Algebraic Topology – Exercises

Najib Idrissi*

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The main file is available at `https://idrissi.eu/teaching/ucas.pdf`.
Exercises marked with $\star$ are to be done with priority.

1 Topological spaces

1.1 Convex bodies $\star$

A convex body is a compact, convex subset $K \subseteq \mathbb{R}^n$ which contains 0 as an interior point and which is symmetric around the origin.

1. Prove that the following formula defines a norm on $\mathbb{R}^n$:

$$N(x) := \inf\{\lambda > 0 \mid x/\lambda \in K\}.$$

2. Prove that K is homeomorphic to $\mathbb{D}^n$ and that ∂K is homeomorphic to $\mathbb{S}^{n-1}$.

1.2 Wedge sum $\star$

Let X, X' be topological spaces and $x \in X$, $x' \in X'$ be base points. Prove that the wedge sum $X \vee X'$ is homeomorphic to the subspace:

$$(X \times \{x'\}) \cup (\{x\} \times X') \subseteq X \times X'.$$

1.3 Torus $\star$

The torus $\mathbb{T}$ is the quotient of $[0,1]^2$ by the equivalence relation generated by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$ for all $x, y \in [0,1]$. Prove that $\mathbb{T}$ is homeomorphic to:

1. The product $\mathbb{S}^1 \times \mathbb{S}^1$.
2. The quotient of $\mathbb{R}^2$ under the action of the (discrete) group $\mathbb{Z}^2$ by translations.

1.4 Quotient and Hausdorff property

1. Let A be a compact subspace of a Hausdorff space X. Prove that X/A is Hausdorff.
2. Let G be a compact Hausdorff group acting on a Hausdorff space X. Prove that the orbit space X/G is Hausdorff.
3. Let $GL_n(\mathbb{C})$ act on $\mathcal{M}_n(\mathbb{C})$ by conjugation. Prove that the quotient $\mathcal{M}_n(\mathbb{C})/GL_n(\mathbb{C})$ is Hausdorff.

*Université Paris Cité, Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France. najib.idrissi-kaitouni@u-paris.fr

1.5 Spheres

Consider the orthogonal group $O_{n-1}(\mathbb{R})$ as the subgroup of $O_n(\mathbb{R})$ of matrices of the form:

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in O_n(\mathbb{R}), \; A \in O_{n-1}(\mathbb{R}).$$

Let $O_{n-1}(\mathbb{R})$ act on $O_n(\mathbb{R})$ by left multiplication. Prove that the orbit space $O_n(\mathbb{R})/O_{n-1}(\mathbb{R})$ is homeomorphic to $\mathbb{S}^{n-1}$.

1.6 Projective spaces ⋆

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The n-dimensional projective space $\mathbb{KP}^n$ is the orbit space of $\mathbb{K}^{n+1} \setminus \{0\}$ under the action of $\mathbb{K}^*$ by rescaling.

1. Prove that $\mathbb{RP}^n$ is homeomorphic to the orbit space $\mathbb{S}^n/\{\pm 1\}$, where $\{\pm 1\}$ acts by multiplication on $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$.
2. Prove that $\mathbb{CP}^n$ is homeomorphic to the orbit space $\mathbb{S}^{2n+1}/\mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ acts on $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ by multiplication.
3. Prove that $\mathbb{RP}^1$ is homeomorphic to $\mathbb{S}^1$ and that $\mathbb{CP}^1$ is homeomorphic to $\mathbb{S}^2$.

1.7 The line with two origins

Let X be the quotient of $\mathbb{R} \times \{-1, 1\}$ under the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Prove that any point of X admits a neighborhood homeomorphic to $\mathbb{R}$ but that X is not a topological manifold.

1.8 Connectedness of manifolds

Let X be a topological manifold.

1. Prove that X is connected if and only if it is path connected.
2. Prove that if X is connected, then for all $x, y \in X$, there exists a homeomorphism $f : X \to X$ such that $f(x) = y$. (Start with the case of an open disk.)

2 Homotopy

2.1 Möbius band ⋆

Let M be the Möbius band, that is, the quotient of $[0,1]^2$ under the equivalence relation generated by $(x, 0) \sim (1 - x, 1)$ for all $x \in [0,1]$. Prove that M is homotopy equivalent to $\mathbb{S}^1$.

2.2 Latin alphabet ⋆

Classify the uppercase letters of the Latin alphabet (A, B, C...) by homotopy type.

2.3 Homotopy type of basic spaces

1. If $X \simeq X'$ and $Y \simeq Y'$, prove that $X \times Y \simeq X' \times Y'$.
2. Let E be a vector subspace of $\mathbb{R}^n$ of dimension $k < n$. Prove that $\mathbb{R}^n \setminus E \simeq \mathbb{S}^{n-k-1}$.
3. Let $C \subseteq \mathbb{R}^n$ be a nonempty bounded convex subset. Prove that $\mathbb{R}^n \setminus C \simeq \mathbb{S}^{n-1}$.
4. Find an example of a space X and subspaces $A, B \subseteq X$ such that $A \simeq B$ but $X \setminus A \not\simeq X \setminus B$.

2.4 Cones ⋆

Let X be space. The cone CX is the quotient space:

$$CX = (X \times [0,1])/(X \times \{0\}).$$

Let $\pi : X \times [0,1] \to CX$ be the quotient map and $\iota : X \to CX, x \mapsto \pi(x,1)$.

1. Let $f : X \to Y$ be a map. Prove that f is homotopic to a constant map if and only if the exists $f' : CX \to Y$ such that $f' \circ \iota = f$.
2. Prove that $C\mathbb{S}^n$ is homeomorphic to $\mathbb{D}^{n+1}$.
3. Prove that for all X, CX is contractible.

2.5 Linear groups ⋆

1. Prove that the inclusion $O_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is a homotopy equivalence. (Use the Gram–Schmidt orthonormalization procedure to construct the reverse map).
2. Among the following matrix groups, determine which ones are compact, and determine their π_0:

$$GL_n(\mathbb{C}),\ GL_n(\mathbb{R}),\ O_n(\mathbb{R}),\ SO_n(\mathbb{R}),\ U_n(\mathbb{C}),\ SU_n(\mathbb{C}).$$

2.6 Path components of functional spaces ⋆

Let X be a compact Hausdorff space and Y be a metric space. We consider the space $\mathcal{C}(X,Y)$ of continuous maps $X \to Y$ endowed with the metric

$$d_\infty(f,g) = \inf\{d_Y(f(x),g(x)) \mid x \in X\}.$$

Prove that two maps $f,g \in \mathcal{C}(X,Y)$ are homotopic if and only if they belong to the same path component.

2.7 Fundamental group of a product ⋆

Let X, Y be spaces and $x_0 \in X$, $y_0 \in Y$ be base points. Prove that there is an isomorphism:

$$\pi_1(X,x_0) \times \pi_1(Y,y_0) \cong \pi_1(X \times Y,(x_0,y_0)).$$

2.8 Eckmann–Hilton principle

Let X be a set equipped with two group structures $(X, *, 1)$ and $(X, \bullet, 1)$ which are compatible, that is, their unit is the same and for all $a,b,c,d \in X$:

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d).$$

1. Prove that for all $a,b \in X$, $a * b = a \bullet b$ and that $a * b = b * a$.
2. Let G be a topological group. Prove that $\pi_1(G,1)$ is abelian.

2.9 Degree of a map on the circle ⋆

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map and let $x \in \mathbb{S}^1$. We let $n_x \in \mathbb{Z}$ be the integer such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1,x) & \xrightarrow{\pi_1(f)} & \pi_1(\mathbb{S}^1,f(x)) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z} & \xrightarrow{\times n_x} & \mathbb{Z} \end{array}$$

1. Prove that for any path $\gamma \in \Omega_{x,y}X$ starting at x and ending at some y, and for all $[\alpha] \in \pi_1(\mathbb{S}^1, x)$, if we let $\phi_\gamma([\alpha]) := [\gamma^{-1}\alpha\gamma]$, then $\deg(\phi_\gamma([\alpha])) = \deg(\alpha)$.
2. Prove that n_x is independent of x. We call it the degree of f, denoted $\deg(f)$.
3. Prove that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
4. Prove that $f \simeq g$ if and only if $\deg(f) = \deg(g)$.
5. Prove that if $\deg(f) \neq 0$, then f is surjective. Find a counterexample for the converse.
6. Prove that if f is injective, then $\deg(f) = \pm 1$. Find a counterexample for the converse.

2.10 Borsuk–Ulam theorem

We'd like to prove the Borsuk-Ulam theorem for $n = 1$ and $n = 2$: if $f : \mathbb{S}^n \to \mathbb{R}^n$ is a continuous map, then there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$.

1. Prove the case $n = 1$.
2. We now assume that $n = 2$ and (by contradiction) assume that f is a continuous map such that for all $x, f(x) \neq f(-x)$.
   a) Prove that if $g : \mathbb{S}^1 \to \mathbb{S}^1$ satisfies $g(-x) = -g(x)$, then $\deg(g)$ is odd.
   b) Construct a map $\phi : \mathbb{S}^2 \to \mathbb{S}^1$ such that for all x, $\phi(-x) = -\phi(x)$.
   c) Let $i : \mathbb{S}^1 \to \mathbb{S}^2$ be the inclusion of the circle as the equator of the sphere. Prove that i is homotopic to a constant map, whereas $g \circ i$ is not. Conclude.

2.11 Fundamental group of a suspension ⋆

Let X be a space and let ΣX be its suspension, that is, the quotient of $X \times [0,1]$ by the equivalence relation generated by $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

1. Prove that if X is path connected, then ΣX is simply connected.
2. Find a counterexample when X is not path connected.

2.12 Klein bottle

The Klein bottle K is the quotient of $[0,1]^2$ by the equivalence relation generated by $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, y)$ for all $x, y \in [0, 1]$.

1. Prove that K is homeomorphic to two Möbius bands glued along their boundary.
2. Prove that the fundamental group of K is isomorphic to a free group on two generators a, b modulo the relation $a^2 = b^2$.
3. Compute the abelianization of $\pi_1(K)$ and deduce that K is not homotopy equivalent to the torus.

2.13 Manifold with a point removed ⋆

Let M be a manifold of dimension ≥ 3 and let $P \subseteq M$ be a finite subset of M. Prove that the inclusion of $M \setminus P$ into M is an isomorphism on π_1.

2.14 Oriented surfaces

Let S_g be the closed oriented surface of genus g. Let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_l\}$ be disjoint sets of pairwise distinct points of S_g.

1. Compute the fundamental group of the complement $S_g \setminus X$.
2. Compute the fundamental group of the quotient S_g/Y.
3. Compute the fundamental group of the quotient of the complement $(S_g \setminus X)/Y$.

2.15 Complex projective space ⋆

1. Prove that $\mathbb{CP}^{n+1}$ is obtained from $\mathbb{CP}^n$ by gluing a cell of dimension $2n+2$.
2. Compute the fundamental group of $\mathbb{CP}^n$ for $n \geq 1$.

2.16 Linear groups of size 2 ⋆

1. Prove that $SU_2(\mathbb{C}) \to \mathbb{C}^2$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a,c)$ induces a homeomorphism $SU_2(\mathbb{C}) \to \mathbb{S}^3$.
2. Compute the fundamental groups of $SU_2(\mathbb{C})$, $U_2(\mathbb{C})$ and $GL_2(\mathbb{C})$.
3. Compute the fundamental groups of $SO_2(\mathbb{R})$, $GL_2^+(\mathbb{R})$ and $GL_2^-(\mathbb{R})$, where $GL_2^\pm(\mathbb{R}) = \{M \in GL_2(\mathbb{R}) \mid \pm\det(M) > 0\}$.

2.17 Free homotopies

1. Let $\gamma \in \Omega_x X$ be a loop in X. Prove that γ is homotopic with fixed extremities to a constant loop if and only if it is homotopic (without necessarily fixing extremities) to a constant loop.
2. Recall that $\pi_1(\mathbb{S}^1) = \langle a, b\rangle$. Prove that ab and ba are homotopic without fixing extremities, but that they are not homotopic with fixed extremities.

2.18 On the free product of groups ⋆

Let G, H be two nontrivial groups.

1. Show that the center $Z(G * H)$ is trivial.
2. Suppose that $x \in G * H$ has finite order. Show that x is conjugate to an element of G or H.

2.19 Infinite dihedral group

Let $G = \mathbb{Z}/2\mathbb{Z}$ be the cyclic group of order 2. The group $G * G$ is called the infinite dihedral group.

Consider the map $G * G \to G$ given by the identity of G on each factor. Prove that the kernel of this map is isomorphic to $\mathbb{Z}$.

3 Homology

3.1 Homotopy of pairs ⋆

Let (X,A) and (Y,B) be pairs of spaces. A map of pairs $f : (X,A) \to (Y,B)$ is a map $f : X \to Y$ such that $f(A) \subseteq B$. Two maps of pairs $f, g : (X,A) \to (Y,B)$ are pair-homotopic if there exists a map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H(a,t) \in B$ for all $x \in X$, $a \in A$ and $t \in [0,1]$.

1. Give an example two maps of pairs $f, g : (X,A) \to (Y,B)$ which are homotopic but not pair-homotopic.
2. Prove that a map of pairs $f : (X,A) \to (Y,B)$ induces a natural map $f_* : H_i(X,A) \to H_i(Y,B)$ for all $i \geq 0$.

3.2 Five lemma ⋆

1. Suppose that in the following commutative diagram, the rows are exact and f_1, f_2, f_4, f_5 are isomorphisms. Prove that f_3 is an isomorphism.

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

2. Suppose that in the following commutative diagram of chain complex, the rows are exact. Prove that if two chain maps out of $\{f, g, h\}$ induce an isomorphism on homology, then so does the third one.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & D_3 & \longrightarrow & 0 \end{array}$$

3.3 Long exact sequence of a triplet ⋆

Suppose that $A \subseteq B \subseteq X$ are subspaces of X. Prove that there is a long exact sequence:

$$\cdots \to H_n(B, A) \to H_n(X, A) \to H_n(X, B) \to H_{n-1}(B, A) \to \cdots$$

3.4 Reduced homology ⋆

Let X be a nonempty space and $x_0 \in X$ be a base point.

1. Prove that $\widetilde{H}_*(X) = H_*(X, \{x_0\})$ is isomorphic to the kernel of $\epsilon_* : H_*(X) \to H_*(\{x_0\})$, where $\epsilon_* : X \to \{x_0\}$ is the unique map.
2. Let $X = U \cup V$ where U and V are open. Prove that there is a long exact sequence:

$$\cdots \to \widetilde{H}_n(U \cap V) \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(U \cap V) \to \cdots$$
$$\cdots \to \widetilde{H}_0(U \cap V) \to \widetilde{H}_0(U) \oplus \widetilde{H}_0(V) \to \widetilde{H}_0(X) \to 0.$$

3. Prove that $\widetilde{H}_*(X \vee Y) = \widetilde{H}_*(X) \oplus \widetilde{H}_*(Y)$ if X, Y are well-pointed spaces (i.e., the pairs $(X, \{x_0\})$ and $(Y, \{y_0\})$ are good pairs).

3.5 Cofibrations ⋆

Let (X, A) be a pair of spaces. We say that the inclusion $A \to X$ is a cofibration if, whenever $f : X \to Y$ and $H : A \times [0,1] \to Y$ are maps such that $H(a, 0) = f(a)$ for all $a \in A$, there exists $\widetilde{H} : X \times [0,1] \to Y$ such that $\widetilde{H}(a, t) = H(a, t)$ for all $(a, t) \in A \times [0,1]$ and $\widetilde{H}(x, 0) = f(x)$ for all $x \in X$.

1. Prove that the inclusion $(X, A) \to (X \cup CA, CA)$ induces an isomorphism on relative homology.
2. Prove that if $A \to X$ is a cofibration and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.
3. Prove that if X is obtained from A by gluing cells, then $A \to X$ is a cofibration.
4. Prove that if $A \to X$ is a cofibration, then $(X \cup CA, CA)$ is a cofibration.
5. Let $A \to X$ be a cofibration. Prove that the quotient map induces an isomorphism $H_i(X, A) \cong H_i(X/A, A/A) = \widetilde{H}_i(X/A)$.

3.6 Homology of a suspension ⋆

Let X be a space. Compute $H_*(\Sigma X)$ in terms of $H_*(X)$.

3.7 Parachute

Compute the homology of the "parachute" obtained by gluing together the three vertices of Δ^2.

3.8 Projective complex space ⋆

Compute the homology of $\mathbb{CP}^n$.

3.9 Smash product

Let (X, x_0) be a based space and $n \geq 1$.

1. Compute $H_*(X \times \mathbb{S}^n)$ in terms of $H_*(X)$.
2. Compute the dimension of $H_i((\mathbb{S}^n)^k)$ for all $i, k \geq 0$.
3. Let the smash product $\mathbb{S}^d \wedge X$ be the quotient $(\mathbb{S}^d \times X)/(\mathbb{S}^d \vee X)$. Compute the homology of $\mathbb{S}^d \wedge X$ in terms of the homology of X.

3.10 Oriented surfaces

Let S_g be the closed oriented surface of genus g. Let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_l\}$ be disjoint sets of pairwise distinct points of S_g.

1. Compute the homology of the complement $S_g \setminus X$.
2. Compute the homology of the quotient S_g/Y.
3. Compute the fundamental homology of the quotient of the complement $(S_g \setminus X)/Y$.

3.11 Klein bottle

Compute the homology of the Klein bottle K.

3.12 Torus vs wedge

Prove that the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ has the same homology as the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ but that they are not homotopy equivalent.

3.13 Pathological spaces

1. Compute the homology of the line with two origins, then of the line with n origins.
2. Compute the homology of the closure of $\{(x, \sin(1/x)) \mid x > 0\} \subset \mathbb{R}^2$.
3. Find a topological space X and an increasing sequence of subsets $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots \subseteq X$ such that $X = \bigcup_i X_i$ but $H_*(X) \neq \lim H_*(X_i)$.

3.14 Degree ⋆

Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a map. Its degree $\deg(f) \in \mathbb{Z}$ is the integer such that for all $x \in H_n(\mathbb{S}^n; \mathbb{Z})$, we have $f_*(x) = \deg(f) \cdot x$.

1. Prove that this definition matches with the definition with fundamental groups for $n = 1$.
2. Prove that if $\deg(f) \neq 0$ then f is surjective. Find a counterexample for the converse.
3. Prove that if f is injective, then $\deg(f) = \pm 1$. Find a counterexample for the converse.
4. Prove that the degree of a reflection is -1 (use Mayer–Vietoris). Given $A \in O_n(\mathbb{R})$, prove that the degree of $x \mapsto Ax$ is equal to $\det(A)$. Compute the degree of the antipodal map $x \mapsto -x$.
5. Prove that $\Sigma\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$, then prove that the degree of f is equal to the degree of its suspension $\Sigma f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$.

3.15 Applications of degree ⋆

1. Prove that any map $f : \mathbb{S}^n \to \mathbb{S}^n$ without fixpoints is homotopic to the antipodal map. (Use that for all $x \in \mathbb{S}^n$, the origin $0 \in \mathbb{R}^{n+1}$ is not on the segment $[f(x), -x]$).
2. Prove that if n is even, then any $f : \mathbb{S}^n \to \mathbb{S}^n$ admits a fixpoint.
3. Prove that if a group G acts freely on $\mathbb{S}^n$ with n even, then $\#G \leq 2$.
4. Prove that $\mathbb{S}^n$, for n odd, admits a nonvanishing vector field.
5. Prove that any vector field on $\mathbb{S}^n$, for n even, vanishes at some point. (Given such a vector field X, consider the map $u \mapsto X(u)/\|X(u)\|$).

3.16 Projective vector space

For $\mathbb{K} = \mathbb{F}_2$ and $\mathbb{K} = \mathbb{Q}$, compute the (cellular) homology of $\mathbb{R}\mathbb{P}^n$.

3.17 Euler characteristic

Let C_* be a bounded chain complex of finite type, that is, there exists $N \geq 0$ such that $C_n = 0$ for $n > N$, and $\dim C_n < \infty$ for all n. The Euler characteristic of C_* is the integer:

$$\chi(C_*) := \sum_{n\geq 0} (-1)^n \dim(C_n).$$

1. Prove that $\chi(C_*) = \chi(H_*(C))$.
2. Suppose that there is a long exact sequence, for some chain complexes A_*, B_*, C_*:

$$\cdots \to A_n \to B_n \to C_n \to A_{n-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

   Prove that $\chi(B_*) = \chi(A_*) + \chi(C_*)$.

For a finite CW complex X, we let $\chi(X) := \chi(H_*(X; \mathbb{K}))$ for some field $\mathbb{K}$.

3. Prove that $\chi(X)$ does not depend on the choice of $\mathbb{K}$. (Use cellular homology.)
4. Let X, Y be finite CW complexes. Prove that $X \times Y$ is a finite CW complex and compute its Euler characteristic.
5. Let X be a finite CW complex and $A \subseteq X$ be a subcomplex. Prove that

$$\chi(X/A) = \chi(X) - \chi(A) + 1.$$

6. Let X be a finite CW complex and $A, B \subseteq X$ be subcomplexes such that $X = A \cup B$. Prove that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

3.18 Brouwer's theorem

Prove the following assertions.

1. If $K \subseteq \mathbb{R}^n$ is a compact convex subset with nonempty interior, then every map $K \to K$ admits a fixpoint.
2. The identity $\mathbb{S}^n \to \mathbb{S}^n$ is not homotopic to a constant map.
3. If $f : \mathbb{D}^n \to \mathbb{R}^n$ is such that $f(x) = x$ for all $x \in \mathbb{S}^{n-1}$, then the image of f contains $\mathbb{D}^n$.
4. Let $f : \mathbb{D}^n \to \mathbb{R}^n$ be a continuous map. Then g admits a fixpoint, or there exists $x \in \mathbb{S}^{n-1}$ and $\lambda \geq 1$ such that $g(x) = \lambda x$.

3.19 Application of Brouwer's theorem

1. Prove that any map $f : \mathbb{S}^n \to \mathbb{S}^n$ homotopic to a constant map admits a fixpoint.
2. (Perron–Frobenius) Let $A = (a_{i,j}) \in GL_n(\mathbb{R})$ be a square invertible real matrix and assume that $a_{i,j} \geq 0$ for all i, j. Prove that A admits a positive eigenvalue associated to an eigenvector with nonnegative entries. (Use the map $\Delta^{n-1} \to \mathbb{R}^n$, $x \mapsto Ax$.)
3. Let $f = (f_1, \dots, f_n) : [0,1]^n \to \mathbb{R}^n$ be a continuous map. Suppose that for all $(t_1, \dots, t_n) \in [0,1]^n$,

$$f_i(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_n) \geq 0, \text{ and } f_i(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \leq 0.$$

   Prove that there exists $(t_1, \dots, t_n) \in [0,1]^n$ such that $f(t_1, \dots, t_n) = 0$.
   Hint: given $x \in \mathbb{R}$, let $r(x) = \min(1, \max(0, a))$ and consider the map:

$$F(t_1, \dots, t_n) := (r(f_1(x_1, \dots, x_n) + x_1), \dots, r(f_n(x_1, \dots, x_n) + x_n)).$$