varepsilon : \Lambda^1_\varepsilon \overset{\sim}{\hookrightarrow} \Delta^1$;
- hence all maps $i \square j \square f_\varepsilon$ are acyclic cofibrations for a cofibration i, $j \in \mathcal{J}$;
- now, all acyclic cofibrations are $(\mathcal{J} \square \{f_0, f_1\})$-cellular complexes ;
- hence all maps $i \square j$ are acyclic cofibrations for a cofibration i and an acyclic cofibration j. $\square$

2.5.2 Simplicial homotopy groups

Let us make explicit the (left) homotopy relation in $s\mathsf{Set}$. If A is a simplicial set, a natural cylinder is given by $A \sqcup A = A \times \partial\Delta^1 \hookrightarrow A \times \Delta^1 \overset{\sim}{\to} A$. We deduce the following definition :

Definition 2.5.12. Two simplicial maps $f, g : A \to X$ are *homotopic* (on the left) if there exists $h : A \times \Delta^1 \to X$ making the following diagram commute :

$$\begin{tikzcd}
A = A \times \Delta^0 \arrow[d, hook] \arrow[dr, "f"] & \\
A \times \Delta^1 \arrow[r, "h"] & X \\
A = A \times \Delta^0 \arrow[u, hook] \arrow[ur, "g"'] &
\end{tikzcd}$$

We already know that if X is fibrant, this gives an equivalence relation on $\mathrm{Hom}_{s\mathsf{Set}}(A, X)$. We can refine this notion slightly.

Definition 2.5.13. Let A be a simplicial set and $B \subset A$ a simplicial subset. Two simplicial maps $f, g : A \to X$ whose restrictions to B coincide are *homotopic relative to* B if there exists a homotopy h as above satisfying $h|_{B\times\Delta^1}(b, w) = f(b) = g(b)$.

It is not difficult to see that if X is fibrant, this induces an equivalence relation on $\{f \in \mathrm{Hom}_{s\mathsf{Set}}(A, X) \mid f|_B = \varphi\}$ for fixed $\varphi : B \to X$.

Definition 2.5.14. Let $X_\bullet$ be a fibrant simplicial set, $v \in X_0$ a vertex and $n \geq 1$. The *nth simplicial homotopy group* $\pi_n(X_\bullet, v)$ is the set of maps $\Delta^n_\bullet \to X_\bullet$ that are constantly equal to v on $\partial\Delta^n$, modulo the homotopy relation rel $\partial\Delta^n$.

Remark 2.5.15. If we define $S^n = \partial\Delta^{n+1}$, then $\pi_n(X, v)$ is the set of pointed homotopy classes of pointed maps $\mathrm{Hom}_{s\mathsf{Set}_*}(S^n, X)/\!\sim$.

Definition 2.5.16. We also define $\pi_0(X_\bullet)$ as the set of homotopy classes of maps $\Delta^0 \to X$ (i.e. the vertices of X modulo the relation "being connected by a path").

Remark 2.5.17 (Exercise). Concretely, $\pi_0(X)$ is the quotient of X_0 by the following relation :

$$x \sim y \iff \exists e \in X_1 \text{ such that } d_0 e = x \text{ and } d_1 e = y.$$

Remark 2.5.18. If X is not fibrant, we define $\pi_n(X, v)$ and $\pi_0(X)$ by first replacing X by a fibrant simplicial set. In other words, we consider the total left derived functor of π_n.

Proposition 2.5.19. *Let X be a fibrant simplicial set, $v \in X_0$ a vertex and $n \geq 1$ an integer. Then $\pi_n(X, v)$ is a group, which is abelian for $n \geq 2$.*

Démonstration. Let $\alpha, \beta : \Delta^n \to X$ be two maps representing classes in $\pi_n(X, v)$. We define $[\alpha] \cdot [\beta] \in \pi_n(X, v)$ as follows. We construct a map $\gamma : \Lambda^{n+1}_n \to X$ by setting $\gamma_i = v$ for $0 \leq i \leq n-1$, $\gamma_n = \alpha$ and $\gamma_{n+1} = \beta$. (In compact notation : $\gamma = (v, \ldots, v, -, \alpha, \beta)$.) We indeed have $d_i \gamma_j = d_{j-1} \gamma_i$. We can therefore find an extension $\omega : \Delta^{n+1} \to X$ satisfying $d_i \omega = \gamma_i$. We then set $[\alpha] \cdot [\beta] \coloneqq [d_n \omega]$.

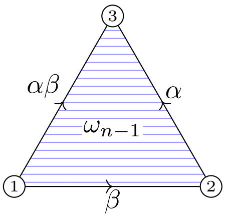
Let us show that this operation is associative. The remaining properties are proved exactly as in Top (exercise). Let $\alpha, \beta, \gamma : \Delta^n \to X$ be maps representing elements of $\pi_n(X, v)$. We can find simplices $\omega_{n-1}, \omega_{n+1}, \omega_{n+2}$ satisfying :

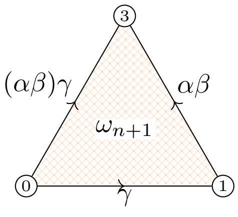
$$\begin{aligned}
\partial \omega_{n-1} &= (v, \ldots, v, \alpha, d_n \omega_{n-1}, \beta), \\
\partial \omega_{n+1} &= (v, \ldots, v, d_n \omega_{n-1}, d_n \omega_{n+1} \gamma), \\
\partial \omega_{n+2} &= (v, \ldots, v, \beta, d_n \omega_{n+2}, \gamma).
\end{aligned}$$

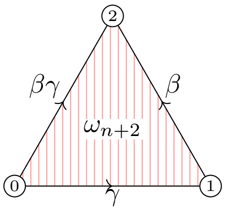
This defines a map $(v, \ldots, v, \omega_{n-1}, -, \omega_{n+1}, \omega_{n+2}) : \Lambda^{n+2}_{n+1} \to X$, which we extend to Δ^{n+2}. We denote by ζ the nth face of the simplex found. Then $\partial \zeta = (v, \ldots, v, x, d_n \omega_{n+1}, d_n \omega_{n+2})$, so :

$$\begin{aligned}
([\alpha][\beta])[\gamma] &= [d_n \omega_{n-1}][\gamma] \\
&= [d_n \omega_{n+1}] \\
&= [d_n \zeta] \\
&= [\alpha][d_n \omega_{n+2}] \\
&= [\alpha]([\beta][\gamma]).
\end{aligned}$$

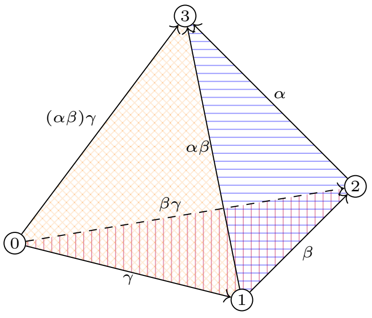
With pictures, for $n = 1$:







and :



□

Lemma 2.5.20. *Let X be a fibrant simplicial set. Then $\pi_0(X) \cong \pi_0(|X|)$.*

Démonstration. There is a natural map $\pi_0(X) \to \pi_0(|X|)$ sending a vertex to the path-connected component of $|X|$ containing it. It passes to the quotient by definition of $|X|$. It is clearly surjective (since Δ^n is path-connected). Let us show that it is also injective. For $\alpha \in \pi_0(X)$, we denote by X_α the simplicial subset formed by the simplices of X all of whose vertices lie in α. One easily checks that $X = \bigsqcup_{\alpha \in \pi_0(X)} X_\alpha$. Geometric realization is a left adjoint, so it preserves coproducts. Now the coproduct in Top is the disjoint union, so two vertices in different classes of $\pi_0(X)$ are sent to different connected components of $|X|$. ☐

Proposition 2.5.21. *Let X be a fibrant simplicial set, $v \in X_0$ a vertex and $n \geq 1$ an integer. Then $\pi_n(X, v) \cong \pi_n(|X|, |v|)$.*

Démonstration. The proof proceeds by induction. We just proved the case $n = 0$. Suppose now that we have shown that $\pi_n(X, v) \cong \pi_n(|X|, |v|)$ for every fibrant simplicial set. The trick is to translate the fact that $\pi_{n-1}(\Omega X) \cong \pi_n(X)$ into the simplicial setting. We need a few lemmas on fibrations and simplicial homotopy groups.

Lemma 2.5.22. *Let $p : X \to Y$ be an acyclic fibration between fibrant simplicial sets. Then $\pi_0(X) \to \pi_0(Y)$ is a bijection, and for all $v \in X_0$ and $n \geq 1$, $\pi_n(X, v) \to \pi_n(Y, f(v))$ is an isomorphism.*

[REVIEWER NOTE : A proof is missing here ; the claimed isomorphism on homotopy groups should be justified.]

Lemma 2.5.23. *Let $p : X \to Y$ be a Kan fibration between fibrant simplicial sets. Let $v \in X_0$ be a vertex and $F = Y \times_X \{p(v)\}$ the fiber of p. Then we have a long exact sequence :*

$$\cdots \to \pi_n(F, v) \to \pi_n(X, v) \to \pi_n(Y, p(v)) \to \pi_{n-1}(F, v) \to \dots$$

[REVIEWER NOTE : A proof is missing here ; the long exact sequence should be established or cited.]

Lemma 2.5.24 (Useful elsewhere)**.** *The geometric realization of a Kan fibration is a Serre fibration.*

Démonstration. See [Hov99, Corollary 3.6.2]. ☐

We can now finish proving the proposition. Let $v \in X_0$ be fixed. We define the path space at v by the pullback :

$$\begin{tikzcd}
PX \arrow[r, dashed] \arrow[d, dashed, "\pi"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}(\Delta^1, X) \arrow[d, "(d_0^*, d_1^*)"] \\
X \arrow[r, "(v \times \mathrm{id}_X)"] & X \times X
\end{tikzcd}$$

We verify that $PX \to X \to \{v\}$ is an acyclic fibration, since it is pulled back from $\mathrm{Map}(\Delta^1, X) \to \mathrm{Map}(\Delta^0, X)$ which is one. We deduce that $\pi_n(PX) = 0$ for $n > 0$. Set $\Omega X = PX \times_X \{v\}$; then the long exact sequence in homotopy tells us that $\pi_{n-1}(\Omega X) \cong \pi_n(X)$ for $n > 0$. Moreover, $|PX| \to *$ is the geometric realization of an acyclic fibration, and we saw in Lemma 2.4.8 that it is therefore an acyclic fibration, in particular a weak equivalence. Since $|\pi|$ is a Serre fibration, we recover a long exact sequence, which gives (combined with the previous isomorphism) $\pi_{n-1}(|\Omega X|, |v|) \cong \pi_n(|X|, |v|)$. We conclude by the induction hypothesis. ☐

2.5.3 End of the proof

Let us finish proving that Top and $s\mathsf{Set}$ are Quillen equivalent.

Proof of Theorem 2.5.1. The geometric realization of the generating cofibrations (resp. generating acyclic cofibrations) are cofibrations (resp. acyclic cofibrations), so the adjunction is indeed a Quillen adjunction.

We will use the criterion of Proposition 1.6.17 to show that it is an equivalence. The first thing to verify is that geometric realization reflects weak equivalences, which holds by definition. The second thing to verify is that the derived counit is a weak equivalence. Now, all simplicial sets (and in particular $S_\bullet(X)$) are cofibrant, so the derived counit is simply the counit. It therefore remains to show that for an arbitrary topological space $X \in \mathsf{Top}$ (necessarily fibrant), $\varepsilon : |S_\bullet(X)| \to X$ is a weak homotopy equivalence. In other words, we need to show that $\pi_0(|S_\bullet(X)|) \to \pi_0(X)$ is a bijection and that $\pi_n(|S_\bullet(X)|, |x|) \to \pi_n(X, x)$ is an isomorphism for all $x \in X$.

Now, the simplicial set $S_\bullet(X)$ is fibrant. Indeed, X is obviously fibrant, i.e. $X \to *$ is a Serre fibration. Therefore by Lemma 2.4.3, $S_\bullet(X)$ is a Kan fibration. We thus have $\pi_n(|S_\bullet(X)|, |v|) \cong \pi_n(S_\bullet(X), v)$ by the results of the previous section.

A class $[\alpha] \in \pi_n(S_\bullet X, v)$ is represented by a simplicial map $\alpha : \Delta^n \to S_\bullet X$ such that $\alpha(\partial\Delta^n) = v$. By adjunction, the set of such maps is in bijection with the set of maps $\bar{\alpha} : \mathbb{\Delta}^n \to X$ satisfying $\bar{\alpha}(\partial\mathbb{\Delta}^n) = |v|$. These $\bar{\alpha}$ correspond exactly to the continuous maps representing elements of $\pi_n(X, x)$. We further verify that $\mathrm{Hom}_{s\mathsf{Set}}(\Delta^n \times \Delta^1, S_\bullet(X)) \cong \mathrm{Hom}_{\mathsf{Top}}(\mathbb{\Delta}^n \times [0,1], X)$, so the equivalence relations are exactly the same and we have the desired isomorphism. ☐

2.6 Dold–Kan Correspondence

As mentioned at the beginning of the chapter, one can define simplicial and cosimplicial objects in any category C (respectively as the category of functors $\Delta^{\mathrm{op}} \to \mathsf{C}$ and $\Delta \to \mathsf{C}$). One can for example consider the category Ab of abelian groups. A simplicial abelian group $A_\bullet$ is nothing other than a sequence of abelian groups $\{A_n\}_{n \geq 0}$ equipped with group morphisms $d_i : A_n \to A_{n-1}$ and $s_j : A_n \to A_{n+1}$ satisfying the simplicial relations.

Theorem 2.6.1 (Dold–Kan [Dol58; Kan58])**.** *The category of simplicial abelian groups* $s\mathsf{Ab}$ *is equivalent to the category of positively graded chain complexes* $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$.

Démonstration. We will define an equivalence of categories

$$N_* : s\mathsf{Ab} \leftrightarrows \mathsf{Ch}_{\geq 0}(\mathbb{Z}) : \Gamma_\bullet.$$

Let $A_\bullet \in s\mathsf{Ab}$ be a simplicial abelian group. We define the chain complex $N_*(A_\bullet)$ of « normalized chains » as follows. In degree n, $N_n(A_\bullet)$ is the abelian group

$$N_n(A_\bullet) \coloneqq A_n / \bigsqcup_{j=0}^{n-1} s_j(A_{n-1}).$$

The differential $d : N_n(A_\bullet) \to N_{n-1}(A_\bullet)$ is given by the sum $d = \sum_{i=0}^{n}(-1)^i d_i$. One verifies that d passes to the quotient and that $d \circ d = 0$ thanks to the simplicial identities.

Remark 2.6.2. Let X be a topological space. Then the complex of singular chains of X is none other than $N_*(S_\bullet(X))$.

Conversely, let $C_* \in \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ be a chain complex. We define the simplicial abelian group $\Gamma_\bullet(C_*)$ as follows. In dimension n, we have :

$$\Gamma_n(C_*) = \bigoplus_{\varphi:[n]\twoheadrightarrow[p]} C_p^{(\varphi)}.$$

Let $f \in \Delta_m^n$ be an order-preserving map $[m] \to [n]$. Let us describe the structure map

$$f^* : \bigoplus_{\varphi:[n]\twoheadrightarrow[p]} C_p^{(\varphi)} \to \bigoplus_{\psi:[m]\twoheadrightarrow[q]} C_q^{(\psi)}$$

on the factor corresponding to $\varphi : [n] \twoheadrightarrow [p]$. The morphism $\varphi \circ f$ factors uniquely as a surjection followed by an injection $[m] \xrightarrow{\psi} [q] \overset{\partial}{\hookrightarrow} [p]$.

- If $p = q$ (and thus $\partial = \mathrm{id}_{[p]}$), then f^* sends $C_p^{(\varphi)}$ to $C_q^{(\psi)} = C_p^{(\psi)}$.
- If $p = q + 1$ and $\partial = \partial^p$, then f^* sends $C_p^{(\varphi)}$ to $C_q^{(\psi)} = C_{p-1}^{(\psi)}$ via the differential.
- In all other cases, f^* vanishes on $C_p^{(\varphi)}$.

One must then verify that these two constructions define functors, and that these two functors are inverses of one another (exercise). ☐

Example 2.6.3. Let us try to understand $d_2 : \Gamma_2 C \to \Gamma_1 C$. Recall that $d_0 = (\partial^0)^*$ where $\partial^0 = 1 \to 2 \in \Delta_1^2$.

- We have $\Gamma_1 C = C_1^{(\psi_1)} \oplus C_0^{(\psi_0)}$ where the first factor is indexed by $\psi_1 = 0 \to 1 \in \Delta_1^1$ and the second by $\psi_0 = 0 \to 0 \in \Delta_1^0$.
- We also have $\Gamma_2 C = C_2^{(\varphi_2)} \oplus C_1^{(\varphi_1)} \oplus C_1^{(\varphi_1')} \oplus C_0^{(\varphi_0)}$, where the factors are indexed by $\varphi_2 = 0 \to 1 \to 2 \in \Delta_2^2$, $\varphi_1 = 0 \to 1 \to 1 \in \Delta_2^1$, $\varphi_1' = 0 \to 0 \to 1 \in \Delta_2^1$, and $\varphi_0 = 0 \to 0 \to 0 \in \Delta_2^0$.

We can then describe d_2 on each factor of $\Gamma_2 C$:

- On $x \in C_2^{(\varphi_2)}$, we have $\varphi_2 \circ \partial^2 = 0 \to 1 \in \Delta_1^2$. This map factors as $[1] \xrightarrow{\psi_1} [1] \xrightarrow{\partial^2} [2]$. Thus $d_2 x = dx \in C_1^{(\psi_1)}$.

- On $x \in C_1^{(\varphi_1)}$, we have $\varphi_1 \circ \partial^2 = 0 \to 1 \in \Delta_1^1$. This map again factors as $[1] \xrightarrow{\psi_1} [1] \xrightarrow{\mathrm{id}_{[1]}} [1]$, so $d_2 x = x \in C_1^{(\psi_1)}$.
- On $x \in C_1^{\varphi'_1}$, we have $\varphi'_1 \circ \partial^2 = 0 \to 0 \in \Delta_1^1$. This map factors as $[1] \xrightarrow{\psi_0} [0] \xrightarrow{\partial^1} [1]$. Thus $d_2 x = dx \in C_0^{(\psi_0)}$.
- On $x \in C_0^{(\varphi_0)}$, we have $\varphi_0 \circ \partial^2 = 0 \to 0 \in \Delta_0^2$. This map factors as $[1] \xrightarrow{\psi_0} [0] \xrightarrow{\mathrm{id}_{[0]}} [0]$, so $d_2 x = x \in C_0^{(\psi_0)}$.

With a diagram :

$$\begin{tikzcd}
\Gamma_2 C \arrow[d, "d_2"] & = & C_2^{(\varphi_2)} \arrow[drr, "d"] & \oplus & C_1^{(\varphi_1)} \arrow[d, "\mathrm{id}"] & \oplus & C_1^{(\varphi'_1)} \arrow[d, "d"] & \oplus & C_0^{(\varphi_0)} \arrow[dll, "\mathrm{id}"] \\
\Gamma_1 C & = & & & C_1^{(\psi_1)} & \oplus & C_0^{(\psi_0)} & &
\end{tikzcd}$$

One can also easily compute that $s_0 : \Gamma_1 C \to \Gamma_2 C$ identifies $C_0^{(\psi_0)}$ with $C_0^{(\varphi_0)}$ and $C_1^{(\psi_1)}$ with $C_1^{(\varphi_1)}$, while s_1 identifies $C_0^{(\psi_0)}$ with $C_0^{(\varphi_0)}$ and $C_1^{(\psi_1)}$ with $C_1^{(\varphi'_1)}$.

Theorem 2.6.4 (Quillen [Qui67])**.** *The category* $s\mathsf{Ab}$ *admits a model category structure where the weak equivalences and fibrations are defined by considering the underlying simplicial sets, and the cofibrations are the morphisms having the left lifting property with respect to acyclic fibrations. With this structure on* $s\mathsf{Ab}$ *and the projective structure on* $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$*, the adjunction* $N \dashv \Gamma$ *is a Quillen equivalence.*

We refer for example to [GJ99, Section III.2] for the proof. This Quillen equivalence has nice properties. For example :

Proposition 2.6.5. *Let* $A_\bullet$ *be a simplicial abelian group. Then we have isomorphisms for all* $n \geq 0$ *:*

$$\pi_n(A, 0) \cong H_n(N_*(A_\bullet)).$$

One can therefore easily construct a space of type $K(A, n)$: it suffices to consider the chain complex $D^n(A)$ with $D^n(A)_n = A$ and $D^n(A)_k = 0$ for $k \neq n$; then $|\Gamma_\bullet(D^n(A))|$ is a topological space of type $K(A, n)$. By playing with the adjunctions and the fact that $C_*(X) = C_*(S_\bullet(X))$, one can also show that

$$\Big[X,\ |\Gamma_\bullet(D^n(A))|\Big] \cong H^n(X; A).$$

3 Rational homotopy theory

Homotopy theory is a powerful theory, but computations can prove to be extremely complex. For example, computing the homotopy groups of simple spaces (e.g. spheres) remains an inaccessible task to this day.

Rational homotopy theory offers a compromise between computability and the amount of information given about a space. In this theory, one « forgets » the torsion and non-commutativity in the homotopy groups of a space. In doing so, one loses information (the projective plane becomes contractible over $\mathbb{Q}$, for example) but gains in computability : the rational homotopy groups of spheres are completely determined, which is not the case over $\mathbb{Z}$.

In this chapter, we will focus on the rational homotopy theory of Sullivan [Sul77]. The idea is as follows. For simplicity, we consider simply connected spaces. One can detect whether a continuous map $f : X \to Y$ is a weak homotopy equivalence by considering the induced maps on $\pi_*(-)$, or else (since the spaces are simply connected) on $H_*(-;\mathbb{Z})$. Now, the homotopy groups $\pi_{\geq 2}$ and the homology groups $H_*(-;\mathbb{Z})$ are abelian, and one can therefore « rationalize » them by applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$. This operation kills the torsion and retains only the ranks of the abelian groups (if they are of finite type). One can therefore introduce a new class « of equivalences », the rational homotopy equivalences, which are the maps inducing an isomorphism on $\pi_*(-) \otimes_{\mathbb{Z}} \mathbb{Q}$ or equivalently on $H_*(-;\mathbb{Q})$.

Rational homotopy theory is concerned with the question of when two spaces are rationally equivalent. To this end, one asks which invariants are preserved by rational homotopy equivalences. For various reasons, one restricts to simply connected spaces. Sullivan's foundational theory gives the answer : all possible rational invariants of X are contained in the commutative differential graded algebra (CDGA) of piecewise polynomial forms $\Omega^*_{\mathrm{PL}}(X)$. This CDGA is analogous to that of de Rham differential forms on a smooth manifold, $\Omega^*_{\mathrm{dR}}(X)$, the latter being moreover quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$. More precisely, the main theorem of this chapter states that there exists a model category structure on simply connected topological spaces whose weak equivalences are the rational homotopy equivalences, and a Quillen equivalence between this category and simply connected CDGAs embodied by the functor Ω^*_{PL}.

Convention 3.0.1. Throughout this chapter, the base field will be $\mathbb{Q}$.

3.1 Bousfield Localization

In this section, we localize $s\mathsf{Set}$ with respect to the field $\mathbb{Q}$.

Definition 3.1.1. Let $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ be a model category. A *left Bousfield localization* of M is a model category structure $(\mathsf{M}, \mathscr{W}_{\mathrm{loc}}, \mathscr{C}_{\mathrm{loc}}, \mathscr{F}_{\mathrm{loc}})$ having the same cofibrations ($\mathscr{C}_{\mathrm{loc}} = \mathscr{C}$) and more weak equivalences ($\mathscr{W}_{\mathrm{loc}} \supset \mathscr{W}$).[1]

Remark 3.1.2. Given a model category $(\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ and a class of weak equivalences $\mathscr{W}_{\mathrm{loc}} \supset \mathscr{W}$, it is not always true that the associated left Bousfield localization exists. One may refer to [Hir03, Theorem 4.1.1] for an example of an existence theorem.

Remark 3.1.3. Since fibrations are determined as the morphisms having the RLP with respect to acyclic cofibrations, we deduce that a left Bousfield localization has fewer fibrations than the original category, $\mathscr{F}_{\mathrm{loc}} \subset \mathscr{F}$ (since the condition is more restrictive). On the other hand, the acyclic fibrations are the same ($\mathscr{F}_{\mathrm{loc}} \cap \mathscr{W}_{\mathrm{loc}} = \mathscr{F} \cap \mathscr{W}$) since these are given by the RLP with respect to cofibrations.

Proposition 3.1.4. *Let* $\mathsf{M} = (\mathsf{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ *be a model category and* $\mathsf{M}_{\mathrm{loc}} = (\mathsf{M}, \mathscr{W}_{\mathrm{loc}}, \mathscr{C}, \mathscr{F}_{\mathrm{loc}})$ *a left Bousfield localization. Then* $\mathrm{id}_{\mathsf{M}} : \mathsf{M}_{\mathrm{loc}} \leftrightarrows \mathsf{M} : \mathrm{id}_{\mathsf{M}}$ *is a Quillen adjunction, and the derived adjunction* $\mathbb{L}\,\mathrm{id} : \mathrm{Ho}(\mathsf{M}_{\mathrm{loc}}) \leftrightarrows \mathrm{Ho}(\mathsf{M}) : \mathbb{R}\,\mathrm{id}$ *exhibits* $\mathrm{Ho}(\mathsf{M}_{\mathrm{loc}})$ *as a reflective subcategory.*[2]

Definition 3.1.5. A simplicial set $X_\bullet$ is called 1-*reduced* if $X_0 = X_1 = *$ are singletons. We denote by $s\mathsf{Set}_{\geq 2} \subset s\mathsf{Set}$ the full subcategory of 1-reduced simplicial sets.

Remark 3.1.6. The geometric realization of a 1-reduced simplicial set is a simply connected topological space.

Proposition 3.1.7. *The category* $s\mathsf{Set}_{\geq 2}$ *inherits a model category structure from* $s\mathsf{Set}$.

Démonstration. One needs to verify that the construction of functorial factorizations by the small object argument preserves the subcategory of 1-reduced spaces. ☐

Theorem 3.1.8 (Serre [Ser53])**.** *Let* $f : X_\bullet \to Y_\bullet$ *be a simplicial map between* 1*-reduced simplicial sets. The following statements are equivalent :*

1. $f_* : H_*(|X|; \mathbb{Q}) \to H_*(|Y|; \mathbb{Q})$ *is an isomorphism ;*
2. $f^* : H^*(|Y|; \mathbb{Q}) \to H^*(|X|; \mathbb{Q})$ *is an isomorphism ;*
3. $f_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is an isomorphism.*

Definition 3.1.9. A simplicial map satisfying the conditions of the preceding theorem is called a *rational equivalence.* We denote by $\sim_{\mathbb{Q}}$ the rational equivalences.

Theorem 3.1.10 (Quillen [Qui69])**.** *Let* $\mathscr{W}_{\mathbb{Q}}$ *be the class of rational equivalences in* $s\mathsf{Set}_{\geq 2}$*. Then the left Bousfield localization of* $s\mathsf{Set}_{\geq 2}$ *with respect to* $\mathscr{W}_{\mathbb{Q}}$ *exists. We denote it* $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$.

We can describe the fibrant objects of $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$.

---

[1] A right Bousfield localization is a model category structure having the same fibrations and more weak equivalences. We will not use this notion here.

[2] It is a full subcategory and the inclusion has a left adjoint.

Definition 3.1.11. A simplicial set is *rational* if all its homotopy groups $\pi_n(X)$ are $\mathbb{Q}$-vector spaces.

Proposition 3.1.12. *A* 1*-reduced simplicial set is fibrant in* $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$ *if and only if it is fibrant in* $s\mathsf{Set}_{\geq 2}$ *(i.e. it is a Kan complex) and it is rational.*

Démonstration. Let X be a 1-reduced simplicial set.

Suppose that $X \to *$ is a fibration in $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$. Since $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$ has fewer fibrations than $s\mathsf{Set}_{\geq 2}$, we deduce that $X \to *$ is a Kan fibration. Let us show that moreover X is rational. Let $n \geq 2$. The canonical map $k_* : S^n \to S^n$ of degree $k \geq 2$ is a rational equivalence, which we can replace by a cofibration $i : S^n \xhookrightarrow{\sim_{\mathbb{Q}}} A$ with $A \simeq S^n$. The map i is an acyclic cofibration in $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$. We can therefore find a lift in the following diagram :

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d, hook, "\sim_{\mathbb{Q}}"'] & X \arrow[d, two heads] \\
A \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

which shows that $\pi_n(X) \xrightarrow{\cdot k} \pi_n(X)$ is a bijection. The converse uses the theory of minimal fibrations (see Section 2.4) and we refer to [Qui69, Proposition 2.4] ☐

Corollary 3.1.13. *Every* 1*-reduced simplicial set has a rational replacement* $X \xrightarrow{\sim} X_{\mathbb{Q}}$*, where* $X_{\mathbb{Q}}$ *is a rational simplicial set.* ☐

3.2 Commutative Differential Graded Algebras

3.2.1 Definitions

Definition 3.2.1. A *cochain complex* (positively graded) is a graded vector space $V = \bigoplus_{n \geq 0} V^n$ equipped with differentials $d : A^n \to A^{n+1}$ satisfying $d \circ d = 0$. We denote by $\mathsf{Ch}^{\geq 0}(\mathbb{Q})$ the category of cochain complexes and their morphisms.

We write $\deg a = p$ for the degree of a homogeneous element $a \in V^p$. If the notation $\deg a$ appears in an equation, we assume by default that the element is homogeneous, extending linearly to the whole graded space as needed.

Definition 3.2.2. A *differential graded algebra* (*DGA*) is a cochain complex A equipped with a bilinear map $\mu : A \otimes A \to A$ and a unit $\eta : \mathbb{Q} \to A$ that is associative and unital :

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu \otimes \mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1 \otimes \eta"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes A \arrow[d, "\mu"] & A \arrow[l, "\eta \otimes 1"'] \arrow[dl, "\mathrm{id}_A"] \\
& A &
\end{tikzcd}$$

and which satisfies the Leibniz rule :

$$d(\mu(a \otimes b)) = \mu(da \otimes b) + (-1)^{\deg a} \mu(a \otimes db).$$

A morphism of DGAs $f : A \to A$ is a linear map that preserves the product, the unit, and the differential.

In the following, we will simply write $a \cdot b$ (or even ab) for the product $\mu(a \otimes b)$, and $1 = \eta(1) \in A^0$ will be the unit. Associativity then reads $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, unitality reads $a \cdot 1 = a = 1 \cdot a$, and the Leibniz rule reads $d(a \cdot b) = da \cdot b + (-1)^{\deg a} a \cdot db$.

Proposition 3.2.3. *Let A be a DGA. Then the cohomology $H^*(A)$ is a graded algebra.*

Démonstration. This follows immediately from the Leibniz rule. □

Definition 3.2.4. A *commutative differential graded algebra* (*CDGA*) is a DGA satisfying the following property : if $a \in A^p$ and $b \in A^q$ are homogeneous elements, then $b \cdot a = (-1)^{(\deg a)(\deg b)}(a \cdot b)$. We denote by CDGA the category of CDGAs and their morphisms.

Proposition 3.2.5. *The cohomology of a CDGA is a commutative graded algebra.* □

Remark 3.2.6. If $a \in A$ has odd degree, then the equation $a \cdot a = -a \cdot a$ implies $a^2 = 0$. On the other hand, if $b \in A$ has even degree, then it commutes with all elements of a (including those of odd degree).

Let V be a cochain complex. The symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ via the "Koszul sign rule" (inspired by the definition of a commutative graded algebra). Concretely, if $\sigma_i = (i\ i{+}1) \in \mathfrak{S}_n$ is a transposition, then :

$$(v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) \cdot \sigma_i = (-1)^{|v_i|\,|v_{i+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n.$$

This action is extended to $\mathfrak{S}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i^2 = 1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$ by verifying compatibility with the relations. We will sometimes abusively denote the sign by ±.

Definition 3.2.7. Let G be a group and V a representation of G. We define the *coinvariants* V_G as the quotient $V/{\sim}$ where $x \sim g \cdot x$. The *invariants* V^G are the subspace $\{v \in V \mid g \cdot x = x\}$.

Definition 3.2.8. Let V be a cochain complex. The *symmetric algebra* on V, denoted $S(V)$, is given by :

$$S(V) \coloneqq \bigoplus_{r \geq 0} S^{(r)}(V) \coloneqq \bigoplus_{r \geq 0} (V^{\otimes r})_{\mathfrak{S}_r}.$$

The differential is given by the Leibniz rule :

$$d(v_1 \dots v_n) = \sum_{i=1}^{n} (-1)^{|v_1| + \dots + |v_{i-1}|} v_1 \dots v_{i-1} (dv_i) v_{i+1} \dots v_n.$$

Concretely, $S(V)$ is the tensor product $\mathbb{Q}[V^{\text{even}}] \otimes \Lambda(V^{\text{odd}})$ of the polynomial algebra on even-degree elements and the exterior algebra on odd-degree elements. To compute the differential, one uses the Koszul sign rule by assuming that the symbol d has degree 1.

Proposition 3.2.9. *There is an adjunction :*

$$S : \mathsf{Ch}^{\geq 0}(\mathbb{Q}) \leftrightarrows \mathsf{CDGA} : U,$$

where S is the "symmetric algebra" functor and U is the "forgetful functor" which associates to a CDGA its underlying cochain complex. ☐

Concretely, this means that a morphism of CDGAs $f : S(V) \to A$ is uniquely determined by its restriction to $V \subset S(V)$.

Corollary 3.2.10. *There is a natural isomorphism $S(V \oplus W) \cong S(V) \otimes S(W)$.*

Démonstration. Since S is a left adjoint functor, it preserves colimits, which are respectively $\oplus$ in $\mathsf{Ch}^{\geq 0}(\mathbb{Q})$ and $\otimes$ in CDGA. ☐

Proposition 3.2.11. *Let $A = S(V)$ be the free CDGA on a cochain complex V. Then $H^*(A)$ is the free symmetric algebra on $H^*(V)$.*

Démonstration. The proof uses the Künneth formula and the fact that if G is a finite group acting on a cochain complex C (in $\mathbb{Q}$-vector spaces), then $H^*(C_G) \cong H^*(C)_G$. Note that the proof only works in characteristic zero. In characteristic p, the fact that the characteristic divides the order of $\mathfrak{S}_n$ for $n \geq p$ causes problems. ☐

Definition 3.2.12. Let A be a CDGA and $k \in \mathbb{Z}$ an integer. A *derivation* (of degree k) $\delta : A \to A$ is a linear map of degree k satisfying the Leibniz rule $\delta(ab) = \delta a \cdot b + (-1)^{k \cdot \deg a} a \cdot \delta b$.

Example 3.2.13. The differential of a CDGA is a derivation of degree 1.

Proposition 3.2.14. *Let $A = S(V)$ be a free CDGA on a cochain complex V. A derivation $\delta : A \to A$ is uniquely determined by its restriction to $V \subset A$.*

Démonstration. Almost identical to the proof that S is adjoint to U. ☐

Let us now describe a generalization of free CDGAs. This generalization will be used to describe the cofibrant objects of CDGA in the next section.

Definition 3.2.15. A CDGA A is called *quasi-free* if its underlying commutative graded algebra is free.

This means that $A = (S(V), d)$ where V is a graded vector space and $d : S(V) \to S(V)$ is a derivation of degree 1 satisfying $d \circ d = 0$. As in the case of free CDGAs, this derivation d is uniquely determined by its restriction $d|_V$ to $V \subset S(V)$. However, the image of $d|_V$ is not necessarily contained in V and may involve polynomials of higher weight. This restricted differential $d|_V$ in fact decomposes as $d_0 + d_1 + \dots$ where $d_i : V \to S^{(i)}(V) = (V^{\otimes i})_{\mathfrak{S}_i}$ is the term of weight i.

Example 3.2.16. Consider the graded vector space V generated by two variables $x = x_2$ and $y = y_3$, of respective degrees 2 and 3. The symmetric algebra on V is the tensor product $S(V) = \mathbb{Q}[x] \otimes \Lambda(y)$. We define a derivation $d : S(V) \to S(V)$ by $dx = 0$ and $dy = x^2$, extended by the Leibniz rule. More precisely, we compute that $d(x^k) = 0$ and $d(x^k y) = x^{k+2}$. One then easily verifies that $d \circ d = 0$, so $(S(V), d)$ is a quasi-free CDGA. It is not free on V, since $dy = x^2$ is not linear.

The equation $d \circ d$ transforms into a complicated series of equations in terms of the d_i. We note in particular that d_1 is a differential on V. A morphism $f : (S(V), d) \to (S(W), d)$ between two quasi-free CDGAs is entirely determined by its restriction $f|_V : V \to S(W)$, which decomposes as $f_0 + f_1 + \dots$ where $f_i : V \to S^{(i)}(W)$. The equation $f \circ d = d \circ f$ also becomes a complicated series of equations. One of these relations states that $f_1 d_1 = d_1 f_1$. We can therefore define :

Definition 3.2.17. Let $f : (S(V), d) \to (S(W), d)$ be a morphism between quasi-free CDGAs. The *linear part* of f is the induced map :

$$f_1 : (V, d_1) \to (W, d_1).$$

An analogue of this linear part can be defined for augmented CDGAs.

Definition 3.2.18. An *augmentation* of a CDGA A is a morphism of CDGAs $\varepsilon : A \to \mathbb{Q}$ (which therefore satisfies $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$, $\varepsilon(1) = 1$, and $\varepsilon(da) = 0$). An *augmented* CDGA is a CDGA A equipped with an augmentation ε. If A is an augmented CDGA, we denote by $\bar{A} = \ker \varepsilon$ its *augmentation ideal.*

Definition 3.2.19. Let A be an augmented CDGA. Its *indecomposables complex* QA is the quotient $QA = \bar{A}/\bar{A} \cdot \bar{A}$ equipped with the induced differential. We then call the "homotopy groups of A"[3]

$$\pi_n(A) = H^n(QA).$$

Example 3.2.20. If $A = (S(V), d)$ is quasi-free, then $QA \cong (V, d_1)$.

3.2.2 Transfer of the model category structure

We will seek to apply the following theorem to Proposition 3.2.9.

Definition 3.2.21. Let $F : \mathsf{D} \leftrightarrows \mathsf{M} : U$ be an adjunction, where D is a model category. We say that a morphism $f \in \mathrm{Hom}_{\mathsf{M}}(X, Y)$ is a fibration (resp. a weak equivalence) if $U(f)$ is one, and that it is a cofibration if f has the LLP with respect to all acyclic fibrations. The model category structure thus defined on M – if it exists – is called the *right transferred structure.*

Theorem 3.2.22 (Quillen [Qui67])**.** *Let D be a model category cofibrantly generated by $(\mathcal{I}, \mathcal{J})$. Suppose that the sources of the morphisms of $\mathcal{I}$ and $\mathcal{J}$ are κ-compact for some given cardinal κ. Let $F : \mathsf{D} \leftrightarrows \mathsf{M} : U$ be an adjunction, where M is a complete and cocomplete category.*

If the following conditions are satisfied :

1. *the functor U preserves κ-sequential colimits ;*
2. *one or the other of the following conditions holds :*

---

[3] These are rather its André–Quillen homology. We will see in Section 3.3 that this indeed corresponds to the homotopy groups of a topological space.

a) *if a morphism of* M *has the LLP with respect to all fibrations, then it is a weak equivalence ;*

b) or *for every* $A \in \mathsf{M}$*, every* $X \xrightarrow{f} Y \in \mathcal{J}$*, and every* $F(X) \to A$*, the canonical map* $A \to A \sqcup_{F(X)} F(Y)$ *is sent to a weak equivalence by the functor* U *;*

then the right transferred structure on M *defines a model category structure, cofibrantly generated by* $(F(\mathcal{I}), F(\mathcal{J}))$ *(whose sources are* κ*-compact), and the adjunction* $F \dashv U$ *is a Quillen adjunction.*

Before proving the theorem, let us first give the application. Recall that $S^n(\mathbb{Q}) \in \mathsf{Ch}^{\geq 0}(\mathbb{Q})$ is the complex given by $\mathbb{Q}$ concentrated in degree n, and $D^n(\mathbb{Q}) = \cdots \to 0 \to \mathbb{Q} \xrightarrow{\mathrm{id}_{\mathbb{Q}}} \mathbb{Q} \to 0 \to \dots$ concentrated in degrees $n-1$ and n.

Proposition 3.2.23. *There exists a model category structure on* $\mathsf{Ch}^{\geq 0}(\mathbb{Q})$ *(called the* projective structure*) whose weak equivalences are the quasi-isomorphisms, whose fibrations are the surjective morphisms in every degree, and whose cofibrations are the injective morphisms in strictly positive degree. It is cofibrantly generated by* $\mathcal{I} = \{S^n(\mathbb{Q}) \to D^n(\mathbb{Q})\}$ *and* $\mathcal{J} = \{0 \to D^n(\mathbb{Q})\}$*.*

Démonstration. Identical to what can be found in Section 1.5.1. □

Corollary 3.2.24. *The adjunction* $S : \mathsf{Ch}^{\geq 0}(\mathbb{Q}) \leftrightarrows \mathsf{CDGA} : U$ *satisfies the hypotheses of Theorem 3.2.22. The right transferred model category structure on* CDGA *therefore exists.*

Démonstration. A κ-sequential colimit in CDGA is computed as a κ-sequential colimit in $\mathsf{Ch}^{\geq 0}(\mathbb{Q})$ equipped with a canonical CDGA structure. Condition 1. of the theorem is therefore satisfied.

We will now verify condition 2.b of the theorem. We must therefore verify that for every CDGA A and every $n \geq 0$, the canonical map (in the category of CDGAs) $A \to A \otimes S(D^n(\mathbb{Q}))$ is a quasi-isomorphism. By the Künneth formula, $H^*(A \otimes S(D^n(\mathbb{Q}))) = H^*(A) \otimes H^*(S(D^n(\mathbb{Q})))$. By Proposition 3.2.11, $H^*(S(D^n(\mathbb{Q}))) = S(H^*(D^n(\mathbb{Q})))$ and it is immediate that $D^n(\mathbb{Q})$ is acyclic. We therefore deduce that $H^*(A \otimes S(D^n(\mathbb{Q}))) = H^*(A)$, and one easily verifies that $A \to A \otimes S(D^n(\mathbb{Q}))$ induces the identity in cohomology. □

Remark 3.2.25. We used the hypothesis that the base field was $\mathbb{Q}$ (or more generally a field of characteristic zero) twice :

- to apply the Künneth formula, it is necessary to work over a field ;
- for $H^*(S(V)) = S(H^*(V))$, it is necessary to have a field of characteristic zero.

Proof of Theorem 3.2.22. Let us verify that M is a model category with the considered classes. Axiom (MC1) – complete + cocomplete – holds by hypothesis. Axiom (MC2) – 2 out of 3 – holds because U is a functor and D satisfies (MC2). Similarly, the fibrations and weak equivalences of M are stable under retracts ; since the classes of morphisms defined by a lifting property are stable under retracts, we deduce that the cofibrations of M are also stable under retracts and that axiom (MC3) is satisfied.

Axiom (MC4)(i) – $\mathscr{C} \perp \mathscr{F} \cap \mathscr{W}$ – holds by definition of the cofibrations of M. It remains to verify (MC4)(ii) – $\mathscr{C} \cap \mathscr{W} \perp \mathscr{F}$ – and (MC5) – the factorizations. We will need a few lemmas.

Lemma 3.2.26. *A morphism $f \in \mathrm{Hom}_{\mathsf{M}}(X,Y)$ is a fibration (resp. acyclic fibration) if and only if it has the RLP with respect to $F(\mathcal{J})$ (resp. $F(\mathcal{I})$).*

Démonstration. Follows from the fact that $F \dashv U$ and that D is cofibrantly generated by $(\mathcal{I},\mathcal{J})$. ☐

Lemma 3.2.27. *Retracts of $F(\mathcal{I})$-cell complexes (resp. $F(\mathcal{J})$-cell complexes) have the LLP with respect to acyclic fibrations (resp. fibrations).*

Démonstration. Follows from the previous point and the stability of classes of type ${}^{\perp}(-)$ under pushouts and retracts. ☐

Lemma 3.2.28. *Every retract of a $F(\mathcal{J})$-cell complex is a weak equivalence in* M.

Démonstration. Corollary of the previous lemma and condition 2. of the theorem. ☐

We can now prove (MC5) using the small object argument. Given $f :\in \mathrm{Hom}_{\mathsf{M}}(X,Y)$, we can factor it as $X \to G^\infty(F(\mathcal{I}),f) \to Y$. In the first case, $X \to G^\infty(F(\mathcal{I}),f)$ is an $F(\mathcal{I})$-cell complex, hence a cofibration ; moreover $G^\infty(F(\mathcal{I}),f) \to Y$ has the RLP with respect to $F(\mathcal{I})$, so it is an acyclic fibration. The other part of (MC5) is similar, replacing $\mathcal{I}$ by $\mathcal{J}$.

Let us finally prove (MC4)(ii). Let $i : A \to B$ be an acyclic cofibration, i.e., i has the LLP with respect to acyclic fibrations and $U(i)$ is a weak equivalence. Let us show that i has the LLP with respect to fibrations. Using the small object argument, we can factor i as $A \xrightarrow{i_\infty} G^\infty(F(\mathcal{J}),i) \xrightarrow{p_\infty} B$, where i_∞ is a $\mathcal{J}$-cell complex and p_∞ is a fibration. By 2 out of 3, p is in fact an acyclic fibration. We can therefore find a lifting in the following diagram :

$$\begin{tikzcd}
A \arrow[r, hook, "j", "\sim"'] \arrow[d, hook, "i_\infty"] & G^\infty(F(\mathcal{J}),i) \arrow[d, "p_\infty"] \\
B \arrow[ur, dashed, "l"] \arrow[r, equal] & B
\end{tikzcd}$$

We then obtain that i is a retract of i_∞. Now i_∞ is a $\mathcal{J}$-cell complex and therefore has the LLP with respect to fibrations by one of the lemmas above, which allows us to conclude. ☐

Example 3.2.29 (Exercise)**.** Another example of an application of the transfer theorem is the following. Let A be a CDGA. An A-module is a cochain complex M equipped with a bilinear map $A \otimes_{\mathbb{Q}} M \to M$ satisfying $a \cdot (b \cdot m) = ab \cdot m$ and $1 \cdot m = m$. We denote by Mod_A the category of A-modules and their morphisms. There is a forgetful functor $\mathsf{Mod}_A \to \mathsf{Ch}^{\geq}(\mathbb{Q})$ which forgets the action of A. One can then verify that this functor satisfies the axioms of the transfer theorem, and that Mod_A is equipped with a projective model category structure.

If $f : A \to B$ is a morphism of CDGAs, then there is a Quillen adjunction :

$$f_! : \mathsf{Mod}_A \leftrightarrows \mathsf{Mod}_B : f^*.$$

The functor f^* is defined by f^*M with an action given by $a \cdot m = f(a) \cdot m$. The functor $f_!$ is given by $f_!N = B \otimes_A N$, where A acts on B via f, and the action is $b \cdot (x \otimes m) = bx \otimes m$. If f is a quasi-isomorphism, then this adjunction is a Quillen equivalence.

3.2.3 Sullivan theory

All CDGAs are fibrant. In this section, we will describe the cofibrant objects. We will also explicitly describe how to represent homotopies between morphisms of CDGAs.

Definition 3.2.30. A *Sullivan algebra* is a quasi-free CDGA $A = (S(V), d)$ where $V = V^{\geq 1}$ and which is equipped with a filtration of the cochain complex V by subcomplexes :

$$0 \subset V(0) \subset V(1) \subset V(2) \subset \cdots \subset V,$$

such that $d(V(0)) = 0$ and $d(V(k+1)) \subset S(V(k))$ for $k \geq 0$.

A *relative Sullivan algebra* is an inclusion $A \to (A \otimes S(V), d)$ where $V = V^{\geq 1}$ is filtered $V(0) \subset V(1) \subset \dots V$ such that $d(V(0)) \subset A$ and $d(V(k+1)) \subset A \otimes S(V(k))$.

Definition 3.2.31. A *minimal algebra* is a Sullivan algebra $A = (S(V), d)$ such that $d(V(k)) \subset S^{(\geq 2)}(V(k-1))$, i.e., the differential is decomposable. A *relative minimal algebra* is a relative Sullivan algebra satisfying an analogous condition.

Example 3.2.32. The CDGA $A = (S(x_2, y_3), dy = x^2)$ is a minimal algebra. Indeed, one can set $V(0) = \langle x \rangle$, $V(1) = \langle x, y \rangle$.

Example 3.2.33. The CDGAs $S(D^n(\mathbb{Q}))$ and $S(S^n(\mathbb{Q}))$ are Sullivan algebras. The first is not minimal but the second is. The inclusion $S(S^n(\mathbb{Q})) \to S(D^n(\mathbb{Q}))$ is a relative minimal algebra.

Example 3.2.34. The algebra $A = (S(x_1, y_1, z_1), dx = yz, dy = zx, dz = xy)$ is not a Sullivan algebra.

Proposition 3.2.35. *A CDGA A is cofibrant if and only if it is a retract of a Sullivan algebra. Similarly, $i : A \to B$ is a cofibration if and only if it is a retract of a relative Sullivan algebra.*

Démonstration. The generating cofibrations $\mathcal{I} = \{S(S^n(\mathbb{Q})) \to S(D^n(\mathbb{Q}))\}$ are relative Sullivan algebras, so all cofibrations are relative Sullivan algebras.[4] □

Remark 3.2.36. A minimal algebra $A = (S(V), d)$ is automatically augmented, and its homotopy groups (Definition 3.2.19) are given by $\pi_n(A) = V^n$.

Proposition 3.2.37. *Let $f : A = (S(V), d) \to B = (S(W), d)$ be a morphism between Sullivan algebras. Then f is a quasi-isomorphism if and only if $f_1 : (V, d_1) \to (W, d_1)$ is one. If moreover both algebras are minimal, then f is a quasi-isomorphism if and only if it is an isomorphism.*

---

[4] To find the filtration, one considers the filtration $G^0(\mathcal{I}, -) \subset G^1(\mathcal{I}, -) \subset \cdots \subset G^\infty(\mathcal{I}, -)$ in the small object argument.

Démonstration. The proof of the first statement relies on spectral sequence arguments that we will not detail here (exercise).

For the second statement, it is clear that if f is an isomorphism then it is a quasi-isomorphism. Conversely, if f is a quasi-isomorphism, then by the first point f_1 is a quasi-isomorphism. Since the algebras are minimal, $d_1 = 0$, so f_1 is in fact an isomorphism. We have a morphism of long exact sequences (associated to short exact sequences of the type $\bar{A}^2 \to \bar{A} \to QA$) :

$$\begin{tikzcd}
0 \arrow[r] & H^1(\bar{A}^2) = 0 \arrow[r] \arrow[d, equal] & H^1(\bar{A}) \arrow[r, phantom, "="] \arrow[d, "\cong"', "f_1"] & H^1(QA) \arrow[r] \arrow[d, "\cong"', "f_1"] & H^2(\bar{A}^2) \arrow[r] \arrow[d] & H^2(\bar{A}) \arrow[r] \arrow[d] & H^2(QA) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & H^1(\bar{B}^2) = 0 \arrow[r] & H^1(\bar{B}) \arrow[r, phantom, "="] & H^1(QB) \arrow[r] & H^2(\bar{B}^2) \arrow[r] & H^2(\bar{B}) \arrow[r] & H^2(QB) \arrow[r] & \dots
\end{tikzcd}$$

We then apply the five lemma by induction and use the fact that $H^n(\bar{A}^2)$ is expressed solely in terms of $V^{<n}$ (thanks to $V = V^{\geq 1}$) to conclude. □

Definition 3.2.38. A *Sullivan model* (resp. *minimal model*) of a CDGA A is a Sullivan algebra (resp. minimal algebra) $(S(V), d)$ equipped with a quasi-isomorphism $(S(V), d) \xrightarrow{\sim} A$. A Sullivan model (resp. minimal model) of a morphism $f : A \to B$ is a factorization $A \to (A \otimes S(V), d) \xrightarrow{\sim} B$ where $A \to (A \otimes S(V), d)$ is a relative Sullivan (resp. minimal) algebra.

Using the axioms of model categories, every CDGA has a Sullivan model, and every morphism has a Sullivan model. Two minimal models of the same CDGA are isomorphic.

Example 3.2.39. Consider the cohomology of the sphere S^2. It is given by $H^*(S^2) = \mathbb{Q} \oplus \mathbb{Q}x_2$, where x is an element of degree 2 with square zero. This algebra is not free on x despite appearances : since $\deg x$ is even, the relation $x^2 = 0$ is non-trivial. To find a minimal model, we “resolve” the relation $x^2 = 0$ by adding a generator y whose differential “kills” x^2. We then obtain $A = (S(x_2, y_3), dy = x^2)$ which is indeed a minimal algebra, and the morphism $A \to H^*(S^2)$ given by $x \mapsto x$ and $y \mapsto 0$ is indeed a quasi-isomorphism.

Example 3.2.40. A fairly interesting case of a minimal model of a morphism is that of the multiplication[5] $\mu : A \otimes A \to A$ where $A = (S(V), d)$ is a minimal algebra. One must find a factorization $(S(V) \otimes S(V), d) \to (S(V) \otimes S(V) \otimes S(W), D) \xrightarrow{\sim} (S(V), d)$ of μ, where the first map is the canonical inclusion and the second is a quasi-isomorphism. For W, we take $V[-1]$, a copy of V shifted in degree by 1 upward (i.e., if $v \in V^k$, then the corresponding element $\bar{v} \in W^{k+1}W$ is in degree $k+1$). The goal is to construct a differential D on $S(V) \otimes S(V) \otimes S(V[-1])$ that extends the differentials of $S(V)$ (i.e., $D(v \otimes 1 \otimes 1) = dv \otimes 1 \otimes 1$ and $D(1 \otimes v \otimes 1) = 1 \otimes dv \otimes 1$) and such that $D(1 \otimes 1 \otimes \bar{v})$ identifies $v \otimes 1 \otimes 1$ and $1 \otimes v \otimes 1$ up to homotopy. This is constructed by induction, see for example [FOT08, Example 2.48].

---

[5] Since A is commutative, one verifies that μ is indeed a morphism of CDGAs.

We now make explicit the (right) homotopy relation between morphisms of CDGAs. As seen in Section 1.4.2, to define a homotopy between $f, g : A \to B$, one must find a path object for the CDGA B. There is a simple (and natural !) way to find one for any CDGA B. It is inspired by topological constructions : in Top, a path object for a space X is simply given by $X \times I$ where $I = [0, 1]$ is an interval. Getting slightly ahead of ourselves (see Section 3.3 : we have $I = \Omega^*_{\mathrm{PL}}(\Delta^1)$), we set :

Definition 3.2.41. The *interval* in CDGA is the quasi-free CDGA $I = (S(t, \mathrm{d}t), d)$ where $\deg t = 0$, $\deg \mathrm{d}t = 1$, and the differential is given by $d(t) = \mathrm{d}t$ and $d(\mathrm{d}t) = 0$.

Lemma 3.2.42. *Let B be any CDGA. A path object for B is given by*

$$B \overset{\sim}{\hookrightarrow} B \otimes I \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} B \oplus B,$$

where $B \to B \otimes I$ is the canonical inclusion and the morphisms $\mathrm{ev}_0, \mathrm{ev}_1$ are defined by :

$$\mathrm{ev}_i(b \otimes P(t)) = P(i) \cdot b, \qquad \mathrm{ev}_i(b \otimes P(t)\mathrm{d}t) = 0.$$

Definition 3.2.43. A *Sullivan homotopy* between two morphisms of CDGAs $f, g : A \to B$ is a morphism of CDGAs $H : A \to B \otimes I$ such that $f = (1 \otimes \mathrm{ev}_0)H$ and $g = (1 \otimes \mathrm{ev}_1)H$.

Concretely, a Sullivan homotopy $H : A \to B \otimes I$ between f and g is of the form

$$H(a) = H_0(a) + H_1(a)t + H_2(a)t^2 + \cdots + H_0'(a)\mathrm{d}t + H_1'(a)t\mathrm{d}t + H_2'(a)t^2\mathrm{d}t + \dots$$

where $H_0(a) = f(a)$, $\sum_{i \geq 0} H_i(a) = g(a)$, and H satisfies a series of equations imposing that H is an algebra morphism and $H \circ d = d \circ H$ (for example $H_1(ab) = f(a)H_1(b) + H_1(a)f(b)$, $H_0'(da) = f(a)$, etc).

Proposition 3.2.44. *Let $A = (S(V), d)$ be a Sullivan algebra. Then two morphisms $f, g : A \to B$ are right homotopic if and only if they are Sullivan homotopic.*

Démonstration. Since $B \otimes I$ is a path object for B, by definition a Sullivan homotopy is a right homotopy. Conversely, suppose that f and g are right homotopic, i.e., there exists a path object $B \overset{\sim}{\hookrightarrow} P \to B \times B$ and a right homotopy $H : A \to P$ between f and g. We can find a lifting in the commutative square :

$$\begin{tikzcd}
B \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"'] & B \otimes I \arrow[d, two heads, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
P \arrow[r, two heads] \arrow[ur, dashed, "l"] & B \times B
\end{tikzcd}$$

and then $l \circ H$ is a Sullivan homotopy between f and g. □

We conclude with a definition that will be useful in the next section :

Definition 3.2.45. A CDGA A is called 1-*reduced* if $A^0 = \mathbb{Q}$ (necessarily generated by the unit) and $A^1 = 0$. We denote by $\mathsf{CDGA}_{\geq 2} \subset \mathsf{CDGA}$ the full subcategory of 1-connected CDGAs.

Proposition 3.2.46. *The subcategory $\mathsf{CDGA}_{\geq 2} \subset \mathsf{CDGA}$ inherits a model category structure.*

Démonstration. As for $s\mathsf{Set}_{\geq 2} \subset s\mathsf{Set}$, one must verify that the liftings and factorizations remain in the subcategory. □

3.3 Comparison between CDGA and rational homotopy

We will show that there exists a Quillen equivalence between $\mathsf{CDGA}_{\geq 2}$ and $s\mathsf{Set}^{\mathbb{Q}}_{\geq 2}$.

Remark 3.3.1. It has been known for a short time that this equivalence can be generalized to the setting of connected and nilpotent simplicial sets [FHT15].

3.3.1 The adjunction

The adjunction is inspired by the CDGA of de Rham differential forms on a smooth manifold, adapted for arbitrary simplicial sets and with rational rather than real coefficients. The idea is that a simplicial set is formed of simplices glued together ; a form on a simplicial set is therefore simply a form on each simplex, compatible with the faces and degeneracies.

We recall that the geometric realization of Δ^n is homeomorphic to the topological space $\mathbb{\Delta}^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, t_0 + \dots + t_n = 1\}$.

Definition 3.3.2. Let $n \geq 0$ be an integer. The CDGA of *polynomial forms on the simplex* Δ^n is :

$$\Omega^*_{\mathrm{PL}}(\Delta^n) := \frac{S(t_0, \dots, t_n, \mathrm{d}t_0, \dots, \mathrm{d}t_n)}{(t_0 + \dots + t_n = 1,\ \mathrm{d}t_0 + \dots + \mathrm{d}t_n = 0)}.$$

We have $|t_i| = 0$, $|\mathrm{d}t_i| = 1$, and the differential is given by $d(t_i) = \mathrm{d}t_i$ and $d(\mathrm{d}t_i) = 0$.

Remark 3.3.3. This CDGA is of course isomorphic to $S(t_1, \dots, t_n, \mathrm{d}t_1, \dots, \mathrm{d}t_n)$, but this isomorphism “breaks” the symmetry of Δ^n. In particular $\Omega^*(\Delta^1)$ is the interval I of Section 3.2.3.

The simplices Δ^n form a cosimplicial object $\Delta^\bullet$ in simplicial sets. We can use this structure to define a simplicial object (by contravariance) from the $\Omega^*_{\mathrm{PL}}(\Delta^n)$:

Lemma 3.3.4. *The CDGAs $\Omega^*_{\mathrm{PL}}(\Delta^n)$ assemble to form a simplicial CDGA $\Omega^*_{\mathrm{PL}}(\Delta^\bullet)$, with structure maps :*

$$d_i(t_k) = \begin{cases} t_k, & \text{if } k < i, \\ 0, & \text{if } k = i, \\ t_{k-1}, & \text{if } k > i; \end{cases} \qquad s_j(t_k) = \begin{cases} t_k, & \text{if } k < i, \\ t_k + t_{k+1}, & \text{if } k = i, \\ t_{k+1}, & \text{if } k > i. \end{cases}$$

This simplicial CDGA has several useful properties, which will allow us for example to show that $\Omega^*_{\mathrm{PL}}(X)$ has the same cohomology as X.

Lemma 3.3.5 (Poincaré Lemma)**.** *For all $n \geq 0$, the unit $\eta : \mathbb{Q} \to \Omega^*_{\mathrm{PL}}(\Delta^n)$ is a quasi-isomorphism.*

Démonstration. We have an isomorphism $\Omega^*_{\mathrm{PL}}(\Delta^n) \cong \bigotimes_{i=1}^n S(t_i, \mathrm{d}t_i)$. It is clear that each $S(t_i, \mathrm{d}t_i)$ is acyclic (the differential is such that $d(t_i^k) = kt_i^{k-1}\mathrm{d}t_i$), so $\Omega^*_{\mathrm{PL}}(\Delta^n)$ is acyclic by the Künneth formula. □

Lemma 3.3.6. *For all $i \geq 0$, the identity of the simplicial $\mathbb{Q}$-module $\Omega^i(\Delta^\bullet)$ is homotopic to the constant map 0, that is, there exist linear maps $H_n : \Omega^i(\Delta^n) \to \Omega^i(\Delta^{n+1})$ satisfying $d_0 H_n = \mathrm{id}$, $d_1 H_0 = 0$, $d_{j+1} H_n = H_{n-1} d_j$ for $n > 0$, and $s_{j+1} s_j = H_n s_j$. In particular, the geometric realization of $\Omega^i(\Delta^\bullet)$ is contractible.*

Démonstration. Let $n \geq 0$ and N be an integer such that $N > 2i$. One can then define :

$$\begin{aligned} &H_n : \Omega^i(\Delta^n) \to \Omega^i(\Delta^{n+1}), \\ P \cdot \mathrm{d}t_{m_1} \dots dt_{m_i} &\mapsto P \cdot t_{n+1}^n \cdot \frac{\mathrm{d}t_{m_1} t_{n+1} - \mathrm{d}t_{n+1} \cdot t_{m_1}}{t_{n+1}^2} \dots \frac{\mathrm{d}t_{m_i} t_{n+1} - \mathrm{d}t_{n+1} \cdot t_{m_i}}{t_{n+1}^2}. \end{aligned}$$

One then verifies the desired equations manually (see [Fre17, Proposition 7.1.3] for details). ☐

Lemma 3.3.7 (Extendability)**.** *Let $n \geq 1$ be an integer and $I \subset [n]$ a subset. For any family $\{\omega_i \in \Omega_{\mathrm{PL}}^d(\Delta^{n-1})\}_{i \in I}$ satisfying the equations $d_i \omega_j = d_{j-1} \omega_i$ for $i < j \in I$, there exists $\omega \in \Omega_{\mathrm{PL}}^d(\Delta^n)$ such that $d_i \omega = \omega_i$.*

Démonstration. Suppose given forms ω_i as in the statement. We set $\zeta_{-1} = 0 \in \Omega_{\mathrm{PL}}^d(\Delta^n)$ and we will construct by induction forms $\zeta_r \in \Omega_{\mathrm{PL}}^d(\Delta^n)$ satisfying $d_i \zeta_r = \omega_i$ for $I \ni i \leq r$. Suppose ζ_{r-1} has been constructed and satisfies the previous hypothesis. If $r \notin I$, we simply set $\zeta_r = \zeta_{r-1}$. Otherwise, we extend the CDGA $\Omega^*(\Delta^n)$ to

$$B := \left(\Omega^*(\Delta^n)\Big[\frac{1}{1-t_r}\Big], d\Big(\frac{1}{1-t_r}\Big) = \frac{\mathrm{d}t_r}{(1-t_r)^2} \right).$$

We define $\varphi : \Omega^*(\Delta^{n-1}) \to B$ by

$$\varphi(t_i) = \begin{cases} \frac{t_i}{1-t_r}, & \text{if } i < r, \\ \frac{t_{i+1}}{1-t_r}, & \text{if } i \geq r; \end{cases} \qquad \varphi(\mathrm{d}t_i) = d(\varphi(t_i)).$$

We can also extend the face map d_r to $d_r : B \to \Omega_{\mathrm{PL}}^*(\Delta^{n-1})$ by setting $d_r(\frac{1}{1-t_r}) = 1$. We can then write

$$\varphi(\omega_r - d_r \zeta_{r-1}) = \frac{1}{(1-t_r)^n} \psi \text{ for } \psi \in \Omega_{\mathrm{PL}}^d(\Delta^n).$$

We then set $\zeta_r = \zeta_{r-1} + \psi$ and verify that ζ_r satisfies the induction hypothesis (see [FHT01, Lemma 10.7] for details). ☐

We can now define piecewise polynomial forms on a simplicial set.

Definition 3.3.8. The functor of *piecewise polynomial forms* is :

$$\Omega_{\mathrm{PL}}^* : s\mathsf{Set} \to \mathsf{CDGA}^{\mathrm{op}}, \qquad X \mapsto \mathrm{Hom}_{s\mathsf{Set}}(X_\bullet, \Omega_{\mathrm{PL}}^*(\Delta^\bullet)).$$

If X is a topological space, we also define $\Omega_{\mathrm{PL}}^*(X) = \Omega_{\mathrm{PL}}^*(S_\bullet(X))$.

Concretely, an element $\omega \in \Omega^i_{\mathrm{PL}}(X)$ is a collection $\{\omega_\sigma \in \Omega^i(\Delta^n)\}_{\sigma\in X_n, n\geq 0}$ of polynomial forms on the simplices of X satisfying $d_i\omega_\sigma = \omega_{d_i\sigma}$ and $s_j\omega_\sigma = \omega_{s_j\sigma}$. The product and differential are defined termwise : $(d\omega)_\sigma = d(\omega_\sigma)$ and $(\alpha\beta)_\sigma = \alpha_\sigma\beta_\sigma$.

Remark 3.3.9. By the Yoneda lemma, we indeed have that $\Omega^*_{\mathrm{PL}}(\Delta^n)$ from Definition 3.3.2 corresponds to that of Definition 3.3.8.

With a little *abstract nonsense*, we easily find a left adjoint for Ω^*_{PL}.

Lemma 3.3.10 (Exercise)**.** *The functor* $\Omega^*_{\mathrm{PL}} : s\mathsf{Set}^{\mathrm{op}} \to \mathsf{CDGA}$ *admits a left adjoint, called the* realization *functor :*

$$\langle - \rangle : \mathsf{CDGA} \to s\mathsf{Set}^{\mathrm{op}}, \qquad A \mapsto \langle A\rangle_\bullet := \mathrm{Hom}_{\mathsf{CDGA}}(A, \Omega^*_{\mathrm{PL}}(\Delta^\bullet)).$$

Let us conclude with a PL analogue of the de Rham theorem. Let X be a simplicial set. We can linearize it degree by degree to obtain a simplicial vector space $\mathbb{Q}[X]$ (a basis in degree n being given by the set X_n). We construct a chain complex from $\mathbb{Q}[X]$ by setting $C_n(X) = \mathbb{Q}[X_n]$ and the differential is given by $d = \sum(-1)^i d_i$ (compare with N_*X in Section 2.6). Finally, we can dualize degree by degree to obtain a cochain complex $C^*(X)$. The cup product defines a DGA structure on $C^*(X)$ (which is not commutative in general).

Theorem 3.3.11 (Sullivan)**.** *There exists a natural quasi-isomorphism of cochain complexes, which induces an isomorphism of CDGAs in cohomology :*

$$\int : \Omega^*_{\mathrm{PL}}(X) \to C^*(X).$$

For an element $\omega = \{\omega_\sigma \in \Omega^n(\Delta^d)\}_{\sigma\in X_d}$ *and* $\sigma \in X_n$*, we write* $\omega_\sigma = f_\sigma \mathrm{d}t_1 \ldots \mathrm{d}t_n$*, then*

$$\int \omega : X_n \to \mathbb{Q}, \quad \sigma \mapsto \int_{\Delta^n} f_\sigma \mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

Moreover, the isomorphism in cohomology is induced by a zigzag of natural quasi-isomorphisms of DGAs between Ω^*_{PL} *and* C^**.*

Démonstration. The fact that $\int$ is well defined follows from manual verification. The fact that it is a quasi-isomorphism follows from the Poincaré lemma and Stokes' formula. For the last point, we use the zigzag of quasi-isomorphisms of simplicial CDGAs $C^*(\Delta^\bullet) \to C^*(\Delta^\bullet) \otimes \Omega^*_{\mathrm{PL}}(\Delta^n) \leftarrow \Omega^*_{\mathrm{PL}}(\Delta^\bullet)$. We use this to define functors on $s\mathsf{Set}$ defined analogously to Ω^*_{PL} and we note that $C^*(X) \cong \mathrm{Hom}_{s\mathsf{Set}}(X, C^*(\Delta^\bullet))$. The extendability of $\Omega^*_{\mathrm{PL}}(\Delta^\bullet)$ and $C^*(\Delta^\bullet)$ allows us to show that $\mathrm{Hom}_{s\mathsf{Set}}(X, -)$ preserves quasi-isomorphisms of simplicial CDGAs, and we can conclude. ☐

Proposition 3.3.12. *The adjunction* $\langle - \rangle : \mathsf{CDGA} \leftrightarrows s\mathsf{Set}^{\mathrm{op}} : \Omega^*_{\mathrm{PL}}$ *of Lemma 3.3.10 is a Quillen adjunction.*

Démonstration. We will verify that the realization of a generating cofibration of CDGAs is a fibration, and that the realization of a generating acyclic cofibration is an acyclic fibration. By adjunction and by the Yoneda lemma (compare with Lemma 1.5.8), we have simplicial bijections :

$$\langle S(D^n(\mathbb{Q}))\rangle \cong \Omega_{\mathrm{PL}}^{n-1}(\Delta^\bullet) \text{ and } \langle S(S^n(\mathbb{Q}))\rangle \cong Z^m\Big(\Omega_{\mathrm{PL}}^*(\Delta^\bullet)\Big).$$

The map induced by $i_n : S(S^n(\mathbb{Q})) \to S(D^n(\mathbb{Q}))$ is simply the differential $i_n^* = d : \Omega_{\mathrm{PL}}^{n-1}(\Delta^\bullet) \to Z^m\Big(\Omega_{\mathrm{PL}}^*(\Delta^\bullet)\Big)$. By Lemma 3.3.6, this map is surjective. In the same way, the map induced by the generating acyclic cofibration $j_n : S(0) = \mathbb{Q} \to S(D^n(\mathbb{Q}))$ is identified with $\Omega_{\mathrm{PL}}^n(\Delta^\bullet) \to *$, which is a weak equivalence by Lemma 3.3.6. It only remains to show that these are fibrations, which follows from Lemma 3.3.14. □

Lemma 3.3.13 (Moore [Moo54])**.** *Let $G_\bullet$ be a simplicial group, that is, a simplicial object in the category of groups. Then the underlying simplicial set of $G_\bullet$ is a Kan complex.*

Démonstration. Let $x = (x_0, \dots, \hat{x}_k, \dots, x_n) : \Lambda_k^n \to G$ be a horn of G, i.e. a collection of $(n-1)$-simplices satisfying $d_i x_j = d_{j-1} x_i$ (whenever both expressions are well defined). There then exists an explicit algorithm that fills the n-simplex.

- If $k = 0$, we set $w_n = s_{n-1}x_n$ then, incrementing the indices one by one, $w_i = w_{i+1} \cdot (s_{i-1}d_i w_{i+1})^{-1} \cdot s_{i-1}x_i$. Then $x = w_1 \in G_n$ fills the horn in the sense that $d_i x = x_i$.
- If $k = n$, we reason in reverse. We set $w_0 = s_0 x_0$ then $w_i = w_{i-1} \cdot (s_i d_i w_{i-1})^{-1} \cdot s_i x_i$. Then w_n fills the horn.
- Finally, if $0 < k < n$, we begin by setting $w_0 = s_0 x_0$, then increment by defining $w_i = w_{i-1}(s_i d_i w_{i-1})^{-1} s_i x_i$ up to defining w_{k-1}. We then set $w_n = w_{k-1}(s_{n-1}d_n w_{k-1})^{-1} s_{n-1}x_n$, and decrement down to w_{k+1} by setting $w_i = w_{i+1}(s_{i-1}d_i w_{i+1})^{-1} s_{i-1}x_i$. We then verify that w_{k+1} indeed fills the simplex.

□

Lemma 3.3.14. *Let $f : A_\bullet \to B_\bullet$ be a morphism of simplicial abelian groups. If f is surjective in each degree, then it is a Kan fibration.*

Démonstration. We consider a commutative diagram :

$$\begin{tikzcd}
\Lambda_l^n \arrow[r, "\alpha"] \arrow[d, hook, "i_n"'] & A \arrow[d, "f"] \\
\Delta^n \arrow[r, "\beta"] \arrow[ur, dashed, "l"] & B
\end{tikzcd}$$

We view β as an n-simplex $\beta \in B_n$. Since f is surjective, there exists $\theta \in A_n$ such that $f(\theta) = \beta$, but of course we do not necessarily have $d_i\theta = \alpha_i$. However, $i_n^*\theta - \alpha$ defines a horn $\Lambda_k^n \to \ker f$. Since $\ker f$ is a simplicial group, it is fibrant by Lemma 3.3.13, so there exists $x \in (\ker f)_n$ whose restriction to Λ_k^n is $i_n^*\theta - \alpha$. We then verify that $l := \theta - x : \Delta^n \to B$ indeed defines a lift in the diagram. □

We can also define an analogue of the mapping space of Definition 2.5.2.

Definition 3.3.15. Let A, B be two CDGAs. We define the mapping space by :

$$\mathrm{Map}_\bullet(A,B) = \mathrm{Hom}_{\mathsf{CDGA}}(A \otimes \Omega^*_{\mathrm{PL}}(\Delta^\bullet), B).$$

In particular, $\mathrm{Map}_0(A,B) = \mathrm{Hom}_{\mathsf{CDGA}}(A \otimes \Omega^*_{\mathrm{PL}}(\Delta^0), B) = \mathrm{Hom}_{\mathsf{CDGA}}(A,B)$.

Proposition 3.3.16. *Let $f, g : A \to B$ be two morphisms of CDGAs. Then f is Sullivan homotopic to g if and only if f and g are in the same connected component of* $\mathrm{Map}_\bullet(A,B)$.

Démonstration. This is immediate from the definition of Sullivan homotopy (where we note that $I = \Omega^*_{\mathrm{PL}}(\Delta^1)$). ☐

3.3.2 The equivalence

We arrive at the key theorem of this chapter.

Proposition 3.3.17. *The adjunction* $\langle - \rangle : \mathsf{CDGA} \leftrightarrows s\mathsf{Set}^{\mathrm{op}} : \Omega^*_{\mathrm{PL}}$ *restricts to a Quillen adjunction* $\mathsf{CDGA}_{\geq 2} \leftrightarrows s\mathsf{Set}^{\mathbb{Q},\mathrm{op}}_{\geq 2}$.

Démonstration. The adjunction passes to the 1-reduced (resp. 1-connected) categories. Indeed, it is clear that if X is 1-reduced then $\Omega^*_{\mathrm{PL}}(X)$ is 1-connected, and that if A is 1-connected then $\langle A \rangle$ is 1-reduced. To show that the adjunction is compatible with the Bousfield localization, we must show that Ω^*_{PL} sends fibrations (resp. acyclic fibrations) of $s\mathsf{Set}^{op,\mathbb{Q}}_{\geq 2}$ to fibrations (resp. acyclic fibrations) of CDGA. By Proposition 3.3.12, Ω^*_{PL} preserves fibrations of $s\mathsf{Set}^{\mathrm{op}}$ (which are in fact cofibrations), and the fibrations of $s\mathsf{Set}^{\mathrm{op},\mathbb{Q}}_{\geq 2}$ are the same as before localization. Moreover, thanks to Theorems 3.1.8 and 3.3.11, Ω^*_{PL} preserves weak equivalences, which allows us to conclude. ☐

Theorem 3.3.18. *The preceding adjunction induces an equivalence between*

- $\mathrm{Ho}(s\mathsf{Set}^{\mathbb{Q},\mathrm{op}}_{\geq})_{tf}$ *: the full subcategory of the homotopy category* $\mathrm{Ho}(s\mathsf{Set}^{\mathbb{Q}}_{\geq 2})$ *of* 1-*reduced spaces whose homology is of finite type*[6] *; and*
- $\mathrm{Ho}(\mathsf{CDGA}_{\geq 2})_{tf}$ *: the full subcategory of the homotopy category of* $\mathsf{CDGA}^{\mathrm{op}}_{\geq 2}$ *of* 1-*connected CDGAs of finite type.*

Remark 3.3.19. One cannot really speak of a Quillen equivalence between $s\mathsf{Set}^{\mathbb{Q},\mathrm{op}}_{\geq 2,tf}$ and CDGA_{tf} : these categories are neither complete nor cocomplete.

Remark 3.3.20. One can replace simplicial sets by topological spaces (cf. Section 2.5).

To prove the theorem, we will study the unit and counit of the derived adjunction $\mathbb{L}\langle - \rangle \dashv \mathbb{R}\Omega^*_{\mathrm{PL}}$. In particular, we must study the behavior of Ω^*_{PL} on cofibrant CDGAs.

Lemma 3.3.21. *The cofibrant objects of* $\mathsf{CDGA}_{\geq 2}$ *are the minimal algebras.*

Démonstration. We have seen that the cofibrant objects of CDGA are the retracts of Sullivan algebras (Proposition 3.2.35). In $\mathsf{CDGA}_{\geq 2}$, a Sullivan algebra is automatically minimal. Moreover, any retract of a connected Sullivan algebra is again a Sullivan algebra. ☐

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[6] That is, finite-dimensional in each degree

Lemma 3.3.22. *Let $A = (S(V), d)$ be a minimal 1-connected CDGA. Then for all $n \geq 2$, there exists a non-degenerate pairing :*

$$\pi_n(\langle A\rangle) \times V^n \to \mathbb{Q}$$

which induces, if V^n is finite-dimensional, a natural isomorphism[7] *in A :*

$$V^n \cong \mathrm{Hom}_{\mathbb{Q}}(\pi_n(\langle A\rangle), \mathbb{Q}).$$

Sketch of proof (see [FHT01, Theorem 15.11] for more details). To simplify notation, we set $X := \langle A\rangle$ in what follows. Let us first define the pairing. Since X is simply connected, we have $\pi_n(X) \cong \mathrm{Hom}_{\mathrm{Ho}(s\mathsf{Set}_{\geq 2})}(S^n, X)$ where $S^n = \partial\Delta^{n+1}$. The rational homotopy class of $\gamma \in \pi_n(X)$ therefore corresponds (by adjunction) to an element of $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{CDGA})}(A, \Omega^*_{\mathrm{PL}}(S^n))$. We verify (Example 3.4.5) that $\Omega^*_{\mathrm{PL}}(S^n) \simeq H^*(S^n) = S(x)/(x^2)$ where $\deg x = n$. The class γ thus induces an element $\gamma^* \in \mathrm{Hom}_{\mathrm{Ho}(\mathsf{CDGA})}(A, S(x)/(x^2))$. We then define the pairing of γ with $v \in V^n \subset A^n$ as the coefficient of x in $\gamma^*(v)$. We verify that this defines a bilinear pairing that does not depend on the choices of homotopy classes.

This pairing induces a linear map $\varphi_n : V^n \to \mathrm{Hom}_{\mathbb{Q}}(\pi_n(X), \mathbb{Q})$. Let us verify that it is an isomorphism. Let r be the smallest integer such that $\pi_r(X) \neq 0$. For $n < r$, we have $H^n(X) = 0$ by the Hurewicz theorem, so necessarily $V^n = 0$ (since the algebra is minimal and 1-connected), and thus φ_n is an isomorphism. We deduce that φ_r is also an isomorphism, again by the Hurewicz theorem (since $\mathrm{Hom}_{\mathbb{Q}}(\pi_r(X), \mathbb{Q}) \cong H^r(X; \mathbb{Q}) \cong V^r$).

We have thus shown that if X is an $(r-1)$-connected space then φ_n is an isomorphism for $n \leq r$. We can find a fibration $F \to X \xrightarrow{p} K(\pi_r(X), r)$ such that $\pi_r(p)$ is the identity ; the space F is therefore r-connected. We can thus apply[8] the result to F and apply the five lemma to deduce that φ_{r+1} is an isomorphism. We conclude by induction. □

Lemma 3.3.23. *Let $A = (S(V), d)$ be a minimal 1-connected CDGA of finite type. Then the counit of the derived adjunction of Theorem 3.3.18 at A is a quasi-isomorphism :*

$$A \xrightarrow{\sim} \Omega^*_{\mathrm{PL}}(\langle A\rangle).$$

Démonstration. Let us first treat the case where $V = V^n$ is concentrated in a single degree. The previous lemma tells us that $X = \langle A\rangle$ is rationally equivalent to an Eilenberg–MacLane space of type $K(\mathbb{Z}^{\dim V}, n)$, whose rational cohomology we know how to compute (for example with spectral sequences, and using the classical computation of the cohomology of $S^1 = K(\mathbb{Z}, 1)$) : it is precisely the symmetric algebra generated by $\dim V$ generators in degree n. In this case, we therefore indeed have a quasi-isomorphism $A = (S(V^n), 0) \to \Omega^*_{\mathrm{PL}}(\langle A\rangle)$.

For the general case, we filter V by setting $V^{<n} = \bigoplus_{i<n} V^i$. By minimality, the differential of A restricts to $S(V^{<n})$, and the quotient $S(V^{<n+1})/(S(V^{<n}))$ is isomorphic

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[7] We use the isomorphism $(QA)^n \cong V^n$ to define the functoriality of V with respect to A.

[8] We implicitly use the fact that we know the minimal models of fibrations, see Section 3.4.3. A careful reader can verify that there is no circular reasoning.

to $S(V^n)$ with zero differential. Since $V^{<2} = 0$, we can apply the result of the previous paragraph to show that the lemma holds for $(S(V^{<3}), d)$. We have commutative diagrams :

$$\begin{tikzcd}
(S(V^{<n}), d) \arrow[r, hook] \arrow[d] & (S(V^{<n+1}), d) \arrow[r, two heads] \arrow[d] & (S(V^n), 0) \arrow[d] \\
\Omega^*_{\mathrm{PL}}(\langle S(V^{<n}), d\rangle) \arrow[r] & \Omega^*_{\mathrm{PL}}(\langle S(V^{<n+1}), d\rangle) \arrow[r] & \Omega^*_{\mathrm{PL}}(\langle S(V^n, 0)\rangle)
\end{tikzcd}$$

By induction, the lemma holds for the CDGA on the left, and we showed in the previous paragraph that it holds for the one on the right. By the five lemma, it therefore holds for the one in the middle. We conclude by passing to the colimit $A = \mathrm{colim}_n(S(V^{<n}), d)$. ☐

Lemma 3.3.24. *Let X be a fibrant 1-reduced simplicial set. Then the unit of the derived adjunction of Theorem 3.3.18 at X is a rational equivalence :*

$$X \xrightarrow{\sim_{\mathbb{Q}}} \langle \Omega^*_{\mathrm{PL}}(X)\rangle.$$

Démonstration. By definition, we must show that the following morphism of DGAs is a quasi-isomorphism :

$$C^*(\langle \Omega^*_{\mathrm{PL}}(X)\rangle) \to C^*(X).$$

Thanks to Theorem 3.3.11, it suffices in fact to show that the following morphism is a quasi-isomorphism :

$$\Omega^*_{\mathrm{PL}}(\langle \Omega^*_{\mathrm{PL}}(X)\rangle) \to \Omega^*_{\mathrm{PL}}(X).$$

Let us choose a cofibrant (hence minimal) replacement $A = (S(V), d) \xrightarrow{\sim} \Omega^*_{\mathrm{PL}}(X)$. We then obtain a commutative diagram :

$$\begin{tikzcd}
\Omega^*_{\mathrm{PL}}(\langle \Omega^*_{\mathrm{PL}}(X)\rangle) \arrow[rr] & & \Omega^*_{\mathrm{PL}}(X) \\
& A \arrow[ul, "\sim"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We then deduce from the previous lemma that f is a quasi-isomorphism. ☐

Remark 3.3.25. One can show that $\langle A\rangle$ is a rational space for any CDGA A. We thus obtain that the rationalization $X_{\mathbb{Q}}$ of a fibrant 1-reduced simplicial set is $\langle \Omega^*_{\mathrm{PL}}(X)\rangle$.

Proof of Theorem 3.3.18. We have just shown that the unit and counit of the adjunction between the corresponding homotopy categories are isomorphisms (i.e. quasi-isomorphisms and rational equivalences, respectively). ☐

3.4 Applications

3.4.1 Models

Proposition 3.4.1. *The rational equivalence classes of 1-reduced simplicial sets of finite type correspond to the isomorphism classes of minimal 1-connected CDGAs of finite type.*

Démonstration. This follows directly from Theorem 3.3.18 and the description of cofibrant CDGAs in $\mathsf{CDGA}_{\geq 2}$ (Lemma 3.3.21). ☐

Definition 3.4.2. Let X be a 1-reduced simplicial set. A *(Sullivan) model* of X is a 1-connected CDGA quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(X)$. A *minimal model* of X is a model that is a minimal CDGA. If Y is a topological space, a (minimal) model of Y is a (minimal) model of $S_\bullet(Y)$.

Corollary 3.4.3. *Every* 1*-reduced simplicial set of finite type admits a unique minimal model of finite type up to isomorphism.*

This corollary is fundamental : the rational homotopy type of a simply connected space of finite type is thus entirely determined by a purely algebraic datum : a minimal 1-connected CDGA of finite type. One can compute “everything” about this rational homotopy type in a purely combinatorial manner (provided one manages to determine a minimal model, which is far from trivial in general.) For example :

Corollary 3.4.4. *Let X be a* 1*-reduced simplicial set of finite type and $A = (S(V), d)$ its minimal model.*

- *There is an isomorphism of graded commutative algebras $H^*(X; \mathbb{Q}) \cong H^*(A)$.*
- *For all $n \geq 2$, there are isomorphisms of vector spaces $V^n \cong \mathrm{Hom}(\pi_n(X), \mathbb{Q})$.*
- *The Whitehead bracket $\pi_{n+1}(X) \times \pi_{m+1}(X) \to \pi_{n+m+1}(X)$ is dual (under the preceding isomorphism) to the quadratic part of the differential, $d_2 : V^{n+m+1} \to V^{n+1} \otimes V^{m+1} \subset S^{(2)}(V)^{n+m+2}$.*

Example 3.4.5 (Spheres)**.** Let $n \geq 2$. Consider the sphere $S^n = \Delta^n / \partial \Delta^n$. Its rational cohomology is $A = H^*(S^n) = S(x)/(x^2)$ where $\deg x = n$.

- If $n = 2k+1$ is odd, then $H^*(S^{2k+1})$ is free (hence minimal) on the generator x. There exists a direct quasi-isomorphism $H^*(S^{2k+1}) \to \Omega^*_{\mathrm{PL}}(S^{2k+1})$, obtained by sending x to any volume form. We thus obtain that the minimal model of S^{2k+1} is $S(x)$. We immediately find that $\pi_{2k+1}(S^{2k+1}) = \mathbb{Q}$ and $\pi_i(S^{2k+1}) = 0$ for $2 \leq i \neq 2k+1$.
- If $n = 2k$ is even, then $H^*(S^{2k})$ is not free. A minimal model is given by $M = (S(x, y), dy = x^2)$ (where $\deg y = 4k-1$). There is a quasi-isomorphism $M \to \Omega^*_{\mathrm{PL}}(S^{2k})$, obtained by sending x to a volume form and y to a form satisfying $dy = x^2$ (which necessarily exists since $[x^2] = 0$ in cohomology). We thus obtain that $H^*(S^{2k})$ is indeed a model for S^{2k}. We can also compute the rational homotopy groups : since there are two generators in the minimal model, $\pi_{2k}(S^{2k}) = \pi_{4k-1}(S^{2k}) = \mathbb{Q}$ and $\pi_i(S^{2k}) = 0$ for $2 \leq i \neq 2k, 4k-1$.

We thus recover a theorem of Serre on the homotopy groups of spheres.

Example 3.4.6. Consider the complex projective space $\mathbb{CP}^n$ (with $n \geq 1$). Its cohomology is $H^*(\mathbb{CP}^n) = S(x)/(x^{n+1})$ where $\deg x = 2$. As before, we find as minimal model $A = (S(x, y), dy = x^{n+1})$.

Example 3.4.7. Let A be the minimal model of X and B the minimal model of Y. Then the product $X \times Y$ has minimal model $A \otimes B$. The wedge $X \vee Y$ has minimal model $A \oplus_{\mathbb{Q}} B$, the quotient of $A \oplus B$ by the relation $(1,0) = (0,1)$ with multiplication $a \cdot b = 0$ if $a \in \bar{A}$ and $b \in \bar{B}$.

Example 3.4.8. Let G be a finite-dimensional Lie group. One can show that the minimal model of G is a free algebra on generators of odd degree. In particular, all homotopy groups of even degree of a Lie group are torsion.

3.4.2 Formality

Spheres, complex projective spaces and Lie groups belong to a very special class of spaces : their cohomology is a (not necessarily minimal) model for their rational homotopy type.

Definition 3.4.9. A simplicial set X (1-reduced of finite type) is *formal* if its rational cohomology $H^*(X;\mathbb{Q})$ is quasi-isomorphic as a CDGA to $\Omega^*_{\mathrm{PL}}(X)$.

Remark 3.4.10. Over a field, a cochain complex is always quasi-isomorphic to its cohomology. In the preceding definition, it is essential to take into account the CDGA structure.

Example 3.4.11. Spheres, complex projective spaces and Lie groups are formal.

Example 3.4.12. The suspension ΣX of any space is formal.

Example 3.4.13. The product and the wedge of two formal spaces are formal.

Example 3.4.14 ([DGMS75])**.** A compact Kähler manifold[9] is formal.

Example 3.4.15 ([FOT08, Proposition 2.99])**.** Let X be a $(p-1)$-connected space ($p \geq 2$) of dimension $\leq 3p-2$. Then X is formal.

Example 3.4.16 ([Hes07, p.13])**.** One can find an example of a non-formal space as follows. Consider

$$A = (S(x_3, y_3, z_5), dz = xy).$$

One easily computes that $H^0(A) = \mathbb{Q}1$, $H^3(A) = \mathbb{Q}x \oplus \mathbb{Q}y$, $H^8(A) = \mathbb{Q}xz \oplus \mathbb{Q}yz$ and $H^{11}(A) = \mathbb{Q}xyz$ (and all other cohomology groups are zero). Since A is minimal (hence cofibrant), if there existed a zigzag of quasi-isomorphisms between A and $H^*(A)$, then there would exist a direct quasi-isomorphism $f : A \to H^*(A)$. For degree reasons, we would necessarily have $f(z) = 0$, hence $f(xyz) = 0$. The morphism f therefore cannot be a quasi-isomorphism.

Example 3.4.17 ([FOT08])**.** One can find a non-formal manifold as follows. Let $q : S^2 \times S^2 \to S^4$ be the map that collapses $S^2 \vee S^2$ to a point. Let $M = S^7 \times_{S^4} (S^2 \times S^2)$, where $S^7 \to S^4$ is the Hopf fibration. The minimal model of M is then given by

$$\Big(S(a_2, b_2, u_3, v_3, t_3), da = 0, db = 0, du = a^2, dv = b^2, dt = ab\Big).$$

One can verify that this CDGA is not formal.

---

[9] This is a symplectic manifold (X, ω) equipped with an integrable almost complex structure J such that $(u,v) \mapsto \omega(u, Jv)$ is a symmetric positive definite bilinear form on each tangent space T_xX.

3.4.3 Models of fibrations

Sullivan models are particularly well-suited for studying fibrations :

Theorem 3.4.18 ([FHT01, Theorem 15.3])**.** *Let $p : E \to B$ be a fibration with fiber F, with E and B simply connected of finite type. Assume furthermore that F is of finite type. Let $A = (S(V), d)$ be the minimal model of B and $i : (S(V), d) \to (S(V \oplus W), D)$ the minimal model of p. Then the quotient $(S(W), \bar{d}) = (S(V \oplus W), D)/(S^{(\geq 1)}(V) \otimes S(W))$ is the minimal model of F.*

This theorem is often used "in reverse" : if one knows the minimal model $(S(V), d)$ of B and the minimal model $(S(W), \bar{d})$ of F, one knows that the minimal model of E will be of the form $(S(V \oplus W), D)$ where D extends d and induces $\bar{d}$ on the quotient.

Example 3.4.19 (Path space)**.** Let (X, x_0) be a pointed space (simply connected of finite type). We define its path space, equipped with the compact-open topology :

$$PX \coloneqq \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\}.$$

This space is contractible, a homotopy being given by $H(\gamma, t) : s \mapsto \gamma(ts)$. There is a fibration $\pi : PX \to X$ given by $\pi(\gamma) = \gamma(1)$, and the fiber $\pi^{-1}(x_0)$ is none other than the loop space ΩX. We thus obtain a fibration $\Omega X \to PX \to X$.

Let $(S(V), d)$ be the minimal model of X. Since PX is contractible, we find that the minimal model of π is $(S(V \oplus V[-1]), D)$ where $V[-1]$ is a copy of V shifted in degree by 1 (whose elements we denote $\bar{v}$ with $\deg \bar{v} = \deg v + 1$), with differential $D(\bar{v}) = v \pm \bar{dv}$. The minimal model of ΩX is therefore given by $(S(V[-1]), \bar{d} = 0)$: since d is decomposable, $\bar{dv}$ vanishes in the quotient. We thus "recover" the fact that the homotopy groups of ΩX are the same as those of X, shifted in degree (which is obvious from the long exact sequence of the fibration). We also note that ΩX is formal (which can be deduced from the fact that it is an H-space[10]).

3.4.4 Real homotopy type

In certain cases, the rational homotopy type of a space X is still too "strong" an invariant. Sometimes one cannot avoid the real numbers, for example when one wants to compute integrals on a manifold. In this section, we adapt the notion of rational model to the notion of "real model". Since both fields $\mathbb{Q}$ and $\mathbb{R}$ appear, we shall be careful to use indices $(-)_\mathbb{Q}$ or $(-)_\mathbb{R}$ to indicate in which category the objects or morphisms lie.

Definition 3.4.20. A *real model* of a 1-reduced simplicial set of finite type is a 1-connected CDGA quasi-isomorphic to $\Omega^*_{\mathrm{PL}}(X) \otimes_\mathbb{Q} \mathbb{R}$. A *minimal real model* is a real model that is a minimal CDGA.

Proposition 3.4.21. *If A is a rational model of X, then it is a real model of X.*

---

[10] An H-space is a topological space X equipped with a map $\mu : X \times X \to X$ and an element $e \in X$ such that $\mu(e, -)$ and $\mu(-, e)$ are homotopic to the identity.

Démonstration. The $\mathbb{Q}$-module $\mathbb{R}$ is flat, since it is free (as all $\mathbb{Q}$-vector spaces, assuming the axiom of choice). The functor $-\otimes_{\mathbb{Q}}\mathbb{R}$ is therefore exact. If we have a quasi-isomorphism $A \xleftarrow{\sim_{\mathbb{Q}}} \cdot \xrightarrow{\sim_{\mathbb{Q}}} \Omega^*_{\mathrm{PL}}(X)$, then $A\otimes_{\mathbb{Q}}\mathbb{R} \xleftarrow{\sim_{\mathbb{R}}} \cdot\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim_{\mathbb{R}}} \Omega^*_{\mathrm{PL}}(X)\otimes_{\mathbb{Q}}\mathbb{R}$ is a quasi-isomorphism. ☐

Example 3.4.22. The converse is false. For a counterexample, it suffices to exhibit two 1-connected CDGAs with real coefficients that are quasi-isomorphic over $\mathbb{R}$ but not over $\mathbb{Q}$. An example is given by [FOT08, Example 2.38]. Let $a > 0$ be a rational number. We define :

$$A_a \coloneqq (S(e_2, x_4, y_7, z_9), d_a),$$

where the differential is given by :

$$d_a(e) = 0,\ d_a(x) = 0,\ d_a(y) = x^2 + ae^4, d_a(z) = e^5.$$

Then $A_a \simeq_{\mathbb{Q}} A_{a'} \iff a/a'$ is the square of a rational number :

- If $a/a' = \tau^2$ where $\tau \in \mathbb{Q}$, then we define a quasi-isomorphism $\varphi : A_a \to A_{a'}$ by $\varphi(e) = e$, $\varphi(x) = \tau x$, $\varphi(y) = \tau^2 y$ and $\varphi(z) = z$.
- Conversely, if A_a and $A_{a'}$ are quasi-isomorphic, then since they are minimal there exists a direct quasi-isomorphism $\varphi : A_a \to A_{a'}$. By reasoning about the coefficients of φ one deduces that a/a' is a square (exercise).

By reusing the proof of the first point and the fact that every rational number is the square of a real number, we deduce that A_a and $A_{a'}$ are always quasi-isomorphic over $\mathbb{R}$. Thus for example $A_{\sqrt{2}}$ and A_1 are quasi-isomorphic over $\mathbb{R}$ but not over $\mathbb{Q}$.

However, there is a partial converse to the proposition. Recall that a space X is formal (over $\mathbb{Q}$) if $\Omega^*_{\mathrm{PL}}(X) \simeq_{\mathbb{Q}} H^*(X;\mathbb{Q})$. Analogously, a space is said to be formal over $\mathbb{R}$ if $\Omega^*_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \mathbb{R} \simeq_{\mathbb{R}} H^*(X;\mathbb{R})$.

Theorem 3.4.23 ([FOT08, Proposition 2.101])**.** *A* 1*-reduced simplicial set of finite type is formal over* $\mathbb{Q}$ *if and only if it is formal over* $\mathbb{R}$*.*

Sketch. The proof uses obstruction theory. Let X be a space as in the statement. For a field $\mathbb{K}$, there exists a chain complex $D_X^{\mathbb{K}}$ and a sequence of obstructions $d_2, d_3, \dots \in H_*(D_X^{\mathbb{K}})$ such that X is formal if and only if $d_2 = d_3 = \dots = 0$. One can show that the complex $D_X^{\mathbb{R}}$ is obtained as $D_X^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ and that the obstructions over $\mathbb{R}$ arise from the obstructions over $\mathbb{Q}$ ($d_i^{\mathbb{R}} = d_i^{\mathbb{Q}} \otimes 1$). One deduces that they vanish over $\mathbb{Q}$ if and only if they vanish over $\mathbb{R}$. ☐

When a space is a manifold, one can use the tools of differential geometry to study its real homotopy type :

Theorem 3.4.24 (Real models and differential forms)**.** *Let M be a simply connected smooth manifold of finite type, and $\Omega^*_{\mathrm{dR}}(M)$ its CDGA of de Rham differential forms. Then any* 1*-connected CDGA of finite type quasi-isomorphic to $\Omega^*_{\mathrm{dR}}(M)$ is a real model of M.*

Sketch. This follows essentially from the de Rham theorem : $\Omega^*_{\mathrm{dR}}(M) \simeq_{\mathbb{R}} C^*(X;\mathbb{R})$ (to be precise, one would need to reuse the properties of Ω^*_{PL} seen in Section 3.3.2). ☐

3.4.5 Dichotomy theorem

Let us conclude with an interesting example of an application of Sullivan's theory. (We will not state the theorem in full as this would require certain notions that we have not covered.)

Definition 3.4.25. Let X be a simply connected space of finite type. It is called *rationally elliptic* if $\sum_{k\geq 2} \operatorname{rang} \pi_k(X) < +\infty$, and *rationally hyperbolic* otherwise.

Definition 3.4.26. Let X be a simply connected space of finite type. If the rational cohomology of X is bounded, we call its *dimension* the maximal integer n such that $H^n(X;\mathbb{Q}) \neq 0$.

Theorem 3.4.27 (Dichotomy, hyperbolic case [FHT01])**.** *Let X be a rationally hyperbolic space of dimension n. Then :*

1. *The series $\sum_{k\leq m} \operatorname{rang} \pi_k(X)$ has exponential growth : there exist constants $A > 1$ and $C > 0$ such that for $m \gg 0$,*

$$\sum_{k\leq m} \operatorname{rang} \pi_k(X) \geq CA^m.$$

2. *The sequence of Betti numbers of ΩX has exponential growth.*
3. *There are no "large gaps" in the rational homotopy groups : for every integer q, there exists an integer $q < p < q + n$ such that* $\operatorname{rang} \pi_p(X) > 0$.
4. *(etc, see [FHT01]).*

For the elliptic case, we define the homotopy Euler characteristic :

$$\chi_\pi(X) = \sum_{k\geq 0} (-1)^k \operatorname{rang} \pi_k(X).$$

Theorem 3.4.28 (Dichotomy, elliptic case [FHT01])**.** *Let X be a rationally elliptic space of dimension n. Then :*

1. *The homotopy groups $\pi_k(X)$ are finite for $k \geq 2n$.*
2. *The homotopy Euler characteristic $\chi_\pi(X)$ is negative (hence* $\operatorname{rang} \pi_{\text{even}}(X) \leq \operatorname{rang} \pi_{\text{odd}}(X)$*) while the Euler characteristic $\chi(X)$ is positive.*
3. *The rational cohomology of X satisfies Poincaré duality.*
4. *(etc, see [FHT01]).*

Example 3.4.29. Tori $T^n = (S^1)^n$, spheres S^n, complex projective spaces $\mathbb{CP}^n$ and compact Lie groups are rationally elliptic. The connected sum of k copies of $S^3 \times S^2$ is rationally hyperbolic for $k \geq 2$ (since its Euler characteristic is strictly negative).

3.5 Quillen Models

There exists another way to approach rational homotopy theory due to Quillen [Qui69] which we will briefly discuss here. This theory uses differential graded Lie algebras rather than CDGAs.

Definition 3.5.1. A *differential graded Lie algebra* (DGLA) is a chain complex $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ (the « bracket ») which is antisymmetric and satisfies the Jacobi and Leibniz relations :

$$[x,y] + (-1)^{\deg x \deg y}[y,x] = 0;$$
$$(-1)^{\deg x \deg z}[x,[y,z]] + (-1)^{\deg x \deg y}[y,[z,x]] + (-1)^{\deg y \deg z}[z,[x,y]] = 0;$$
$$d[x,y] = [dx,y] + (-1)^{\deg x}[x,dy].$$

We denote by DGLA the category of DGLAs (graded over $\mathbb{N}$) and their morphisms, and by $\mathsf{DGLA}_{\geq 1}$ the full subcategory of DGLAs $\mathfrak{g}$ satisfying $\mathfrak{g}_0 = 0$.

Example 3.5.2. Let A be a DGA. One can define a DGLA with the *commutator* $[a,b] = ab - \pm ba$.

Example 3.5.3. Let X be a simplicial set. Then we have a graded Lie algebra (with zero differential) given by $\mathfrak{g}_i = \pi_{i+1}(X)$ equipped with the Whitehead bracket, which can be viewed as the commutator of loop concatenation on $\pi_i(\Omega X) \cong \pi_{i+1}(X)$. This defines a functor $\pi : s\mathsf{Set}_{\geq 2} \to \mathsf{DGLA}_{\geq 1}$. (Not to be confused with the « fundamental groupoid » functor of Chapter 4 !)

Theorem 3.5.4 (Quillen [Qui69])**.** *There exists a Quillen equivalence*

$$\lambda : s\mathsf{Set}_{\geq 2} \to \mathsf{DGLA}_{\geq 1}.$$

Moreover, for all $X \in s\mathsf{Set}_{\geq 2}$, we have $\pi X \cong H_*(\lambda X)$.

Definition 3.5.5. A *Quillen model* of a 1-reduced simplicial set X is a DGLA quasi-isomorphic to λX.

Idea of Theorem 3.5.4. The equivalence λ is obtained by composing several equivalences :

$$\begin{tikzcd} s\mathsf{Set}_{\geq 2} \arrow[r, shift left, "G"] & s\mathtt{Gp}_{\geq 1} \arrow[l, shift left, "\bar{W}"] \arrow[r, shift left, "\widehat{\mathbb{Q}}"] & s\mathtt{CHA}_{\geq 1} \arrow[l, shift left, "\mathbb{G}"] \arrow[r, shift left, "\mathrm{Prim}"] & s\mathtt{LA}_{\geq 1} \arrow[l, shift left, "\widehat{U}"] \arrow[r, shift left, "N"] & \mathsf{DGLA}_{\geq 1}. \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

Let us briefly explain the notation and the functors.

- The category $s\mathtt{Gp}_{\geq 1}$ is that of reduced simplicial groups, i.e. simplicial objects in the category of groups $G_\bullet$ satisfying $G_0 = 1$.
- The category $s\mathtt{CHA}_{\geq 1}$ is that of reduced simplicial complete Hopf algebras[11].

---

[11] Recall that a Hopf algebra is a chain complex H equipped with a product $\mu : H^{\otimes 2} \to H$, a « coproduct » $\Delta : H \to H^{\otimes 2}$ (Definition 3.5.6) and an antipode $\sigma : H \to H$ satisfying compatibility relations. It is complete if it has a filtration $H = F_0 H \supset F_1 H \supset F_2 H \supset \dots$ such that $F_1 H = \bar{H}$, the algebra $\operatorname{gr} H$ is generated by $\operatorname{gr}_1 H$, we have $H \cong \lim_i H/F_i H$, and the coproduct takes values in the completed tensor product $H \widehat{\otimes} H$.

- The category $s\mathtt{LA}_{\geq 1}$ is that of 1-reduced simplicial Lie algebras (without differential or grading).
- The functor $G : s\mathsf{Set}_{\geq 2} \to s\mathtt{Gp}_{\geq 1}$ (defined by Kan) is such that if X is a simplicial set, then GX is a simplicial group whose realization is the loop space of the realization of X. Its adjoint $\bar{W}$ represents the classifying space of a simplicial group.
- The functor $\widehat{\mathbb{Q}} : s\mathtt{Gp}_{\geq 1} \to s\mathtt{CHA}_{\geq 1}$ sends a simplicial group $G_\bullet$ to the completion of its group algebra $\widehat{Q}[G_\bullet]$. Its adjoint $\mathbb{G}$ sends a complete Hopf algebra to the group formed by its group-like elements (those satisfying $\Delta x = x \otimes x$ and $\varepsilon(x) = 1$).
- The functor Prim : $s\mathtt{CHA}_{\geq 1} \to s\mathtt{LA}_{\geq 1}$ sends a complete Hopf algebra to the Lie algebra formed by its primitive elements (those satisfying $\Delta x = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$). Its adjoint $\widehat{U}$ is the completion of the enveloping algebra of a Lie algebra (which is a Hopf algebra).
- The functor $N : s\mathtt{LA}_{\geq 1} \to \mathsf{DGLA}_{\geq 1}$ is the analogue of the normalized chains functor from Section 2.6. Its adjoint Γ is the analogue of the right adjoint in the Dold–Kan correspondence.

To prove Quillen's theorem, one verifies that each of these categories is a model category, and that each of these adjunctions is a Quillen adjunction. Composing them all, one obtains a Quillen equivalence. □

Most of the statements obtained in the setting of Sullivan models (Section 3.4.1) have a « dual » counterpart in the setting of Quillen models. We refer to [FHT01, Part IV]. One can for example define free DGLAs $\mathbb{L}(V)$, quasi-cofree DGLAs $(\mathbb{L}(V), d)$, Sullivan DGLAs and minimal DGLAs. Every DGLA is quasi-isomorphic to a unique minimal DGLA up to isomorphism, so in particular every space has a unique minimal Quillen model.

We have for example seen (Section 3.4.3) that Sullivan models are particularly well suited for studying fibrations. Dually, Quillen models are particularly well suited for studying cell attachments. If $\mathfrak{g}_X$ is a Quillen model of a space X and $Y = X \cup_{S^{n-1}} D^n$ is obtained by attaching a cell to X, then a Quillen model of Y can be obtained by adding a generator to $\mathfrak{g}_X$, along with a differential that mimics the way the cell is attached to X.

Koszul duality One may wonder what connection exists between Quillen models and Sullivan models. In « modern » terms, an answer is provided by *Koszul duality between the operad of commutative algebras and the operad of Lie algebras*. Operads are combinatorial objects that encode « types of algebras », for example associative algebras, commutative algebras, or Lie algebras. They were initially used to study loop spaces in algebraic topology [BV68 ; May72] and have since found numerous applications. For an introduction to the theory of operads one may for example read [LV12] or [Fre17, Part I(a)].

This Koszul duality of operads was discovered by Ginzburg and Kapranov [GK94] following foundational ideas of Kontsevich [Kon93]. Briefly, this duality between operads explains certain older duality phenomena in algebra, of which we will give an example (related to rational homotopy theory).

Definition 3.5.6. A differential graded *coalgebra* (*DGC*) is a chain complex C equipped with a « coproduct » $\Delta : C \to C \otimes C$ and a counit $\varepsilon : C \to \mathbb{Q}$ such that the following diagrams commute :

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\Delta\otimes 1"] \\
C \otimes C \arrow[r, "1\otimes\Delta"] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[dr, "1\otimes\varepsilon"'] & C \arrow[l, leftrightarrow, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}_C"] & C \otimes C \arrow[dl, "\varepsilon\otimes 1"] \\
& C &
\end{tikzcd}$$

and satisfying the dual of the Leibniz relation :

$$\Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta.$$

This coalgebra is *cocommutative* (*CDGC*) if $\tau \circ \Delta = \Delta$ where $\tau : C \otimes C \to C \otimes C$ is defined by $\tau(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$.

Definition 3.5.7. The *cofree* (conilpotent) cocommutative coalgebra on a cochain complex V is[12] :

$$S^c(V) := \bigoplus_{n \geq 0} (V^{\otimes n})^{\mathfrak{S}_n}.$$

Its coproduct is given by :

$$\Delta(v_1 \dots v_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} (v_{\sigma(1)} \dots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \dots v_{\sigma(n)}),$$

where $\mathrm{Sh}(p,q) = \{\sigma \in \mathfrak{S}_{p+q} \mid \sigma(1) < \dots < \sigma(p) \text{ and } \sigma(p+1) < \dots < \sigma(p+q)\}$ is the set of (p,q)-shuffles. Its counit is given by the projection onto $(V^{\otimes 0})^{\mathfrak{S}_0} = \mathbb{Q}$. Its differential is defined by :

$$d(v_1 \dots v_n) = \sum_{i=1}^{n} \pm v_1 \dots (dv_i) \dots v_n.$$

It is equipped with a canonical projection $\pi : S^c(V) \to V$.

Proposition 3.5.8. *Let C be a cocommutative coalgebra and V a chain complex. Let $f : C \to V$ be a morphism of chain complexes. Then there exists a unique morphism of CDGCs $\varphi_f : C \to S^c(V)$ making the following diagram commute :*

$$\begin{tikzcd}
& S^c(V) \arrow[d, two heads, "\pi"] \\
C \arrow[ur, dashed, bend left, "\varphi_f"] \arrow[r, "f"] & V
\end{tikzcd}$$

Démonstration. The proof is formally dual to the proof that a morphism of complexes $V \to A$ induces a unique morphism of CDGAs $S(V) \to A$. (Exercise : describe φ_f explicitly.) ☐

---

[12] Recall that V^G denotes the subspace of elements invariant under the action of the group G.

Proposition 3.5.9. *Let V be a chain complex. For any morphism of chain complexes $f : S^c(V) \to V[-1]$ of degree 1, there exists a unique coderivation*[13] $\delta_f : S^c(V) \to S^c(V)$ *making the following diagram commute :*

$$\begin{tikzcd} & S^c(V)[-1] \arrow[d, "\pi"] \\ S^c(V) \arrow[ru, dashed, bend left, "\delta_f"] \arrow[r, "f"] & V[-1] \end{tikzcd}$$

Démonstration. Likewise. (Exercise : describe δ_f explicitly.) ☐

Definition 3.5.10. Let $\mathfrak{g}$ be a DGLA. Its *Chevalley–Eilenberg complex* $C_*^{CE}(\mathfrak{g})$ is the quasi-cofree CDGC :

$$C_*^{CE}(\mathfrak{g}) \coloneqq (S^c(\mathfrak{g}[1]), d + \delta),$$

where the differential is the sum of the internal differential of $S^c(\mathfrak{g}[1])$ and of[14] :

$$\delta(x_1 \dots x_n) = \sum_{1 \le i < j \le n} \pm [x_i, x_j] x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_n.$$

Remark 3.5.11. This additional differential is the unique coderivation induced by $S^c(\mathfrak{g}[1]) \to (\mathfrak{g}[1]^{\otimes 2})^{\mathfrak{S}_2} \xrightarrow{[-,-]} \mathfrak{g}[2]$. One verifies that its sum with the internal differential has square zero, which follows from the Jacobi and Leibniz relations.

Definition 3.5.12. Let C be a CDGC. Its *Harrison complex* is the quasi-free DGLA :

$$C_*^{\mathrm{Harr}}(C) \coloneqq (\mathbb{L}(C[-1]), d + \delta)$$

where the differential is the sum of the internal differential of $\mathbb{L}(C[-1])$ and of the unique Lie algebra derivation induced by $C[-1] \xrightarrow{\Delta} ((C[-1])^{\otimes 2})^{\mathfrak{S}_2}[1] \subset \mathbb{L}(C[-1])[1]$.

Theorem 3.5.13 (Koszul duality)**.** *The functors C_*^{CE} and C_*^{Harr} are adjoint to each other. There exist model category structures on $\mathsf{DGLA}_{\geq 1}$ and $\mathsf{CDGC}_{\geq 2}$ such that this adjunction is a Quillen equivalence.*

Remark 3.5.14. This adjunction is formally similar to the bar/cobar adjunction between differential graded algebras and differential graded coalgebras. The functor C_*^{CE} corresponds to a bar functor, while the functor C_*^{Harr} corresponds to a cobar functor.

Theorem 3.5.15 (Equivalence between Quillen models and Sullivan models)**.** *Let X be a 1-reduced simplicial set of finite type. Then there is a natural zigzag of quasi-isomorphisms :*

$$\Omega^*_{\mathrm{PL}}(X) \simeq \left(C_*^{CE}(\lambda X)\right)^\vee$$

where $\left(C_*^{CE}(\lambda X)\right)^\vee$ is the CDGA dual to the CDGC $C_*^{CE}(\lambda X)$.

---

[13] That is, a map $\delta : C \to C$ such that $\Delta\delta = (\delta \otimes 1 + 1 \otimes \delta)\Delta$.

[14] The notation $\dots \hat{x}_i \dots$ means that the element x_i is omitted from $(\dots)$.

One can thus pass from a Sullivan model to a Quillen model and vice versa.

Remark 3.5.16. One can even do better than that. If $A = (S(V), d)$ is the minimal model of X, then the dual $\mathfrak{g} \coloneqq V^\vee[-1]$ is what is called a *homotopy* Lie (differential graded) algebra, or an L_∞-*algebra*. Consider the restriction $d|_V : V \to S(V)$, which dualizes to a sequence of maps $l_i : \mathfrak{g}^{\otimes i} \to \mathfrak{g}$ of degrees $i - 2$.

- Weight 1 : $l_1 \eqqcolon \delta : \mathfrak{g} \to \mathfrak{g}$ is a differential of degree -1 and makes $\mathfrak{g}$ a chain complex.
- Weight 2 : $l_2 \eqqcolon [-,-]_2 : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ is a bilinear map of degree zero. One verifies that the commutativity of A implies that l_2 is antisymmetric, and the relation $d(ab) = (da)b \pm a(db)$ in A implies that l_2 satisfies the Leibniz relation. However, it does not satisfy the Jacobi relation in general, but...
- Weight 3 : $l_3 \eqqcolon [-,-,-]_3 : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$ is a trilinear map of degree 1 which vanishes on shuffles (generalizing antisymmetry). The compatibility with the differential is expressed as follows :

  $$\begin{aligned}&[x,[y,z]_2]_2 \pm [y,[z,x]_2]_2 \pm [z,[x,y]_2]_2 \\ &\qquad = [dx,y,z]_3 \pm [x,dy,z]_3 \pm [x,y,dz]_3 \pm d[x,y,z]_3.\end{aligned}$$

  The bracket l_3 therefore defines a homotopy between the Jacobi relation for l_2 and the zero map ; in a certain sense, this Jacobi relation is thus satisfied « up to homotopy ».
- Weight 4 : $l_4 \eqqcolon [-,-,-,-]_4 : \mathfrak{g}^{\otimes 4} \to \mathfrak{g}$ is quadrilinear of degree 2 and defines a homotopy for a « higher Jacobi relation » satisfied by l_2 and l_3. One can informally understand this relation as follows. The bracket l_3 defines several homotopies between the Jacobi relation and 0 depending on the order in which one places the variables x, y, z. Concatenating these homotopies, one potentially obtains a homology class in $\mathfrak{g}$ described by this higher Jacobi relation, which is then killed by l_4.
- The sequence continues with $l_5, l_6 \dots$: the differential of l_n kills a higher Jacobi relation involving $l_2, \dots, l_{n-1}$.

The Quillen model λX is a DGLA and is therefore also an L_∞-algebra : the higher brackets $l_3, l_4, \dots$ vanish and the Jacobi relation is strictly satisfied. One can then show that λX is quasi-isomorphic (as an L_∞-algebra) to $\mathfrak{g} = V^\vee[-1]$.

4 Infinity categories

Remark 4.0.1. Higher categories will be treated in much greater detail in the course *Catégories supérieures* by Muriel Livernet. What follows is only a brief overview of the theory.

It sometimes happens that a “natural” definition produces a category where composition is no longer associative. For example, if one naively defines the fundamental groupoid πX of a space X as the category whose objects are the points of X and whose morphisms are paths between two points, one obtains something that strongly resembles a category, except that composition (concatenation of paths) is not strictly associative, but only so up to homotopy. The unit and inverses are not strict either.

One can resolve this problem by “strictifying” : one replaces paths by homotopy classes of paths relative to their endpoints. But in doing so, one loses an enormous amount of information. The naive definition of the fundamental groupoid contains all the homotopy information : $\pi_0(X)$ is the set of connected components of the category, and each connected component is equivalent to the loop space ΩX (based at a point of the component). On the other hand, in the strict definition, only the information of the connected components and the fundamental groups remains ; all the “higher” information (here, in degree ≥ 2) has disappeared. The quotient by the homotopy relation is far too coarse.

A similar phenomenon occurs in the study of model categories. Given a model category C, its homotopy category $\mathrm{Ho}(\mathsf{C})$ no longer contains much information about C. This is particularly apparent when one is interested in homotopy (co)limits : the category $\mathrm{Ho}(\mathsf{C})$ admits in general very few (co)limits, and it is necessary to return to the world of model categories, to consider diagram categories therein, and then to pass back to the homotopy category (schematically, $\mathrm{Ho}(\mathsf{C})^I \neq \mathrm{Ho}(\mathsf{C}^I)$). One of the possible motivations for ∞-categories is to introduce an object halfway between model categories and their homotopy categories, which would lose less information and could be studied with new tools.

Another possible motivation is the following. The definition of a model category involves auxiliary classes of morphisms, the fibrations and the cofibrations. These auxiliary classes are in principle not needed to define the homotopy category $\mathrm{Ho}(\mathsf{C}) = \mathsf{C}[\mathscr{W}^{-1}]$, which depends only on C and the weak equivalences. They serve “only” to be able to explicitly compute morphisms in the homotopy category (which would otherwise be given by unwieldy zigzags). Another motivation for ∞-categories is the inversion of a class of weak equivalences in a given category, without having to introduce the entire machinery of (co)fibrations.

The general philosophy of higher categories (of which ∞-categories are a part) is illustrated by the fundamental groupoid πX of a space X. Rather than taking the quotient by the homotopy relation, one can consider a homotopy as a path between two paths. If $\gamma, \gamma' : I \to X$ are two paths with $\gamma(0) = \gamma'(0) = x_0$, $\gamma(1) = \gamma'(1) = x_1$, and if H is a homotopy between the two, one can represent this by the following diagram :

$$\begin{tikzcd}
x_0 \arrow[r, bend left=50, "\gamma"{name=U}] \arrow[r, bend right=50, "\gamma'"'{name=D}] & x_1 \arrow[Rightarrow, from=U, to=D, "H"]
\end{tikzcd}$$

If $x_0 = x_1$ and $\gamma = \gamma'$ is the constant path, then H represents an element of $\pi_2(X, x_0)$: by adding the information of homotopies (paths between paths) to the fundamental groupoid, one thus recovers homotopy information of degree 2. Equality of paths is replaced by the existence of a path between two paths. One sees in particular that if one is given three composable paths $\gamma, \gamma', \gamma''$, then there exists a path between $\gamma(\gamma'\gamma'')$ and $(\gamma\gamma')\gamma''$, and composition is therefore associative modulo a path between paths.

In categorical terms, if a path is a morphism, then a homotopy is a morphism between morphisms, or a 2-morphism. Of course, there may exist many 2-morphisms between two given morphisms. One can compose 2-morphisms by concatenating paths, but this composition is not associative ! It is only associative modulo a path between homotopies (or a path between paths between paths, which we abbreviate as a 3-morphism).

One will quickly see that once launched in this direction, one must define n-morphisms for all $n \geq 1$ (and for the sake of consistency, we will call objects 0-morphisms). One can compose $(n+1)$-morphisms between two fixed n-morphisms, and all compositions must satisfy compatibility relations among themselves. One then arrives at the notion of a higher category. The higher categories we are going to consider (fundamental groupoids, nerve of a category, simplicial localization of a model category. . .) are such that n-morphisms are invertible as soon as $n \geq 2$; they are called $(\infty, 1)$-categories, or simply ∞-categories. Fundamental groupoids even satisfy the property that all 1-morphisms are invertible ; they are called $(\infty, 0)$-categories, or ∞-groupoids.

Expressing the compatibility relations in an ∞-category quickly turns out to be difficult. There exist many models of ∞-categories and one can refer to [Ber18] for an overview of different models. In these notes, we will be particularly interested in quasi-categories, initially developed by Joyal [Joy08 ; Joy02] and further developed by Lurie [Lur09 ; Lur17] (see also the work of Boardman–Vogt [BV73] where they were called weak Kan complexes). We will also treat the case of simplicial categories. Other models are given for example by topological categories, Segal categories, or complete Segal spaces.

4.1 Nerve of a category

Definition 4.1.1. Let C be a small category. Its *nerve* $N_\bullet\mathsf{C}$ is a simplicial set whose n-simplices are given by sequences of n composable morphisms :

$$N_n\mathsf{C} = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n.\}$$

In particular, $N_0\mathsf{C}$ is the set of objects of C.

The face and degeneracy maps are given by :

$$\begin{aligned}
&d_0(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n) = X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n,\\
&d_i(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n) = X_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_{i+1}\circ f_i} X_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} X_n \ (1 < i < n),\\
&d_n(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n) = X_0 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_{n-1},\\
&s_j(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n) = (X_0 \xrightarrow{f_1} \dots \xrightarrow{f_j} X_j \xrightarrow{\mathrm{id}_{X_j}} X_j \xrightarrow{f_{j+1}} \dots \xrightarrow{f_n} X_n).
\end{aligned}$$

One can verify by hand that this is a simplicial set. However, there is a more "theoretical" point of view that allows for simplification.

Definition 4.1.2. We denote by cat the category of small categories.

Recall that the objects of Δ are the $[n] = \{0 < 1 < \dots < n\}$ for $n \geq 0$ and that the morphisms are the order-preserving maps. Any (partially) ordered set can be viewed as a category : we declare that $\mathrm{Hom}_{[n]}(i, j)$ is a singleton if $i \leq j$, and that it is empty otherwise. A functor $[m] \to [n]$ is the same thing as an order-preserving map.[1] We thus obtain a functor $\Delta \to \mathsf{cat}$ sending the ordered set $[n]$ to the category $[n]$; in other words, we have a cosimplicial object in the category of small categories :

$$[\bullet] : \Delta \to \mathsf{cat},\ [n] \mapsto [n].$$

We then observe that the nerve $N_\bullet\mathsf{C}$ is simply given by :

$$N_\bullet\mathsf{C} = \mathrm{Hom}_{\mathsf{cat}}([\bullet], \mathsf{C})$$

and is therefore indeed a simplicial set. Since $\mathrm{Hom}_{\mathsf{cat}}([\bullet], -)$ is functorial, we also see that :

Proposition 4.1.3. *The nerve defines a functor :*

$$N_\bullet : \mathsf{cat} \to s\mathsf{Set}.$$

This point of view reveals that $N_\bullet$ is analogous to the "singular set" functor $S_\bullet = \mathrm{Hom}_{\mathsf{Top}}(\mathbb{\Delta}^\bullet, -)$. We will see later that $N_\bullet$ has a left adjoint (Proposition 4.2.44), which can be defined in a manner analogous to the geometric realization functor. We can already see immediately :

Proposition 4.1.4. *The functor $N_\bullet$ preserves limits.*

Example 4.1.5. The Yoneda lemma implies that $N_\bullet[n] \cong \Delta^n$.

Let us study the homotopy properties of the nerve.

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[1] In more sophisticated language, the category of partially ordered sets is a full subcategory of the category of categories.

Proposition 4.1.6. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors and $\eta : F \Rightarrow G$ a natural transformation. Then η defines a simplicial homotopy between $N_\bullet F$ and $N_\bullet G$.*

Démonstration. We seek to define a map $H : N_\bullet \mathsf{C} \times \Delta^1 \to N_\bullet \mathsf{D}$ such that $H|_{N_\bullet \mathsf{C} \times \{0\}} = N_\bullet F$ and $H|_{N_\bullet \mathsf{C} \times \{1\}} = N_\bullet G$. The data of a natural transformation $\eta : F \Rightarrow G$ is equivalent (exercise) to the data of a functor $\hat{\eta} : \mathsf{C} \times [1] \to \mathsf{D}$ (where $[1]$ is the category with two objects $0, 1$ and a unique morphism between them) satisfying $\hat{\eta}(c, 0) = F(c)$ and $\hat{\eta}(c, 1) = G(c)$. Applying the nerve functor, we obtain a simplicial map :

$$N_\bullet \hat{\eta} : N_\bullet(\mathsf{C} \times [1]) \to N_\bullet \mathsf{D}.$$

Since $N_\bullet$ preserves limits and $N_\bullet[1] = \Delta^1$, we thus obtain a map $H : N_\bullet \mathsf{C} \times \Delta^1 \to \mathsf{D}$. One then verifies that this is indeed the desired homotopy. ☐

Corollary 4.1.7. *Let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be an adjunction between two small categories. Then $N_\bullet \mathsf{C}$ and $N_\bullet \mathsf{D}$ are homotopy equivalent.*

Démonstration. The unit $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ and the counit $\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ induce homotopies showing that $N_\bullet F$ and $N_\bullet G$ are inverses of each other up to homotopy. ☐

Example 4.1.8. If a small category C has an initial object, then $N_\bullet \mathsf{C}$ is contractible : indeed, in this case, the inclusion $[0] \to \mathsf{C}$ sending 0 to the initial object has a right adjoint. Likewise, if C has a terminal object then $N_\bullet \mathsf{C}$ is contractible (the inclusion then has a left adjoint).

These examples might suggest that the nerve is of little interest ; this is not the case. We will see that the nerve defines a fully faithful inclusion of cat into $s\mathsf{Set}$. In a certain sense, the nerve of C therefore contains all of the category structure of C. It is simply that the homotopy type is too coarse an invariant to recover this category structure. We will see in Section 4.2.6 that one can define a model category structure on $s\mathsf{Set}$ that is better suited to the study of nerves.

Let us answer the following question : when is a simplicial set isomorphic to the nerve of a small category ? The answer is given by conditions resembling those defining a fibrant simplicial set.

Let us first introduce a graphical representation for the nerve. Graphically, we can represent an element of $N_n \mathsf{C} = \mathrm{Hom}_{s\mathsf{Set}}(\Delta^n, N_\bullet \mathsf{C})$ as in Figure 4.1 : each of the vertices $0, 1, \dots, n$ is decorated by an object of C, the edges $i \to i+1$ are decorated by morphisms $f_{i+1} : X_i \to X_{i+1}$, and the other edges are decorated by the appropriate compositions.

We recall the horns $\Lambda^n_k \subset \Delta^n$, defined for $0 \le k \le n$ (Definition 2.3.2).

Definition 4.1.9. A horn $\Lambda^n_k \subset \Delta^n$ is called *inner* if $0 < k < n$. The horns $\Lambda^n_0 \subset \Delta^n$ and $\Lambda^n_n \subset \Delta^n$ are called the *initial horn* and the *final horn*, respectively.

We recall (Lemma 2.3.3) that a simplicial map $\Lambda^n_k \to X$ is equivalent to the data of $(n-1)$-simplices $x_0, \dots, \hat{x}_k, \dots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for all i, j such that the equation makes sense. One can fill the horn if there exists an n-simplex $x \in X_n$ such that $d_i x = x_i$.

$$\begin{tikzcd}
 & X_3 & \\
 & & X_2 \arrow[ul, "f_3"'] \\
X_0 \arrow[uur, "f_3f_2f_1"] \arrow[urr, dashed, "f_2f_1"] \arrow[dr, "f_1"'] & & \\
 & X_1 \arrow[uuu, "f_3f_2"] \arrow[uur, "f_2"'] &
\end{tikzcd}$$

FIGURE 4.1 : Graphical representation of an element of $N_3\mathsf{C}$

$$\begin{tikzcd}
 & X_1 \arrow[dr, "f_2"] & \\
X_0 \arrow[ur, "f_1"] \arrow[rr, dashed, "?"] & & X_2
\end{tikzcd}$$

FIGURE 4.2 : Graphical representation of a horn $\Lambda^2_1 \to N_\bullet\mathsf{C}$

Horns in $N_\bullet\mathsf{C}$ are represented as in Figure 4.1 but with one arrow (and therefore all compositions involving it) missing, see Figure 4.2.

We observe from the preceding figure that one can always fill a horn of type Λ^2_1 : it suffices to put f_2f_1 for the third arrow, and since the diagram commutes, we indeed have a 2-simplex. Moreover, this filling is unique. On the other hand, one cannot always fill a horn of type Λ^2_0 or Λ^2_2 : for that, all morphisms would need to be invertible. More generally :

Lemma 4.1.10. *Let* C *be a small category. The restriction map* $\mathrm{Hom}_{s\mathsf{Set}}(\Delta^n, N_\bullet\mathsf{C}) \to \mathrm{Hom}_{s\mathsf{Set}}(\Lambda^n_k, N_\bullet\mathsf{C})$ *is a bijection for all* $n \geq 0$ *and all* $0 < k < n$. *In other words,* $N_\bullet\mathsf{C} \to *$ *has the* unique *lifting property with respect to the inclusions of inner horns into simplices* $\Lambda^n_k \subset \Delta^n$.

Démonstration. Let $f : \Lambda^n_k \to N_\bullet\mathsf{C}$ be a simplicial map with $0 < k < n$. Let us show that there exists a unique map $\Delta^n \to \mathsf{C}$ that restricts to f. We represent f by $(n-1)$-simplices $x_0, \ldots, \hat{x}_k, \ldots, x_n \in N_{n-1}\mathsf{C}$ satisfying $d_ix_j = d_{j-1}x_i$ for $i < j$. We recall that the horn is inner, so $k \neq 0, n$. Using the relation $d_0x_n = d_{n-1}x_0$, we can write (by applying face maps repeatedly to verify that the objects and morphisms in the middle coincide) :

$$x_0 = X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n,$$
$$x_n = X_0 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_{n-1}.$$

We must then necessarily define $x = X_0 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_n$ and one verifies by hand that $d_ix = x_i$. □

Remark 4.1.11. This lemma means that the canonical map $N_\bullet\mathsf{C} \to \mathrm{cosk}_2 N_\bullet\mathsf{C}$ (where cosk is the coskeleton functor of Remark 2.3.8) is an isomorphism.

Proposition 4.1.12. *The functor* $N_\bullet : \mathsf{cat} \to s\mathsf{Set}$ *is fully faithful, and its image consists of the simplicial sets that have the unique RLP with respect to all inner horns.*

Démonstration. We have seen that if C is a small category, then $N_\bullet\mathsf{C}$ satisfies the desired lifting property. Conversely, let X be a simplicial set that has the RLP with respect to all inner horns. We define a small category C as follows :

- The objects of C are the 0-simplices X_0.
- Let $x, y \in X_0$. The set of morphisms between x and y is given by :

$$\mathrm{Hom}_{\mathsf{C}}(x,y) = \{f \in X_1 \mid d_1 f = x \text{ and } d_0 f = y\}.$$

- The identity of $x \in X_0$ is $s_0(x) \in X_1$.
- Composition is defined as follows. Let x, y, z be three objects and $f : x \to y$, $g : y \to z$ two morphisms. Then f and g together define a horn $\Lambda^2_1 \to X$ (see Figure 4.2). There therefore exists a unique $\gamma \in X_2$ such that $d_0\gamma = g$ and $d_2\gamma = f$. We then define $g \circ f := d_1\gamma$.

One must verify that composition is associative and unital. Associativity is proved exactly as in Proposition 2.5.19. Unitality follows from the uniqueness of liftings and the simplicial identities (exercise).

Let us now show that $N_\bullet$ is fully faithful, i.e. that $N_\bullet : \mathrm{Hom}_{\mathsf{cat}}(\mathsf{C}, \mathsf{D}) \to \mathrm{Hom}_{s\mathsf{Set}}(N_\bullet\mathsf{C}, N_\bullet\mathsf{D})$ is a bijection for all small categories C and D. Let $\varphi : N_\bullet\mathsf{C} \to N_\bullet\mathsf{D}$ be a simplicial map ; let us show that there exists a unique functor F such that $N_\bullet F = \varphi$. Then $\varphi_0 : N_0\mathsf{C} \to N_0\mathsf{D}$ defines a map from the objects of C to the objects of D. We must necessarily define $F : \mathsf{C} \to \mathsf{D}$ by φ_0 on objects. By definition of the nerve, for $x, y \in \mathsf{C} = N_0\mathsf{C}$, there is an equality between $\mathrm{Hom}_{\mathsf{C}}(x,y)$ and $\{f \in N_1\mathsf{C} \mid d_0 f = x \text{ and } d_1 f = y\}$. The map φ_1 sends this subset of $N_1\mathsf{C}$ to the analogously defined subset of $N_1\mathsf{D}$, by virtue of the simplicial identities. We must therefore define F by φ_1 on morphisms, and one verifies that with this definition we obtain a functor $F : \mathsf{C} \to \mathsf{D}$ such that $N_\bullet F = \varphi$. ☐

A nerve is however generally not fibrant :

Definition 4.1.13. A *groupoid* is a category in which all morphisms are invertible.

Lemma 4.1.14. *Let C be a small category. The nerve $N_\bullet\mathsf{C}$ is fibrant (i.e. all horns, inner and outer, can be filled) if and only if C is a groupoid. In this case, the fillings of outer horns are unique.*

Démonstration. Suppose that $N_\bullet\mathsf{C}$ is fibrant. We can therefore fill all horns $\Lambda^2_0 \to N_\bullet\mathsf{C}$ and $\Lambda^2_2 \to N_\bullet\mathsf{C}$. Let $f : x \to y$ be a morphism of C ; we define a horn $\Lambda^2_0 \to N_\bullet\mathsf{C}$ by :

$$\begin{tikzcd}
& x \arrow[rd, dashed] & \\
x \arrow[ru, "\mathrm{id}_x"] \arrow[rr, "f"] & & y
\end{tikzcd}$$

By filling this horn, we obtain a right inverse g of f. By filling a similar horn of type $\Lambda^2_2 \to N_\bullet\mathsf{C}$, we obtain a left inverse g' of f. We have $g = g'fg = g'$ so $g = g'$ is an inverse of f, which is therefore an isomorphism.

The converse and uniqueness are proved as in Lemma 4.1.10, using inverses in C to fill the outer horns. ☐

Example 4.1.15 (Exercise)**.** Let G be a group, viewed as a groupoid with one object. Then $N_\bullet G$ is a Kan complex. This Kan complex has the homotopy type of a classifying space $BG = K(G, 1)$. One can define its (contractible) universal covering by considering the nerve of the "indiscrete" category E_G whose objects are G and with a unique morphism between any pair of objects. There is a functor $E_G \to G$ sending $g \to h$ to hg^{-1}. One verifies that $N_\bullet E_G$ is contractible, and that $N_\bullet E_G \to N_\bullet G$ is a Kan fibration with fiber G.

4.2 Quasi-categories

4.2.1 Definition

Informally, a quasi-category is a simplicial set with properties that mimic those of the nerve of a category. (The nerve of a category will indeed be an example of a quasi-category.) The difference with the nerve of a category lies in uniqueness. In a normal category, there is only one way to compose two morphisms, only one identity per object, only one "associativity equality" between $f(gh)$ and $(fg)h$. In a quasi-category, there are potentially several ways to compose two morphisms ; but two different compositions will be equivalent in a sense to be defined. Since there are several ways to compose morphisms, requiring a strict equality for associativity does not make sense, but one can require that all ways of composing three morphisms be homotopic to each other, and that all homotopies between such homotopies be homotopic to each other, etc. The definition of quasi-categories is a very compact way of encapsulating all these properties.

Definition 4.2.1. A *quasi-category* is a simplicial set that has the RLP with respect to the inclusions of inner horns $\Lambda^n_k \subset \Delta^n$ $(0 < k < n)$:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

A *quasi-functor*[2] is a simplicial map between two quasi-categories. We denote by qcat the category of quasi-categories and their quasi-functors.

Remark 4.2.2. Some authors sometimes use the terminology "∞-categories" and "∞-functors" for quasi-categories and quasi-functors. The model given by quasi-categories is indeed one of the most popular models – at the present time – of ∞-categories. Note however that depending on the applications, some other models may prove more suitable.

Example 4.2.3. A fibrant simplicial set is a quasi-category.

Example 4.2.4. If C is a small category, then its nerve is a quasi-category (Lemma 4.1.10). A quasi-functor $N_\bullet\mathsf{C} \to N_\bullet\mathsf{D}$ is a functor $\mathsf{C} \to \mathsf{D}$ (Proposition 4.1.12). One thus obtains

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[2] The terminology is not standard.

a fully faithful inclusion of the category of small categories into the category of quasi-categories.

Definition 4.2.5. A *quasi-groupoid* is a fibrant simplicial set.

Example 4.2.6. The nerve of a groupoid is a quasi-groupoid (Lemma 4.1.14).

4.2.2 Morphisms

By mimicking the properties of nerves, we arrive at the following definition :

Definition 4.2.7. Let C be a quasi-category. The *objects* are the elements of C_0. Let $x, y \in \mathsf{C}_0$ be two objects. The *morphisms* from x to y are the elements of the set

$$\mathrm{Hom}_{\mathsf{C}}(x,y) \coloneqq \{f \in \mathsf{C}_1 \mid d_1 f = x \text{ and } d_0 f = y\} = \{x\} \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \{y\}.$$

We also write $\mathrm{id}_x \coloneqq s_0(x)$ for the *identity* of x.

The composition of morphisms in a quasi-category is more complex to describe. In this section, we will introduce a *space* of morphisms (rather than a set) between two objects of a quasi-category. Let us briefly explain why this is necessary in order to define composition.

Following the proof of Lemma 4.1.10, one is tempted to define composition in the following way. Let $x, y, z \in \mathsf{C}_0$ be three objects and $f \in \mathrm{Hom}_{\mathsf{C}}(x,y), g \in \mathrm{Hom}_{\mathsf{C}}(y,z)$ two morphisms. The elements $f, g \in \mathsf{C}_1$ together define a horn $\Lambda^2_1 \to \mathsf{C}$ (Figure 4.2). By the lifting property, there exists $\sigma \in \mathsf{C}_2$ such that $d_0\sigma = g$ and $d_2\sigma = f$. One might then want to define $g \circ f = d_1\sigma$.

Unfortunately, this is not a well-defined definition : unlike the nerve of a category, the simplex σ is not unique ! There may potentially exist many simplices satisfying $d_0\sigma = g$ and $d_2\sigma = f$. A priori, the different possible compositions $(d_1\sigma)$ have no reason to be related. However, if we are given two simplices $\sigma, \sigma' \in \mathsf{C}_2$ satisfying the above equations, we obtain a horn $\Lambda^3_1 \to \mathsf{C}$. Using the lifting property, we find $\omega \in \mathsf{C}_3$ such that $d_0\omega = s_1 g$, $d_2\omega = \sigma$, and $d_3\omega = \sigma'$ (see Figure 4.3). We then find that $H = d_1\omega$ is a homotopy between $d_1\sigma$ and $d_1\sigma'$.

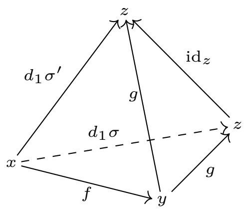

Figure 4.3 : Homotopy between the $g \circ f$

The set of possible compositions of two morphisms is in fact a space. The argument above shows that it is connected. But the homotopies are also related by homotopies between homotopies, etc. One finds that the space of possible compositions is contractible, which is consistent with the intuition behind higher categories from the beginning of the chapter.

We recall the definition of the space of simplicial maps between two simplicial sets (Definition 2.5.2).

Definition 4.2.8. Let C be a quasi-category and $x, y \in \mathsf{C}_0$ objects. The *morphism space* from x to y is :

$$\operatorname{Map}_{\mathsf{C}}(x, y) \coloneqq \{x\} \times_{\mathsf{C}} \operatorname{Map}_{\bullet}(\Delta^1, \mathsf{C}) \times \{y\}.$$

Concretely, this simplicial set is the pullback :

$$\begin{tikzcd}
\operatorname{Map}_{\mathsf{C}}(x, y) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Map}_{\bullet}(\Delta^1, \mathsf{C}) \arrow[d, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"] \\
* \arrow[r, "{(x,y)}"] & \mathsf{C} \times \mathsf{C}
\end{tikzcd}$$

where $* \xrightarrow{(x,y)} \mathsf{C} \times \mathsf{C}$ is the constant map equal to (x, y), and $\mathrm{ev}_0, \mathrm{ev}_1 : \operatorname{Map}_{\bullet}(\Delta^1, \mathsf{C}) \to \operatorname{Map}_{\bullet}(\Delta^0, \mathsf{C}) \cong \mathsf{C}$ are obtained by precomposition by $\partial^0, \partial^1 : \Delta^0 \to \Delta^1$.

Proposition 4.2.9. *Let C be a quasi-category and $x, y \in \mathsf{C}_0$ two objects. The 0-simplices of $\operatorname{Map}_{\mathsf{C}}(x, y)$ are the morphisms from x to y :*

$$\operatorname{Map}_{\mathsf{C}}(x, y)_0 = \operatorname{Hom}_{\mathsf{C}}(x, y).$$

Démonstration. Pullbacks in $s\mathsf{Set}$ are computed dimension by dimension. In dimension 0, we obtain $\operatorname{Map}_{\mathsf{C}}(x, y)_0 = \{x\} \times_{\mathsf{C}_0} \times \operatorname{Map}_0(\Delta^1, \mathsf{C}) \times_{\mathsf{C}_0} \{y\}$. Now $\operatorname{Map}_0(\Delta^1, \mathsf{C}) = \operatorname{Hom}_{s\mathsf{Set}}(\Delta^1, \mathsf{C}) \cong \mathsf{C}_1$ and we recover Definition 4.2.7. ☐

Proposition 4.2.10. *Let C be a quasi-category and $x, y \in \mathsf{C}_0$ two objects. The morphism space $\operatorname{Map}_{\mathsf{C}}(x, y)$ is a quasi-groupoid, i.e. it is a Kan complex.*

[REVIEWER NOTE : The proof of this proposition appears problematic as written, since it uses a fibrancy property of C that has not been established.]

Démonstration. The inclusion $\Delta^0 \sqcup \Delta^0 \xrightarrow{(\partial^0, \partial^1)} \Delta^1$ is a cofibration. It follows that its induced map $\operatorname{Map}_{\bullet}(\Delta^1, \mathsf{C}) \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} \mathsf{C} \times \mathsf{C}$ is a fibration (Proposition 2.5.7). Since $\operatorname{Map}_{\mathsf{C}}(x, y) \to *$ is the pullback of a fibration, it is itself a fibration. The simplicial set $\operatorname{Map}_{\mathsf{C}}(x, y)$ is therefore fibrant, hence by definition a quasi-groupoid. ☐

Proposition 4.2.11 (Exercise)**.** *Let $F : \mathsf{C} \to \mathsf{D}$ be a quasi-functor between two quasi-categories. Then postcomposition $F \circ - : \operatorname{Map}_{\mathsf{C}}(x, y) \to \operatorname{Map}_{\mathsf{D}}(F_0(x), F_0(y))$ is a simplicial map.* ☐

Proposition 4.2.12. *Let X be a simplicial set and C a quasi-category. Then $\mathsf{C}^X \coloneqq \operatorname{Map}_{\bullet}(X, \mathsf{C})$ is a quasi-category.*

Démonstration. Similar to Proposition 2.5.7, except that we only consider lifting properties with respect to inner horns. ☐

Remark 4.2.13. This proposition is to be compared with the discussion in Section 1.7. If C is a model category, it is not straightforward to give a model category structure to the diagram category C^I : one must choose the right fibrations and cofibrations, and impose conditions on I. Here, if C is a quasi-category, then C^I is automatically a quasi-category. (The difference lies of course in the computations : once the model category structure is defined on C^I, computing a homotopy (co)limit follows a clear procedure, which is not necessarily the case in the ∞-categorical setting.)

Definition 4.2.14. Let C, D be two quasi-categories. The quasi-category of quasi-functors from C to D is $\mathrm{Map}_\bullet(\mathsf{C}, \mathsf{D})$. The *quasi-natural transformations* are the 1-simplices of this quasi-category.

Remark 4.2.15. Defining a quasi-functor between two quasi-categories is not an easy task : one must define it on objects and morphisms, but also on 2-simplices (homotopies), 3-simplices (homotopies between homotopies), etc., in a coherent manner.

4.2.3 Composition

Recall that if $f \in \mathrm{Hom}_{\mathsf{C}}(x, y)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(y, z)$ are two composable morphisms, then a possible composition of g and f is a morphism $h : x \to z$ such that there exists $\sigma \in \mathsf{C}_2$ satisfying $d_0\sigma = g$, $d_1\sigma = h$ and $d_2\sigma = f$ (see Figure 4.2). These possible compositions are in fact the 0-simplices of a simplicial set.

Definition 4.2.16. Let C be a quasi-category, $x, y, z \in \mathsf{C}_0$ three objects and $f \in \mathrm{Hom}_{\mathsf{C}}(x, y), g \in \mathrm{Hom}_{\mathsf{C}}(y, z)$ two composable morphisms. The space of *possible compositions* of f with g is defined by the pullback :

$$\begin{tikzcd}
\mathrm{Comp}_\bullet(g; f) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_\bullet(\Delta^2, \mathsf{C}) \arrow[d, "i^*"] \\
* \arrow[r, "{(g,f)}"] & \mathrm{Map}_\bullet(\Lambda^2_1, \mathsf{C})
\end{tikzcd}$$

where $i : \Lambda^2_1 \to \Delta^2$ is the inclusion and $(g, f) : * \to \mathrm{Map}_\bullet(\Lambda^2_1, \mathsf{C})$ is the constant map equal to the horn of Figure 4.2.

Concretely, a possible composition of g and f is a 2-simplex σ satisfying $d_2\sigma = f$ and $d_0\sigma = g$. A possible composition is therefore not simply a morphism $h = d_1\sigma$; it is also a 2-simplex that “witnesses” the fact that h is a candidate for $g \circ f$. For $\sigma \in \mathrm{Comp}_\bullet(g; f)$, we write :

$$g \circ_\sigma f := d_1\sigma$$

for the morphism that is a candidate for $g \circ f$.

The goal of this section is to show that $\mathrm{Comp}_\bullet(g; f)$ is contractible. This will have as a corollary that all possible compositions $g \circ_\sigma f$ of two given morphisms are homotopic, in a sense that remains to be defined.

Theorem 4.2.17 (Joyal [Joy]). *Let* C *be a quasi-category. The restriction map* $i^* : \mathrm{Map}_\bullet(\Delta^2, \mathsf{C}) \to \mathrm{Map}_\bullet(\Lambda^2_1, \mathsf{C})$ *is an acyclic fibration.*

Démonstration. The proof that

$$i^* : \mathrm{Map}_\bullet(\Delta^2, \mathsf{C}) \to \mathrm{Map}_\bullet(\Lambda^2_1, \mathsf{C}) \cong \mathrm{Map}_\bullet(\Lambda^2_1, \mathsf{C}) \times_{\mathrm{Map}(\Lambda^2_1, *)} \mathrm{Map}_\bullet(\Delta^2, *)$$

is a fibration is an adaptation of the proof of Proposition 2.5.7 by considering the RLP with respect to inner horns. ☐

Remark 4.2.18. The converse is also true : if i^* is an acyclic fibration, then C is a quasi-category [Lur09, Corollary 2.3.2.2].

Corollary 4.2.19. *Let* C *be a quasi-category,* $x, y, z \in \mathsf{C}_0$ *three objects and* $f \in \mathrm{Hom}_{\mathsf{C}}(x, y)$*,* $g \in \mathrm{Hom}_{\mathsf{C}}(y, z)$ *two composable morphisms. Then* $\mathrm{Comp}_\bullet(g; f)$ *is contractible.*

Démonstration. The map $\mathrm{Comp}_\bullet(g; f) \to *$ is the pullback of an acyclic fibration, hence is itself an acyclic fibration. ☐

Let us now explain how this implies that all possible compositions of g and f are "equivalent".

Lemma 4.2.20. *Let* C *be a quasi-category and* $f, g \in \mathrm{Hom}_{\mathsf{C}}(x, y)$ *two morphisms of* C*. If there exists a* 2*-simplex filling one of the following diagrams, then there exist* 2*-simplices filling the others :*

$$\begin{tikzcd}
& x \arrow[dr, "g"] & & & y \arrow[dr, "\mathrm{id}_y"] & & & x \arrow[dr, "f"] & & & y \arrow[dr, "\mathrm{id}_y"] & \\
x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"] & & y & x \arrow[ur, "g"] \arrow[rr, "f"] & & y & x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "g"] & & y & x \arrow[ur, "f"] \arrow[rr, "g"] & & y
\end{tikzcd}$$

In other words, we have the equivalences :

$$\begin{aligned}
\exists \sigma, g \circ_\sigma \mathrm{id}_x = f &\iff \exists \sigma, \mathrm{id}_y \circ_\sigma g = f \\
&\iff \exists \sigma, f \circ_\sigma \mathrm{id}_x = g \\
&\iff \exists \sigma, \mathrm{id}_y \circ_\sigma f = g.
\end{aligned}$$

Démonstration. For each of the equivalences, we will use the degeneracies of f and g to define inner horns which we then fill. For example, suppose that $g \circ_\sigma \mathrm{id}_x = f$ for some $\sigma \in \mathsf{C}_2$. Then we have a horn $(s_1 g, -, s_0 g, \sigma) : \Lambda^3_1 \to \mathsf{C}$. We fill it to obtain $\omega \in \mathsf{C}_3$, and then $\tau = d_1 \omega$ satisfies $\mathrm{id}_y \circ_\tau f = g$:

$$\begin{tikzcd}
& y & \\
& y \arrow[u, "\mathrm{id}_y"] & \\
x \arrow[uur, bend left, "g"] \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"] & & x \arrow[uul, bend right, "g"'] \arrow[ul, "g"']
\end{tikzcd}$$

☐

Definition 4.2.21. Let C be a quasi-category. Two morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}}(x, y)$ are *equivalent*[3] (written $f \simeq g$) if they satisfy the conditions of the preceding lemma.

Lemma 4.2.22. *The equivalence of morphisms in a quasi-category is an equivalence relation.*

Démonstration. Reflexivity follows from the degeneracies. Symmetry follows from the definition. Transitivity is proved as in the preceding lemma. ☐

Recall that $\mathrm{Map}_{\mathsf{C}}(x, y)$ is a Kan complex (Proposition 4.2.10), so its connected components are well defined (see Section 2.5.2).

Lemma 4.2.23. *Two morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}}(x, y)$ of a quasi-category C are equivalent if and only if they lie in the same connected component of $\mathrm{Map}_{\mathsf{C}}(x, y)$.*

Démonstration. One can describe $\Delta^1 \times \Delta^1$ as two copies of Δ^2 glued along a common Δ^1 (Figure 4.4). We therefore have :

$$\mathrm{Hom}_{s\mathsf{Set}}(\Delta^1 \times \Delta^1, \mathsf{C}) = \{(\sigma, \tau) \in \mathsf{C}_2 \times \mathsf{C}_2 \mid d_1\sigma = d_1\tau\}.$$

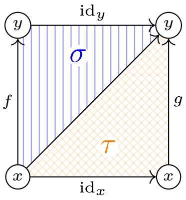

Figure 4.4 : Representation of an element of $\mathrm{Map}_{\mathsf{C}}(x, y)_1$

By definition of the pullback, we also have

$$\begin{aligned}\mathrm{Map}_{\mathsf{C}}(x, y)_1 &= \{(\sigma, \tau) \in \mathrm{Hom}_{s\mathsf{Set}}(\Delta^1 \times \Delta^1, \mathsf{C}) \mid d_0\sigma = s_0 y \text{ and } d_2\tau = s_0 x\} \\ &= \{(\sigma, \tau) \in \mathsf{C}_2 \times \mathsf{C}_2 \mid d_0\sigma = s_0 y,\ d_1\sigma = d_1\tau,\ d_2\tau = s_0 x\}.\end{aligned}$$

Suppose that f and g are in the same connected component. There exists $\varphi = (\sigma, \tau)$ as above such that $d_1\varphi = d_2\sigma = f$ and $d_0\varphi = d_0\tau = g$. Set $h = d_1\sigma = d_1\tau$. Then the simplex σ gives an equivalence between f and h, while the simplex τ gives an equivalence between g and h. We conclude by transitivity and symmetry.

Conversely, if f and g are equivalent, then the equivalence fills one half of the diagram in Figure 4.4 ; we fill the other half by a degeneracy (of f or g) to obtain a path in $\mathrm{Map}_{\mathsf{C}}(x, y)$. ☐

Corollary 4.2.24. *Let C be a quasi-category and $f \in \mathrm{Hom}_{\mathsf{C}}(x, y)$, $g \in \mathrm{Hom}_{\mathsf{C}}(y, z)$ two composable morphisms of C. All morphisms $h \in \mathrm{Comp}(g; f)$ are equivalent in the sense of Definition 4.2.21.*

---

[3] One sometimes says “homotopic”.

Démonstration. The simplicial set $\mathrm{Comp}(g;f)$ is contractible and therefore has only one connected component. ☐

Let us now study the associativity of possible compositions.

Proposition 4.2.25. *Let* C *be a quasi-category and* $f \in \mathrm{Hom}_{\mathsf{C}}(x,y), g \in \mathrm{Hom}_{\mathsf{C}}(y,z), h \in \mathrm{Hom}_{\mathsf{C}}(z,w)$ *three pairwise composable morphisms. Then for all possible compositions we have :*

$$h \circ_\kappa (g \circ_\sigma f) \simeq (h \circ_\tau g) \circ_\lambda f.$$

Démonstration. We fix κ, σ and τ. By the preceding corollary, it suffices to find $\lambda \in \mathsf{C}_2$ such that $d_2\lambda = f$ and $d_0\lambda = h \circ_\tau g (= d_1\tau)$ and $d_1\lambda = h \circ_\kappa (g \circ_\sigma f)$. We define a horn $(\tau, \kappa, -, \sigma) : \Lambda^3_2 \to \mathsf{C}$ (exercise : draw a picture). We fill this horn and take the face d_2 to obtain λ. ☐

4.2.4 Homotopy category

Thanks to all the properties we have just proved, we can define :

Definition 4.2.26. Let C be a quasi-category. The *homotopy category* of C, denoted $\pi\mathsf{C}$, is the category whose objects are the 0-simplices of C and whose morphisms are the equivalence classes (in the sense of Definition 4.2.21) of morphisms of C. The composition of two equivalence classes of morphisms $[f], [g]$ is the unique equivalence class of $\mathrm{Comp}(g;f)$. The units are the classes of the morphisms $\mathrm{id}_x = s_0x$.

Corollary 4.2.27. *There is a functor* $\mathsf{qcat} \to \mathsf{cat}$ *that associates to a quasi-category its homotopy category.* ☐

Remark 4.2.28. A priori, this construction has nothing to do with that of the homotopy category of a model category. We will compare them in Section 4.5.

Proposition 4.2.29. *There are two transformations, natural in* $\mathsf{C} \in \mathsf{qcat}$ *and* $\mathsf{D} \in \mathsf{cat}$ *respectively :*

$$\mathsf{C} \to N_\bullet(\pi\mathsf{C}), \qquad \pi(N_\bullet\mathsf{D}) \xrightarrow{\sim} \mathsf{D}.$$

The natural transformation $\pi(N_\bullet\mathsf{D}) \to \mathsf{D}$ *is an equivalence for every category* D.

Démonstration. The two natural transformations are essentially induced by the identity. The fact that the second natural transformation is an equivalence at each category follows from the fact that $N_\bullet : \mathsf{cat} \to \mathsf{qcat}$ is fully faithful. ☐

Definition 4.2.30. Let C be a quasi-category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(x,y)$ is called an *isomorphism* if $\pi(f) \in \mathrm{Hom}_{\pi\mathsf{C}}(x,y)$ is one.

Proposition 4.2.31 (Exercise)**.** *A quasi-category* C *is a quasi-groupoid if and only if all its morphisms are isomorphisms.*

Example 4.2.32. Let X be a topological space. The singular set $S_\bullet(X)$ being fibrant, it is a quasi-groupoid. By Proposition 2.5.21, $\pi S_\bullet(X)$ is the fundamental groupoid of X. One can therefore view $S_\bullet(X)$ as a "higher" version of the fundamental groupoid, which retains information in dimension > 1.

4.2.5 Non-small categories

Up to now, we have focused on small categories, that is, categories whose objects and morphisms form sets. However, many important examples of model categories are not small : since they are complete and cocomplete by definition, this is almost never the case ! In this section, we briefly explain how to generalize the definition of quasi-categories to encompass non-small quasi-categories.

From a set-theoretic point of view, one solution is to consider a larger "universe", such that all sets of the initial universe form a set in the new universe. For information on this point of view, one may for example refer to [Lur09, Section 1.2.15]. In particular, it is necessary to add an additional axiom to ZFC : the existence of strongly inaccessible cardinals. We denote by Cat the category of not-necessarily-small categories.

One can then define "simplicial classes" as simplicial objects in this new universe. A simplicial class is a sequence of classes[4] $X_\bullet = \{X_n\}_{n\geq 0}$ equipped with maps $d_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ satisfying the simplicial identities. If $\mathsf{C} \in \mathsf{Cat}$ is any category, one can define its nerve $N_\bullet\mathsf{C} = \mathrm{Hom}_{\mathsf{Cat}}([\bullet], \mathsf{C})$ in the usual way. All the results of Section 4.1 still apply, replacing "set" by "class" where necessary.

One can also define quasi-categories as simplicial classes that have the RLP with respect to inclusions of inner horns $\Lambda^n_k \subset \Delta^n$ (for $0 < k < n$). Again, all the results of Section 4.2 apply, replacing "set" by "class". We denote by QCat the category of not-necessarily-small quasi-categories. The nerve defines a functor $N_\bullet : \mathsf{Cat} \to \mathsf{QCat}$, and the homotopy category a functor $\pi : \mathsf{QCat} \to \mathsf{Cat}$.

4.2.6 Joyal model structure

In this section, we will study equivalences between quasi-categories. We saw earlier that these could not simply be given by weak homotopy equivalences : if a category C has an initial or terminal object, then $N_\bullet\mathsf{C}$ is contractible. It is therefore necessary to have a finer invariant. We will also explain how these equivalences can be placed within the framework of a model category on simplicial sets whose fibrant objects are exactly the quasi-categories and whose weak equivalences between quasi-categories model equivalences of categories.

Definition 4.2.33. A *categorical equivalence*[5] is a simplicial map $f : X \to Y$ such that for every quasi-category C, the induced simplicial map $f^* : \mathrm{Map}(Y, \mathsf{C}) \to \mathrm{Map}(X, \mathsf{C})$ is an equivalence.

Definition 4.2.34. The *Joyal structure* on $s\mathsf{Set}$ is given by $s\mathsf{Set}^J := (s\mathsf{Set}, \mathscr{W}^I, \mathscr{C}^I, \mathscr{F}^I)$, where

- the weak equivalences $\mathscr{W}^J$ are the categorical equivalences ;
- the cofibrations $\mathscr{C}^J = \mathscr{C}$ are the inclusions ;

---

[4] This is the name given to sets of the new universe that were not necessarily sets in the old one. An example of a class would be the set of all sets of the old universe.

[5] Joyal called them "weak categorical equivalences".

- the fibrations $\mathscr{F}^J$ are the simplicial maps having the RLP with respect to acyclic cofibrations.

Theorem 4.2.35 (Joyal [Joy], Lurie [Lur09, Theorem 2.2.5.1])**.** *The Joyal structure on* $s\mathsf{Set}$ *defines a cofibrantly generated and left proper model category structure.*

Remark 4.2.36. In Lurie's proof, categorical equivalences are defined differently, but the final result is the same (see [Lur09, Remark 2.2.5.9]).

Proposition 4.2.37 ([Lur09, Theorem 2.4.6.1])**.** *The fibrant objects in the Joyal structure are exactly the quasi-categories.*

Remark 4.2.38. One can show that a categorical equivalence is a weak homotopy equivalence (exercise : do so, using the fact that a Kan complex is a quasi-category). The standard structure on $s\mathsf{Set}$ is therefore a left Bousfield localization of $s\mathsf{Set}^J$; equivalently, $s\mathsf{Set}^J$ is a left Bousfield delocalization of the standard structure. In particular, there is a Quillen adjunction $\mathrm{id} : s\mathsf{Set} \leftrightarrows s\mathsf{Set}^J : \mathrm{id}$.

Example 4.2.39 (Exercise)**.** An acyclic fibration is a categorical equivalence.

Proposition 4.2.40 (Exercise)**.** *Let* $F : \mathsf{C} \to \mathsf{D}$ *be an equivalence of categories. Then* $N_\bullet F : N_\bullet \mathsf{C} \to N_\bullet \mathsf{D}$ *is a categorical equivalence.*

Consider the functor $R : s\mathsf{Set} \to s\mathsf{Set}$ of fibrant replacement in the Quillen structure (which associates a Kan complex to any simplicial set). Intuitively, this functor associates to a quasi-category its underlying quasi-groupoid.

Example 4.2.41 (Exercise)**.** Let C be a category. Then $R(N_\bullet \mathsf{C})$ is isomorphic to the nerve of the underlying groupoid of C.

Proposition 4.2.42. *There exists a model category structure on* Cat*, called the* canonical structure*, such that :*

- *the weak equivalences are the equivalences of categories ;*
- *the fibrations are the* isofibrations *: these are the functors* $p : \mathsf{E} \to \mathsf{B}$ *such that for every object* $e \in \mathsf{E}$ *and every isomorphism* $\varphi : p(e) \xrightarrow{\cong} b \in \mathsf{B}$*, there exists an isomorphism* $\psi : e \xrightarrow{\cong} e'$ *such that* $p(\varphi) = \psi$ *;*
- *the cofibrations are the functors injective on objects.*

Remark 4.2.43. If one assumes the axiom of choice, this model category structure is the unique model category structure such that the weak equivalences are the equivalences of categories, hence the name "canonical" [Sch12]. Its homotopy category $\mathrm{Ho}(\mathsf{C})$ has categories as objects and classes of natural isomorphisms of functors as morphisms.

Proposition 4.2.44. *The nerve is a right Quillen adjoint to the composition* $\pi \circ R$ *:*

$$\pi \circ R : s\mathsf{Set}^J \leftrightarrows \mathsf{Cat} : N_\bullet.$$

Remark 4.2.45. For formal reasons, one could have defined this adjoint in a manner analogous to the geometric realization functor. First note that if X is a set and C a category, then $X \times \mathsf{C}$ is a category : we view X as a discrete category (all morphisms are identities) and form the product in cat. Concretely, $X \times \mathsf{C}$ consists of several disjoint copies of C, one for each element of X.

Now let $X_\bullet \in s\mathsf{Set}$ be a simplicial set. One can define $h(X)$ as the following coequalizer, in the category of categories :

$$h(X) \coloneqq \operatorname{coeq}\left(\bigsqcup_{n,m\geq 0} X_m \times [n] \rightrightarrows \bigsqcup_{n \geq 0} X_n \times [n] \right),$$

where the functors $X_m \times [n] \to X_n \times [n]$ are induced by the simplicial structure of X, while the functors $X_m \times [n] \to X_m \times [m]$ are induced by the cosimplicial structure of $[\bullet]$. (This colimit is an example of a coend : $h(X) = \int_{n \in \Delta} X_n \times [n]$.)

One then verifies, by a purely formal argument, that h is a left adjoint to $N_\bullet = \operatorname{Hom}_{\mathsf{cat}}([\bullet], -)$. Note however that computing colimits in cat is not easy, especially when one identifies morphisms between different objects. One would in particular need to verify that this h is indeed given by the composition $\pi \circ R$.

4.3 Simplicial categories

4.3.1 Definition and model structure

Another model for ∞-categories is given by categories whose morphism sets form simplicial sets. (It is also possible to consider categories with topological spaces of morphisms – see Section 2.5).

Definition 4.3.1. A *simplicial category* C consists of the following data :

- a class of objects, $\operatorname{ob} \mathsf{C}$;
- for each pair of objects (x, y), a simplicial set $\operatorname{Map}_{\mathsf{C}}(x, y)$;
- for each object x, a vertex $\mathrm{id}_x \in \operatorname{Map}_{\mathsf{C}}(x, y)_0$;
- for each triple of objects (x, y, z), a simplicial composition map

$$\operatorname{Map}_{\mathsf{C}}(y, z) \times \operatorname{Map}_{\mathsf{C}}(x, y) \xrightarrow{\circ} \operatorname{Map}_{\mathsf{C}}(x, z);$$

satisfying the usual associativity and unitality axioms. A functor of simplicial categories is defined analogously to ordinary functors. We denote by Cat_Δ the category of simplicial categories.

Remark 4.3.2. Contrary to what the terminology might suggest, a simplicial category *is not* a simplicial object in the category of categories. They are sometimes called “simplicially enriched categories” or “$s\mathsf{Set}$-categories” to mark the difference.

Proposition 4.3.3 (Exercise)**.** *There is an adjunction :*

$$\iota : \mathsf{Cat} \leftrightarrows \mathsf{Cat}_\Delta : \pi_0,$$

where :

- *the functor ι assigns to a category C the simplicial category $\iota\mathsf{C}$ which has the same objects and such that $\mathrm{Map}_{\iota\mathsf{C}}(x,y)$ is the discrete simplicial set on $\mathrm{Hom}_{\mathsf{C}}(x,y)$;*
- *the functor π_0 assigns to a simplicial category D its* homotopy category *$\pi_0\mathsf{D}$ with the same objects and $\mathrm{Hom}_{\pi_0\mathsf{C}}(x,y) = \pi_0 \mathrm{Map}_{\mathsf{C}}(x,y)$.*

Démonstration. This follows essentially from the fact that there is an adjunction $\iota : \mathsf{Set} \leftrightarrows s\mathsf{Set} : \pi_0$ where ι assigns a discrete simplicial set to a set. □

Definition 4.3.4. Let C be a simplicial category. An *equivalence* in C is a morphism f such that $[f]$ is an isomorphism in $\pi_0\mathsf{C}$.

Lemma 4.3.5 (Exercise)**.** *The functor ι is fully faithful.* □

Example 4.3.6. Every category C thus defines a “discrete” simplicial category $\iota\mathsf{C}$.

Definition 4.3.7. A simplicial category C has an *underlying category $U\mathsf{C}$* which has the same objects as C and such that $\mathrm{Hom}_{U\mathsf{C}}(x,y) = \mathrm{Map}_{\mathsf{C}}(x,y)_0$. We say that a simplicial category *extends* its underlying category.

Example 4.3.8. The category $s\mathsf{Set}$ of simplicial sets extends to a simplicial category via the morphism spaces $\mathrm{Map}_\bullet(X,Y)$ of Definition 2.5.2.

Example 4.3.9. The category CDGA of commutative differential graded algebras extends to a simplicial category with the morphism spaces $\mathrm{Map}_\bullet(A,B)$ of Definition 3.3.15.

Example 4.3.10. The category Cat_Δ itself extends to a simplicial category : given two simplicial categories C and D, the set of (simplicial) functors $\mathsf{C} \to \mathsf{D}$ extends to a simplicial set whose 1-simplices are (simplicial) natural transformations, etc.

Theorem 4.3.11 (Bergner [Ber07])**.** *There exists a cofibrantly generated model category structure on Cat_Δ, called the* Bergner structure *where :*

- *the* Dwyer–Kan equivalences *are the functors $F : \mathsf{C} \to \mathsf{D}$ such that*
  1. *the functor $\pi_0 F : \pi_0\mathsf{C} \to \pi_0\mathsf{D}$ is an equivalence of categories ;*
  2. *for all $x, y \in \mathsf{C}$, the induced map $\mathrm{Map}_{\mathsf{C}}(x,y) \to \mathrm{Map}_{\mathsf{D}}(F(x),F(y))$ is a weak equivalence of simplicial sets ;*
- *the* Dwyer–Kan fibrations *are the functors $F : \mathsf{C} \to \mathsf{D}$ such that :*
  1. *for all $x, y \in \mathsf{C}$, the induced map $\mathrm{Map}_{\mathsf{C}}(x,y) \to \mathrm{Map}_{\mathsf{D}}(F(x),F(y))$ is a fibration of simplicial sets ;*
  2. *for all $x \in \mathsf{C}$, $y \in \mathsf{D}$ and for every equivalence $\gamma : F(x) \to y$, there exists an object $x' \in \mathsf{C}$ and an equivalence $\gamma' : x \to x'$ such that $F(\gamma) = \gamma'$;*

- *the* Dwyer–Kan cofibrations *are the functors that have the LLP with respect to acyclic fibrations.*

One can show that a category is fibrant if and only if all its morphism spaces are fibrant. Not all categories are cofibrant.

Proposition 4.3.12 (Exercise)**.** *The following adjunction is a Quillen adjunction, where* Cat_Δ *is equipped with the Quillen structure and* Cat *is equipped with its canonical structure (Proposition 4.2.42) :*

$$\pi_0 : \mathsf{Cat}_\Delta \leftrightarrows \mathsf{Cat} : \iota.$$

4.3.2 Comparison with quasi-categories

In this section, we introduce a functor $\mathsf{cat}_\Delta \to s\mathsf{Set}$ that generalizes the nerve functor $N_\bullet = \mathrm{Hom}_{\mathsf{cat}}([\bullet], -)$. The category $[n]$ extends to a (discrete) simplicial category $\iota[n]$ which is not cofibrant. For our construction to have a homotopical meaning, one possibility is to resolve this simplicial category.

Definition 4.3.13. Let $n \geq 0$ be an integer. For $0 \leq i \leq j \leq n$, define $P_{i,j}$ as the partially ordered set of subsets of $\{i, \dots, j\}$ that contain both i and j. We then define a simplicial category $[[n]]$ whose objects are the integers $0, \dots, n$ and whose morphism spaces are :

$$\mathrm{Map}_{[[n]]}(i,j) = \begin{cases} \varnothing, & \text{if } i > j; \\ N_\bullet P_{i,j}, & \text{if } i \leq j. \end{cases}$$

The composition $\mathrm{Map}_{[[n]]}(i,j) \times \mathrm{Map}_{[[n]]}(j,k) \to \mathrm{Map}_{[[n]]}(i,k)$ (for $i \leq j \leq k$) is induced by the map $P_{i,j} \times P_{j,k} \to P_{i,k}$, $(I,J) \mapsto I \cup J$. The unit is the unique element $\{i\} \in \mathrm{Map}_{[[n]]}(i,i) = N_0 P_{ii} = P_{ii}$.

Lemma 4.3.14 (Exercise)**.** *The collection* $[[\bullet]]$ *defines a cosimplicial object in* Cat_Δ. ☐

Lemma 4.3.15. *For all* $n \geq 0$*, there is a Dwyer–Kan equivalence :*

$$[[n]] \xrightarrow{\sim} \iota[n].$$

Démonstration. For $0 \leq i < j \leq n$, there is an isomorphism $\mathrm{Map}_{[[n]]}(i,j) \cong (\Delta^1)^{j-i-1}$. Moreover, $\mathrm{Map}_{\iota[n]}(i,j) = *$. Each of these morphism spaces is therefore contractible, and one readily verifies that $[[n]] \to \iota[n]$ is indeed a Dwyer–Kan equivalence. ☐

Remark 4.3.16. The simplicial category $[[n]]$ is a "free resolution" of $[n]$ (see [DK80c, Section 2.5] or [Ber07, Definition 3.5.2] where $[[n]]$ is denoted $F_*[n]$). A vertex in $\mathrm{Map}_{[[n]]}(i,j)_0 = P_{ij}$ can be seen as a path between i and j in $[n]$ (represented by the sequence of objects through which it passes).

Definition 4.3.17. The *coherent nerve* (also called the "simplicial nerve") is the functor

$$N^\Delta_\bullet := \mathrm{Hom}_{\mathsf{Cat}_\Delta}([[\bullet]], -) : \mathsf{Cat}_\Delta \to s\mathsf{Set}.$$

Remark 4.3.18 (Exercise). If C is a category, then $N_\bullet^\Delta(\iota\mathsf{C}) \cong N_\bullet(\mathsf{C})$.

Theorem 4.3.19 (Dugger–Spivak [DS11], Lurie [Lur09])**.** *The coherent nerve is part of a Quillen equivalence :*

$$\mathfrak{C} : s\mathsf{Set}^J \leftrightarrows \mathsf{Cat}_\Delta : N_\bullet^\Delta$$

where $s\mathsf{Set}^J$ *is equipped with the Joyal structure (Definition 4.2.34) and* Cat_Δ *with the Bergner structure (Theorem 4.3.11).*

Remark 4.3.20. One can show that a simplicial map $f : X \to Y$ is a categorical equivalence if and only if $\mathfrak{C}(f)$ is a Dwyer–Kan equivalence [Lur09, Proposition 2.2.5.8].

Remark 4.3.21. The functor $\mathfrak{C}$, called the *rigidification functor*, can be computed as follows. By the Yoneda lemma, $\mathrm{Hom}_{\mathsf{Cat}_\Delta}(\mathfrak{C}(\Delta^n), \mathsf{C}) \cong \mathrm{Hom}_{s\mathsf{Set}}(\Delta^n, N_\bullet^\Delta \mathsf{C}) \cong N_n^\Delta \mathsf{C} := \mathrm{Hom}_{\mathsf{Cat}_\Delta}([[n]], \mathsf{C})$. We must therefore have $\mathfrak{C}(\Delta^n) = [[n]]$. Since $\mathfrak{C}$ is a left adjoint, it preserves colimits. Every simplicial set is a colimit of simplices, from which we deduce that

$$\mathfrak{C}(X) = \mathrm{colim}_{f:\Delta^n \to X}[[n]].$$

(This is an example of a left Kan extension.) If C is a quasi-category, then $\mathrm{Map}_{\mathfrak{C}(\mathsf{C})}(x, y) \simeq \mathrm{Map}_{\mathsf{C}}(x, y)$ [DS11, Corollary 5.3].

Quasi-categories and (fibrant) simplicial categories are therefore both models for ∞-categories. One advantage of simplicial categories is that composition is strictly defined, whereas in a quasi-category one only has a (contractible) space of possible compositions. However, the space of functors between two fibrant simplicial categories is not necessarily fibrant (compare with Proposition 4.2.10). It is therefore necessary to replace this functor space by a Kan complex before applying homotopical operations to it.

4.4 Limits and colimits (homotopic)

In this section, we will define homotopy limits and colimits in a quasi-category. Since all notions in a quasi-category are necessarily “homotopic”, we will drop this adjective and simply call them limits and colimits. One can refer to [Lur09, Section 4.2] or [Gro20, Section 2].

In a model category, recall that homotopy (co)limits are defined by considering the total derived functor of the (co)limit functor, which is the right (resp. left) adjoint of the “constant diagram” functor. Here, we adopt a slightly different point of view : given a diagram category C^I, we consider a (co)limit to be the terminal (resp. initial) object in the category of (co)cones of C^I. We therefore need to define (co)cones and terminal and initial objects, as well as “slice categories”.

4.4.1 Joins

Let us first recall a classical notion in topology.

Definition 4.4.1. Let X and Y be two topological spaces. The *join* of X and Y is the quotient space :

$$X \star Y := X \times Y \times [0,1] / \Big((x,y,0) \sim (x,y',0) \text{ and } (x,y,1) \sim (x',y,1) \Big).$$

Concretely, we collapse $X \times Y \times \{0\}$ onto X and collapse $X \times Y \times \{1\}$ onto Y. Intuitively, one can think of the join as follows : we consider the disjoint union $X \sqcup Y$, then attach a segment between each pair (x,y) where $x \in X$ and $y \in Y$.

Example 4.4.2. The join $X \star *$ is the cone CX. The join $X \star S^0$ is the suspension ΣX. We have homeomorphisms $\mathbb{\Delta}^n \star \mathbb{\Delta}^m = \mathbb{\Delta}^{n+m+1}$.

This construction has an analogue for categories. This analogue is not symmetric : one must consider that the segment attached between x and y *goes from x to y*.

Definition 4.4.3. Let C and D be two categories. The *join* $\mathsf{C} \star \mathsf{D}$ is the category defined by :

- The class of objects is the disjoint union of the classes of objects of C and D.
- The morphisms are given by :

$$\mathrm{Hom}_{\mathsf{C}\star\mathsf{D}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathsf{C}}(x,y), & \text{if } x, y \in \mathsf{C}; \\ \mathrm{Hom}_{\mathsf{D}}(x,y), & \text{if } x, y \in \mathsf{D}; \\ *, & \text{if } x \in \mathsf{C} \text{ and } y \in \mathsf{D}; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Example 4.4.4. Let C be a category and $*$ the terminal category.

- The join $\mathsf{C}^{\triangleright} := \mathsf{C} \star *$ is the *cocone* on C. It is obtained from C by adding a new terminal object.
- Dually, the join $\mathsf{C}^{\triangleleft} := * \star \mathsf{C}$ is the *cone* on C. It is obtained from C by adding a new initial object.

These two constructions extend to simplicial sets.

Definition 4.4.5. Let X and Y be two simplicial sets. Their join $X \star Y$ is the simplicial set defined by :

$$(X \star Y)_n := X_n \sqcup \left(\bigsqcup_{i+j+1=n} X_i \times Y_j \right) \sqcup Y_n.$$

Lemma 4.4.6 (Exercise)**.** *The join $X \star Y$ is indeed a simplicial set.*

Example 4.4.7. We have $\Delta^n \star \Delta^m \cong \Delta^{n+m+1}$.

Example 4.4.8. If C is a simplicial set, we define the cone $\mathsf{C}^{\triangleleft}$ and the cocone $\mathsf{C}^{\triangleright}$ as in Example 4.4.4. For instance, $(\Lambda^2_0)^{\triangleright} \cong \Delta^1 \times \Delta^1 \cong (\Lambda^2_2)^{\triangleleft}$

Lemma 4.4.9 (Exercise)**.** *Let X and Y be two simplicial sets, then $|X \star Y| \cong |X| \star |Y|$.*

Lemma 4.4.10 (Exercise)**.** *Let C and D be two categories, then $N_\bullet(\mathsf{C} \star \mathsf{D}) \cong N_\bullet \mathsf{C} \star N_\bullet \mathsf{D}$.*

Proposition 4.4.11. *If C and D are quasi-categories, then $\mathsf{C} \star \mathsf{D}$ is a quasi-category.*

Proposition 4.4.12. *If $F : \mathsf{C} \to \mathsf{C}'$ and $G : \mathsf{D} \to \mathsf{D}'$ are categorical equivalences, then $F \star G : \mathsf{C} \star \mathsf{D} \to \mathsf{C}' \star \mathsf{D}'$ is a categorical equivalence.*

4.4.2 Slices, initial objects, terminal objects

Let us recall some concepts from classical category theory.

Definition 4.4.13. Let C be a category and $x \in \mathsf{C}$ an object. The *slice over x* is the category $\mathsf{C}_{/x}$ whose objects are morphisms $y \xrightarrow{f} x$ where $y \in \mathsf{C}$ and $f \in \mathrm{Hom}_{\mathsf{C}}(y, x)$, and whose morphisms are

$$\mathrm{Hom}_{\mathsf{C}_{/x}}(y \xrightarrow{f} x, z \xrightarrow{g} x) = \{\varphi : y \to z \mid \varphi \circ f = g\}.$$

Dually, we define the *slice under x* as the category $\mathsf{C}_{x/}$ whose objects are morphisms $x \xrightarrow{f} y$.

Example 4.4.14. The category of pointed spaces Top_* is the slice $\mathsf{Top}_{*/}$.

Proposition 4.4.15 (Exercise)**.** *Let C be a category and $x \in \mathsf{C}$ an object. The forgetful functor $\mathsf{C}_{/x} \to \mathsf{C}$ is an equivalence if and only if x is a terminal object of C. Dually, the forgetful functor $\mathsf{C}_{x/} \to \mathsf{C}$ is an equivalence if and only if x is an initial object of C.*

This definition can be generalized as follows :

Definition 4.4.16. Let C be a category and $p : \mathsf{E} \to \mathsf{B}$ a functor.

- The *slice over p* is the category $\mathsf{B}_{/p}$ whose objects are of the form $\mathrm{cst}_b \overset{\eta}{\Rightarrow} p$ where $b \in \mathsf{B}$ and η is a natural transformation between the constant functor $\mathrm{cst}_b : \mathsf{E} \to \mathsf{B}$ equal to b and p.
- The *slice under p* is the category $\mathsf{B}_{p/}$ whose objects are of the form $p \overset{\eta}{\Rightarrow} \mathrm{cst}_b$ where $b \in \mathsf{B}$ and η is a natural transformation between p and the constant functor $\mathrm{cst}_b : \mathsf{E} \to \mathsf{B}$ equal to b.

Concretely, an object of $\mathsf{B}_{/p}$ is a pair (b, η) where b is an object of B and η is a collection of morphisms $\{\eta_e : b \to p(e)\}_{e \in E}$ satisfying, for every $f \in \mathrm{Hom}_{\mathsf{E}}(e, e')$, $p(f) \circ \eta_e = \eta_{e'}$.

Example 4.4.17. Let $x \in \mathsf{C}$ be an object and $\mathrm{cst}_x : * \to \mathsf{C}$ the constant functor equal to x. Then $\mathsf{C}_{/\mathrm{cst}_x} = \mathsf{C}_{/x}$ and $\mathsf{C}_{\mathrm{cst}_x/} = \mathsf{C}_{x/}$.

Proposition 4.4.18 (Exercise)**.** *Let $p : \mathsf{E} \to \mathsf{B}$ be a functor. Then we have natural bijections in C :*

$$\mathrm{Hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{B}_{/p}) \cong \mathrm{Hom}_{\mathsf{Cat}_{\mathsf{E}/}}(\mathsf{C} \star \mathsf{E}, \mathsf{B}), \qquad \mathrm{Hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{B}_{p/}) \cong \mathrm{Hom}_{\mathsf{Cat}_{\mathsf{E}/}}(\mathsf{E} \star \mathsf{C}, \mathsf{B}).$$

Remark 4.4.19. In this proposition (and in what follows) we implicitly view $\mathsf{C} \star \mathsf{E}$ as being under E via the canonical map, and B with p.

This proposition gives the idea for generalizing these constructions to quasi-categories.

Proposition 4.4.20. *Let $p : \mathsf{E} \to \mathsf{C}$ be a simplicial map between two simplicial sets. Then there exists a simplicial set $\mathsf{C}_{/p}$ having the following universal property (in X) :*

$$\mathrm{Hom}_{s\mathsf{Set}}(\mathsf{C}, \mathsf{B}_{/p}) = \mathrm{Hom}_{s\mathsf{Set}_{\mathsf{E}/}}(\mathsf{C} \star \mathsf{E}, \mathsf{B}).$$

Dually, there exists a simplicial set $\mathsf{C}_{p/}$ having the following universal property (in X) :

$$\mathrm{Hom}_{s\mathsf{Set}}(\mathsf{C}, \mathsf{B}_{p/}) = \mathrm{Hom}_{s\mathsf{Set}_{\mathsf{E}/}}(\mathsf{E} \star \mathsf{C}, \mathsf{B}).$$

Démonstration. We define $(\mathsf{C}_{/p})_n = \mathrm{Hom}_{s\mathsf{Set}_{\mathsf{E}/}}(\Delta^n \star \mathsf{E}, \mathsf{C})$ and verify that everything works. ☐

Proposition 4.4.21 (Exercise)**.** *If C is a quasi-category and $p : \mathsf{E} \to \mathsf{C}$ a simplicial map (where $\mathsf{E} \in s\mathsf{Set}$) then $\mathsf{C}_{/p}$ (resp. $\mathsf{C}_{p/}$) is a quasi-category. If moreover $q : \mathsf{C} \to \mathsf{C}'$ is a categorical equivalence then $\mathsf{C}_{/p} \to \mathsf{C}'_{/qp}$ (resp. $\mathsf{C}_{p/} \to \mathsf{C}'_{qp/}$) is a categorical equivalence.*

Definition 4.4.22. We call the quasi-category $\mathsf{C}_{/p}$ (resp. $\mathsf{C}_{p/}$) the *slice over p* (resp. *under p*). If $p = \mathrm{cst}_x : * \to \mathsf{C}$ is a constant map, then we write $\mathsf{C}_{/p} = \mathsf{C}_{/x}$ and $\mathsf{C}_{p/} = \mathsf{C}_{x/}$.

Lemma 4.4.23 (Exercise)**.** *For any quasi-category C and any object $x \in \mathsf{C}_0$, (by the universal property) there are canonical "forgetful" functors :*

$$\mathsf{C}_{/x} \to \mathsf{C}, \quad \mathsf{C}_{x/} \to \mathsf{C}.$$

Definition 4.4.24. Let C be a quasi-category and $x \in \mathsf{C}_0$ an object. The object x is called a *terminal object* (resp. *initial object*) if $\mathsf{C}_{/x} \to \mathsf{C}$ (resp. $\mathsf{C}_{x/} \to \mathsf{C}$) is an acyclic fibration of quasi-categories.

Proposition 4.4.25. *Let C be a quasi-category and $x \in \mathsf{C}_0$ an object. Then x is terminal (resp. initial) if and only if for every $y \in \mathsf{C}_0$, $\mathrm{Map}_{\mathsf{C}}(y, x)$ (resp. $\mathrm{Map}_{\mathsf{C}}(x, y)$) is weakly equivalent to a point.*

Corollary 4.4.26. *Let C be a quasi-category. The full subcategory $\mathsf{C}_{\mathrm{term}}$ (resp. $\mathsf{C}_{\mathrm{init}}$) generated by terminal (resp. initial) objects is either empty or weakly contractible.*

Remark 4.4.27. This is an ∞-categorical version of a statement of the type : "the terminal object, if it exists, is unique up to unique isomorphism".

4.4.3 Limits and colimits

We now arrive at the definition of (homotopy) limits and colimits. Recall that if I is any simplicial set and C a quasi-category, then $\mathsf{C}^I := \mathrm{Map}_\bullet(I, \mathsf{C})$ is a quasi-category (Proposition 4.2.12).

Definition 4.4.28. Let I be a simplicial set and $p : I \to \mathsf{C}$ a diagram, viewed as an object of C^I.

- A *limit* of p is a terminal object of $\mathsf{C}^I_{/p}$.
- A *colimit* of p is an initial object of $\mathsf{C}^I_{p/}$.

A quasi-category is *complete* (resp. *cocomplete*) if it admits all small limits (resp. colimits).

Proposition 4.4.29 (Exercise)**.** *The nerve $N_\bullet$ is compatible with limits and colimits.*

Example 4.4.30. If $I = \Lambda^2_0$, a colimit indexed by I is a pushout ; if $I = \Lambda^2_2$, a limit indexed by I is a pullback.

4.5 Hammock localization

Let us explain how to associate a fibrant simplicial category (hence an ∞-category) to a model category. This ∞-category will only depend on the weak equivalences of the model category.

Definition 4.5.1 ([DK80b]). Let C be a model category.[6] Its *hammock localization* is the simplicial category $L^H\mathsf{C}$ defined as follows. Its objects are those of C. For $x, y \in \mathsf{C}$, the n-simplices of $\mathrm{Map}_{L^H\mathsf{C}}(x, y)$ are the reduced n-hammocks of width k, that is, commutative diagrams of the following form :

$$\begin{tikzcd}
 & z_{0,1} \arrow[r, dash] \arrow[d] & z_{0,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[dd, bend left=0] & z_{0,n} \arrow[d] \arrow[dr, dash] & \\
x \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \vdots \arrow[r, dash] \arrow[d] & \vdots \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & \vdots \arrow[r, dash] \arrow[d] & y \\
 & z_{k,1} \arrow[r, dash] & z_{k,1} \arrow[r, dash] & \cdots \arrow[r, dash] & z_{k,n} \arrow[ur, dash] &
\end{tikzcd}$$

where $n + 1 \geq 1$ is the length of the hammock, all vertical arrows are in $\mathscr{W}$, all horizontal arrows in a fixed column go in the same direction and are in $\mathscr{W}$ if they go to the left, the directions of the columns alternate, and each column contains at least one arrow that is not the identity.

The faces delete rows and compose the relevant maps, while the degeneracies repeat rows. Composition concatenates two hammocks.

Lemma 4.5.2 (Exercise)**.** *The hammock localization is indeed a simplicial category.*

Remark 4.5.3. In the case of model categories, it suffices to consider hammocks of length at most 3 [DK80a].

Remark 4.5.4. There are other ways to associate a simplicial category to a relative category, for example the simplicial localization [DK80c] which involves free resolutions (as when we defined $[[n]]$ from $[n]$, see Section 4.3.2). The category obtained is equivalent (in the sense of Dwyer–Kan) to the hammock localization. One can refer to [Ber18, Section 3.5] for an overview of the different notions.

Proposition 4.5.5 (Exercise)**.** *Let* C *be a model category. There is an equivalence of categories between the homotopy category of* $L^H\mathsf{C}$ *and that of* C *:*

$$\pi_0(L^H\mathsf{C}) \simeq \mathrm{Ho}(\mathsf{C}).$$

Proposition 4.5.6. *Let* C *be a model category. Its hammock localization* $L^H\mathsf{C} \in \mathsf{Cat}_\Delta$ *is fibrant : each* $\mathrm{Map}_{L^H\mathsf{C}}(x, y)$ *is a Kan complex.*

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[6] More generally, one can take a *relative category*, that is, a category equipped with a subclass of morphisms $\mathscr{W}$ closed under composition and containing all identities.

Definition 4.5.7. Let C be a model category. We define its *associated quasi-category* C_∞ as the coherent nerve of its hammock localization, $\mathsf{C}_\infty := N^\Delta L^H \mathsf{C}$.

Lemma 4.5.8. *Let C be a model category. There is a canonical quasi-functor*

$$\lambda_\infty : N_\bullet \mathsf{C} \to \mathsf{C}_\infty.$$

Démonstration. Recall that $N_\bullet \mathsf{C} \cong N_\bullet^\Delta(\iota \mathsf{C})$ where $\iota\mathsf{C}$ is the discrete simplicial category associated to C (Remark 4.3.18). One can define a simplicial functor $\iota\mathsf{C} \to L^H\mathsf{C}$ which sends a morphism to a hammock of length 1. By composing this functor with $N_\bullet^\Delta$, one obtains the desired quasi-functor λ_∞. ☐

Corollary 4.5.9. *Let C be a model category. The functor λ_∞ factors through an equivalence between the homotopy categories :*

$$\pi : \mathrm{Ho}(\mathsf{C}) \xrightarrow{\sim} \pi(\mathsf{C}_\infty).$$

Let us now turn to the functoriality of $(-)_\infty$, the ultimate goal being to define an ∞-category of ∞-categories. In general, a functor between two model categories does not induce a simplicial functor between the hammock localizations : it would need to send weak equivalences to weak equivalences, which is rarely satisfied. However, if the functor in question is a Quillen functor (right or left adjoint), then it sends weak equivalences between bifibrant objects to weak equivalences.

Lemma 4.5.10 (Exercise)**.** *Let C be a model category. The inclusions induce categorical equivalences :*

$$\begin{tikzcd}
 & (\mathsf{C}_c)_\infty \arrow[dr, "\sim"] & \\
(\mathsf{C}_{cf})_\infty \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & \mathsf{C}_\infty \\
 & (\mathsf{C}_f)_\infty . \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Remark 4.5.11. Here we implicitly use the fact that $(-)_\infty$ can be defined for any relative category ; in general, C_c, C_f and C_{cf} are not model categories.

We recall that we denote by $Q : \mathsf{C} \to \mathsf{C}$ and $R : \mathsf{C} \to \mathsf{C}$ the cofibrant and fibrant replacement functors, respectively.

Lemma 4.5.12. *Let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be a Quillen adjunction. The functors $\mathbb{L}F = F \circ Q$ and $\mathbb{R}G = G \circ R$ preserve hammocks.*

Corollary 4.5.13. *Let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be a Quillen adjunction. The functors F and G induce simplicial functors between hammock localizations :*

$$\mathbb{L}F : L^H\mathsf{C} \to L^H\mathsf{D}, \quad \mathbb{R}G : L^H\mathsf{D} \to L^H\mathsf{C}.$$

They also induce functors on their coherent nerves :

$$F_\infty := N_\bullet^\Delta(\mathbb{L}F) : \mathsf{C}_\infty \to \mathsf{D}_\infty, \quad G_\infty := N_\bullet^\Delta(\mathbb{R}G) : \mathsf{D}_\infty \to \mathsf{C}_\infty.$$

These functors commute with the equivalence π of Corollary 4.5.9.

Theorem 4.5.14. *Let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be a Quillen equivalence. The functors F_∞, G_∞ of the preceding corollary are categorical equivalences inverse to one another.*

Definition 4.5.15. The ∞-*category of* ∞-*categories* $\infty\mathsf{Cat}$ is the localization $(s\mathsf{Set}^J)_\infty$ of the Joyal model category.

Using the results of Section 4.3 :

Corollary 4.5.16. *There is a categorical equivalence between* $\infty\mathsf{Cat}$ *and* $(\mathsf{Cat}_\Delta)_\infty$.

4.6 Presentable quasi-categories and simplicial model categories

Let us conclude this chapter with an answer to the following question : which ∞-categories are obtained as localizations of model categories ?

Definition 4.6.1. A quasi-category C is said to be *presentable* if it is cocomplete and if there exists a set S of compact objects[7] such that every object of C is obtained as a filtered colimit of objects of S.

We will see that such a quasi-category can always be obtained as the simplicial nerve of a combinatorial simplicial model category.

Definition 4.6.2. A *simplicial model category* is a model category C equipped with an extension to a simplicial category and two functors

$$\boxtimes : \mathsf{C} \times s\mathsf{Set} \to \mathsf{C}, \quad (-)^{(-)} : \mathsf{C} \times s\mathsf{Set}^{\mathrm{op}} \to \mathsf{C}$$

such that :

1. for every $X \in \mathsf{C}$, the functor $X \boxtimes - : s\mathsf{Set} \to \mathsf{C}$ is left adjoint to $\mathrm{Map}_{\mathsf{C}}(X, -) : \mathsf{C} \to s\mathsf{Set}$:
$$\mathrm{Hom}_{\mathsf{C}}(X \boxtimes K, Y) \cong \mathrm{Hom}_{s\mathsf{Set}}(K, \mathrm{Map}_{\mathsf{C}}(X, Y));$$
2. for every $X \in \mathsf{C}$, the functor $Y^{(-)} : s\mathsf{Set} \to \mathsf{C}^{\mathrm{op}}$ is left adjoint to $\mathrm{Map}_{\mathsf{C}}(-, Y) : \mathsf{C}^{\mathrm{op}} \to s\mathsf{Set}$:
$$\mathrm{Hom}_{\mathsf{C}}(X, Y^K) \cong \mathrm{Hom}_{s\mathsf{Set}}(K, \mathrm{Map}_{\mathsf{C}}(X, Y)).$$
3. there is an isomorphism $X \boxtimes (K \times L) \cong X \boxtimes K \boxtimes L$, natural in $X \in \mathsf{C}$ and $K, L \in s\mathsf{Set}$;
4. for every simplicial inclusion $i : K \hookrightarrow L$ and for every fibration $q : X \to Y$ in C, the canonical map
$$\mathrm{Map}_{\mathsf{C}}(L, X) \to \mathrm{Map}_{\mathsf{C}}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$$
is a simplicial fibration, which is acyclic if i or q is so.

---

[7] In general one can also require κ-small for a fixed cardinal κ.

Remark 4.6.3. The first two conditions mean respectively that C is tensored and cotensored over $s\mathsf{Set}$. Given the first two conditions, the next two can be summarized by the fact that $\boxtimes$ is a left Quillen bifunctor such that any pair of cofibrations, one of which is acyclic, is sent to an acyclic cofibration. The last condition is an analogue of condition (MC4) (draw a commutative diagram).

Example 4.6.4. The category $s\mathsf{Set}$ itself is a simplicial model category, with $\boxtimes = \times$ and $K^L = \mathrm{Map}_\bullet(K, L)$. More generally, model categories of the type $s\mathsf{C}$ (for example simplicial abelian groups) are simplicial model categories.

Simplicial model categories are particularly useful for computations :

Proposition 4.6.5. *Let C be a simplicial model category. Let A be a cofibrant object and X a fibrant object. Then $\mathrm{Map}_{\mathsf{C}}(A, X)$ is a Kan complex, and*

$$\pi_0(\mathrm{Map}_{\mathsf{C}}(A, X)) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(A, X).$$

Démonstration. The fact that $\mathrm{Map}_{\mathsf{C}}(A, X)$ is Kan follows from the last condition in the definition. Since every simplicial set K is cofibrant, we deduce that :

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho}(s\mathsf{Set})}(K, \mathrm{Map}_{\mathsf{C}}(A, X)) &\cong \mathrm{Hom}_{s\mathsf{Set}}(K, \mathrm{Map}_{\mathsf{C}}(A, X))/\sim \\ &\cong \mathrm{Hom}_{\mathsf{C}}(A \boxtimes K, X) \\ &\cong \mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(A \boxtimes K, X).\end{aligned}$$

Applying the above to $K = \Delta^0$, we obtain :

$$\pi_0 \mathrm{Map}_{\mathsf{C}}(A, X) = \mathrm{Hom}_{\mathrm{Ho}(s\mathsf{Set})}(\Delta^0, \mathrm{Map}_{\mathsf{C}}(A, X)) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(A \boxtimes \Delta^0, X).$$

We deduce from the fact that $\mathrm{Map}_{\mathsf{C}}(A, X)_0 = \mathrm{Hom}_{\mathsf{C}}(A, X)$ and from the Yoneda lemma that $A \boxtimes \Delta^0 \cong A$, which allows us to conclude. ☐

If C is a simplicial model category, then one can recover its simplicial structure from its underlying model category :

Proposition 4.6.6 ([DK80b, Proposition 4.8])**.** *Let C be a simplicial model category. There exists a Dwyer–Kan equivalence between C and the hammock localization of its underlying model category, natural in C :*

$$\mathsf{C} \xrightarrow{\sim} L^H\mathsf{C}.$$

Many model categories are equivalent to simplicial model categories :

Theorem 4.6.7 (Dugger [Dug01])**.** *Every combinatorial model category (Definition 1.7.18) is Quillen equivalent to a combinatorial, left proper simplicial model category.*

Remark 4.6.8. Dugger [Dug01] has even shown that if C is a combinatorial and left proper model category, then one can obtain an equivalent simplicial model category by considering a left Bousfield localization of C.

We now come to the theorem.

Theorem 4.6.9 (Lurie [Lur09, Proposition A.3.7.6]). *A quasi-category is presentable if and only if it is equivalent (in the sense of quasi-categories) to the simplicial nerve* $N^{\Delta}(\mathsf{C}_{cf})$ *of the simplicial category formed by the fibrant and cofibrant objects of a combinatorial simplicial model category.*

Combining this result with the previous results, we find that a quasi-category is presentable if and only if it comes from a combinatorial model category. The model category in question becomes in some sense a « presentation », or a « model » of the original quasi-category. This result resolves the ambiguity that exists in English for the terminology « *model category* ». On the one hand, one can consider – as Quillen did – that a « *model category* » is a category that contains models (the fibrant and cofibrant objects) for homotopy types. On the other hand, one can consider a « *model category* » as a « template category », in the sense that it models a homotopy theory, namely its associated quasi-category.

A Categorical Reminders

A.1 Basic Definitions

Definition A.1.1. A *category* C is the data of

- a class of objects $\mathrm{ob}\,\mathsf{C}$;
- for every pair of objects X, Y, a set of morphisms $\mathrm{Hom}_{\mathsf{C}}(X, Y)$;
- for every object X, an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ called « the identity » ;
- for every triple of objects X, Y, Z, a composition map $\circ : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$.

These data satisfy the following axioms : for every morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, we have $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$, and for every triple of composable morphisms f, g, h, we have $f \circ (g \circ h) = (f \circ g) \circ h$.

Definition A.1.2. A category is called *small* if its objects form a set.

Example A.1.3. The category of sets Set, the category of topological spaces Top, the category of chain complexes $\mathsf{Ch}(R)$...

Definition A.1.4. Let C and D be two categories. A *functor* $F : \mathsf{C} \to \mathsf{D}$ associates an object $F(X) \in \mathsf{D}$ to every object $X \in \mathsf{C}$, and a morphism $F(f) \in \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ to every morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$. This association must satisfy $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Example A.1.5. There is a category Cat whose objects are categories and whose morphisms are functors.

Definition A.1.6. Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A *natural transformation* $\eta : F \Rightarrow G$ is the data of a morphism $\eta_X : F(X) \to G(X)$ for every $X \in \mathsf{C}$ such that the following diagrams commute for every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd} .$$

It is called a *natural equivalence* if η_X is a bijection for every X. In this case we write $F \simeq G$.

Definition A.1.7. An *equivalence of categories* is a pair of functors $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ such that $F \circ G \simeq \mathrm{id}_{\mathsf{D}}$ and $G \circ F \simeq \mathrm{id}_{\mathsf{C}}$.

Definition A.1.8. A functor $F : \mathsf{C} \to \mathsf{D}$ is called *full* (resp. *faithful*, resp. *fully faithful*) if, for every pair of objects $X, Y \in \mathsf{C}$, the map $F : \mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ is a surjection (resp. injection, resp. bijection). It is called *essentially surjective* if every object $Z \in \mathsf{D}$ is isomorphic to an object of the form $F(X)$ for some $X \in \mathsf{C}$.

Proposition A.1.9 (Exercise)**.** *A functor is part of an equivalence of categories if and only if it is fully faithful.*

Definition A.1.10. Two functors $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ are called *adjoint* if there exist natural transformations $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ and $\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ inducing bijections $\mathrm{Hom}_{\mathsf{C}}(X, G(Y)) \cong \mathrm{Hom}_{\mathsf{D}}(F(X), Y)$. We say that F is the left adjoint of G and that G is the right adjoint of F. We write $F \dashv G$.

Remark A.1.11. The unit $\eta_X : X \to G(F(X))$ corresponds to $\mathrm{id}_{F(X)}$ under the bijection $\mathrm{Hom}_{\mathsf{C}}(X, G(F(X))) \cong \mathrm{Hom}_{\mathsf{D}}(F(X), F(X))$, while the counit $\varepsilon_Y : F(G(Y)) \to Y$ corresponds to $\mathrm{id}_{G(Y)}$ under the bijection $\mathrm{Hom}_{\mathsf{D}}(F(G(Y)), Y) \cong \mathrm{Hom}_{\mathsf{C}}(G(Y), G(Y))$.

Definition A.1.12. A morphism $f : A \to B$ is a *retract* of $g : X \to Y$ if there exists a commutative diagram :

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "i'"] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r, "r'"] & B
\end{tikzcd}$$

Example A.1.13. In Set, every subset is a retract. In Top, not every subspace is a retract (for example S^{n-1} is not a retract of D^n.)

A.2 Limits and Colimits

Definition A.2.1. Let I be a small category, C any category, and $X = \{X_i\}_{i \in I} : I \to \mathsf{C}$ a functor (also called a diagram in this context). A *limit* of X, if it exists, is the data of

- an object $L \in \mathsf{C}$;
- for every $i \in I$, a morphism $\pi_i : L \to X_i$, such that for every morphism $f \in \mathrm{Hom}_I(i,j)$, the following diagram commutes :

$$\begin{tikzcd}
 & X_i \arrow[dd, "X_f"] \\
L \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"] & \\
 & X_j
\end{tikzcd}$$

satisfying the following universal property : for every object $Y \in \mathsf{C}$ equipped with morphisms $\alpha_j : Y \to X_j$ making similar diagrams commute, there exists a unique morphism $\alpha : Y \to L$ such that $\pi_i \circ \alpha = \alpha_i$.

Proposition A.2.2. *Let* $X : I \to \mathsf{C}$ *be a diagram. If a limit of* X *exists, then it is unique up to unique isomorphism respecting the structure morphisms. It is denoted* $\lim X = \lim_{i \in I} X_i$.

Example A.2.3. Let $I = \varnothing$ be the empty category and $X : \varnothing \to \mathsf{C}$ the unique possible diagram. A limit of I is an object T such that for every object $Y \in \mathsf{C}$, there exists a unique morphism $Y \to T$. Such an object is called *terminal* and is generally denoted $*$. In the category of sets, $*$ is any singleton.

Example A.2.4. Let $I = \bullet \sqcup \bullet$ be the category with two objects and no non-trivial morphisms. A diagram $X : I \to \mathsf{C}$ is the data of a pair of objects (X_1, X_2). A limit of X is an object P equipped with two morphisms $\pi_1 : P \to X_1$ and $\pi_2 : P \to X_2$. It satisfies the property that if Y is an object equipped with two morphisms $\alpha_1 : Y \to X_1$ and $\alpha_2 : Y \to X_2$, then there exists a unique morphism $\alpha : Y \to P$ such that $\pi_1 \circ \alpha = \alpha_1$ and $\pi_2 \circ \alpha = \alpha_2$. Such an object is called a *product* of X_1 and X_2 and is denoted $X_1 \times X_2$. In the category of sets, the product is indeed the Cartesian product.

Example A.2.5. The previous example generalizes. Consider $I = \bullet \to \bullet \leftarrow \bullet$, the category with three objects and two morphisms as indicated. A diagram $I \to \mathsf{C}$ is of the form $X_1 \xrightarrow{f_1} Z \xleftarrow{f_2} X_2$. A limit of this diagram is an object P equipped with $\pi_1 : P \to X_1$ and $\pi_2 : P \to X_2$ such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$ satisfying the universal property. Such an object is called a *fiber product* of X_1 and X_2 over Z and is denoted $X_1 \times_Z X_2$. One also says that the map $\pi_1 : P \to X_1$ is the pullback of $f_2 : X_2 \to Z$ along $f_1 : X_1 \to Z$ (and similarly π_2 is the pullback of f_1 along f_2). In sets, $X_1 \times_Z X_2 = \{(x, y) \in X_1 \times X_2 \mid f_1(x) = f_2(y)\}$.

Example A.2.6. Let $I = \bullet \rightrightarrows \bullet$ be the category with two objects and two non-trivial morphisms as indicated. A diagram $X : I \to \mathsf{C}$ is the data of two parallel morphisms $f, g : X_1 \rightrightarrows X_2$. A limit of I is an object E equipped with a morphism $i : E \to X_1$ such that $f \circ i = g \circ i$. It satisfies the property that if Y is an object equipped with a morphism $\alpha : Y \to X_1$ such that $f \circ \alpha = g \circ \alpha$, then there exists a unique morphism $\alpha' : Y \to P$ such that $i \circ \alpha' = \alpha$. Such an object is called an *equalizer* of f and g and can be denoted $\mathrm{eq}(f, g)$ or $\mathrm{eq}(X_1 \rightrightarrows X_2)$. In the category of sets, the equalizer of f and g is given by $E = \{x \in X_1 \mid f(x) = g(x)\}$.

Let us now turn to the dual case.

Definition A.2.7. Let I be a small category, C any category, and $X = \{X_i\}_{i \in I} : I \to \mathsf{C}$ a functor (also called a diagram in this context). A *colimit* of X, if it exists, is the data of

- an object $C \in \mathsf{C}$;

- for every $i \in I$, a morphism $\iota_i : X_i \to C$, such that for every morphism $f \in \mathrm{Hom}_I(i, j)$, the following diagram commutes :

$$\begin{tikzcd}
X_i \arrow[d, "X_f"] \arrow[dr, "\iota_i"] & \\
 & C \\
X_j \arrow[ur, "\iota_j"] &
\end{tikzcd}$$

satisfying the following universal property : for every object $Y \in \mathsf{C}$ equipped with morphisms $\alpha_i : X_i \to Y$ making similar diagrams commute, there exists a unique morphism $\alpha : C \to Y$ such that $\alpha \circ \iota_i = \alpha_i$.

Proposition A.2.8. *Let $X : I \to \mathsf{C}$ be a diagram. If a colimit of X exists, then it is unique up to unique isomorphism respecting the structure morphisms. It is denoted* $\operatorname{colim} X = \operatorname{colim}_{i \in I} X_i$.

Example A.2.9. Let $I = \varnothing$ be the empty category and $X : \varnothing \to \mathsf{C}$ the unique possible diagram. A colimit of X is an object D such that for every object $Y \in \mathsf{C}$, there exists a unique morphism $D \to Y$. Such an object is called *initial* and is generally denoted $\varnothing$. In the category of sets, $\varnothing$ is the empty set.

Example A.2.10. Let $I = \bullet \sqcup \bullet$ be the category with two objects and no non-trivial morphisms. A diagram $X : I \to \mathsf{C}$ is the data of a pair of objects (X_1, X_2). A colimit of X is an object S equipped with two morphisms $\iota_1 : X_1 \to S$ and $\iota_2 : X_2 \to S$. It satisfies the property that if Y is an object equipped with two morphisms $\alpha_1 : X_1 \to Y$ and $\alpha_2 : X_2 \to Y$, then there exists a unique morphism $\alpha : S \to Y$ such that $\alpha \circ \iota_1 = \alpha_1$ and $\alpha \circ \iota_2 = \alpha_2$. Such an object is called a *coproduct* of X_1 and X_2 and is denoted $X_1 \sqcup X_2$. In the category of sets, the coproduct is the disjoint union.

Example A.2.11. The previous example generalizes. Consider $I = \bullet \leftarrow \bullet \rightarrow \bullet$, the category with three objects and two morphisms as indicated. A diagram $I \to \mathsf{C}$ is of the form $X_1 \xleftarrow{f_1} Z \xrightarrow{f_2} X_2$. A colimit of this diagram is an object S equipped with $\iota_1 : X_1 \to S$ and $\iota_2 : X_2 \to S$ such that $\iota_1 \circ f_1 = \iota_2 \circ f_2$ satisfying the universal property. Such an object is called a *pushout* of X_1 and X_2 along Z and is denoted $X_1 \cup_Z X_2$. One also says that ι_1 is the pushout of f_2 along f_1 (and also that ι_2 is the pushout of f_1 along f_2). In sets, $X_1 \cup_Z X_2 = X_1 \sqcup X_2 / {\sim}$ where $\sim$ is the equivalence relation generated by $f_1(z) \sim f_2(z)$.

Example A.2.12. Let $I = \bullet \rightrightarrows \bullet$ be the category with two objects and two non-trivial morphisms as indicated. A diagram $X : I \to \mathsf{C}$ is the data of two parallel morphisms $f, g : X_1 \rightrightarrows X_2$. A colimit of I is an object C equipped with a morphism $q : X_2 \to C$ such that $q \circ f = q \circ g$. It satisfies the property that if Y is an object equipped with a morphism $\alpha : X_2 \to Y$ such that $\alpha \circ f = \alpha \circ f$, then there exists a unique morphism $\alpha' : C \to Y$ such that $\alpha' \circ q = \alpha$. Such an object is called a *coequalizer* of f and g and can be denoted $\mathrm{coeq}(f, g)$ or $\mathrm{coeq}(X_1 \rightrightarrows X_2)$. In the category of sets, the coequalizer of f and g is given by $C = X_2 / {\sim}$ where $\sim$ is the equivalence relation induced by $f(x) \sim g(x)$.

Definition A.2.13. A category is called *complete* (resp. cocomplete) if every diagram indexed by a small category admits a limit (resp. a colimit).

Proposition A.2.14 (Exercise)**.** *Let I be a small category and C any category. Define the « constant diagram » functor* $\mathrm{cst}_I : \mathsf{C} \to \mathsf{C}^I$.

- *If every diagram indexed by I admits a limit, then* cst_I *admits a right adjoint given by* $\{X_i\} \mapsto \lim_{i\in I} X_i$.
- *If every diagram indexed by I admits a colimit, then* cst_I *admits a left adjoint given by* $\{X_i\} \mapsto \operatorname{colim}_{i\in I} X_i$.

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