Algebraic Topology

Najib Idrissi*

Last updated: March 21, 2025

These are the lecture notes for a course given at a summer school at the School of Mathematical Sciences at the University of Chinese Academy of Sciences in 2024. I would like to thank the school (and especially Prof. Ping Zhang and Prof. Fusheng Deng) for their hospitality.

I would also like to thank Catherine Gille and Muriel Livernet for helpful discussions. These notes are inspired by several sources, including *Algebraic Topology* by Hatcher [2] and the course given by Antoine Touzé at the École Normale Supérieure [4]. Other references include [1, 3].

Contents

*Université Paris Cité and Sorbonne Université, IMJ-PRG, F-75013 Paris, France. najib.idrissi-kaitouni@u-paris.fr

1 Topological spaces

This section contains mainly basic reminders on topological spaces.

1.1 Basic definitions

Definition 1.1. A *topological space* is a pair $(X, \mathcal{O})$ where X is a set and $\mathcal{O} \subseteq \mathcal{P}(X)$ is a set of subsets of X which contains $\varnothing$ and X and which is stable under finite intersections and arbitrary unions. Elements of $\mathcal{O}$ are called *open subsets*.

Example 1.2. Discrete topology, trivial topology, metric spaces, subspaces, product spaces. . .

Definition 1.3. A map $f : X \to Y$ between topological spaces is *continuous* if for every open subset $V \subseteq Y$, the subset $f^{-1}(V) \subseteq X$ is open. It is a *homeomorphism* if it admits a continuous inverse. If there exists a homeomorphism between X and Y, we write $X \cong Y$.

Definition 1.4. A topological space X is *Hausdorff* (also known as T_2) if for every $x \neq y \in X$, there exist opens $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Example 1.5. The trivial topology is not Hausdorff. Discrete spaces and metric spaces are Hausdorff. Subspaces of Hausdorff spaces are themselves Hausdorff.

1.2 Compact spaces

Definition 1.6. A topological space X is *compact* if every open cover (i.e., a collection of open subsets $(U_i)_{i\in I}$ such that $X = \bigcup_{i\in I} U_i$) admits a finite subcover.

Example 1.7. Closed bounded subsets of $\mathbb{R}^n$ are compact. If X is compact, then a subspace $A \subseteq X$ is compact if and only if it is closed. A product of compact spaces is compact.

Proposition 1.8. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proposition 1.9. *If $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Definition 1.10. A space X is *locally compact* if every point of X admits a compact neighborhood.

1.3 Connected spaces

Definition 1.11. A space X is *connected* if, there does not exist disjoint open subsets $U, V \subseteq X$ such that $X = U \cup V$.

Example 1.12. The empty set is disconnected. A nonempty trivial space is connected. A discrete space with more than one point is disconnected. Products of connected spaces are connected. The image of a connected space by a continuous map is connected. The connected subspaces of $\mathbb{R}$ are nonempty intervals.

Proposition 1.13. *If $A \subseteq X$ is a connected subspace of a space X, then $\bar{A}$ is connected.*

Definition 1.14. A space X is *path connected* if, there exists $x_0 \in X$ such that, for all $y \in X$, there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = y$.

Proposition 1.15. *Path connected spaces are connected.*

Example 1.16. The topologist's sine curve, i.e., the closure of $\{(x, \sin(\frac{1}{x})) \mid x > 0\} \subset \mathbb{R}^2$, is connected but not path connected.

1.4 Quotient spaces

Definition 1.17. Let X be a topological space and $\sim$ be an equivalence relation on X. Let $\pi : X \to X/{\sim}$ be the quotient map. The quotient topology on $X/{\sim}$ is defined by:

$$U \subseteq X/{\sim} \text{ is open } \iff \pi^{-1}(U) \subseteq X \text{ is open.}$$

Remark 1.18. The quotient map π is clearly continuous.

Remark 1.19. It is possible for the quotient of a Hausdorff space to not be Hausdorff.

Proposition 1.20. *Let X, Y be topological spaces and $\sim$ an equivalence relation on X. For every continuous map $f : X \to Y$ which is constant on equivalence classes, there exists a unique continuous map $\bar{f} : X/{\sim} \to Y$ such that $\bar{f} \circ \pi = f$.*

1.4.1 Subspace collapse

Definition 1.21. Let X be a topological space and $A \subseteq X$ be a subspace. The *quotient* of X by A, denoted X/A, is the quotient space $X/\sim$ where $\sim$ is generated by $a \sim a'$ for all $a, a' \in A$.

Example 1.22. Let $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the n- disk and let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ be the $(n-1)$-sphere. Then $\mathbb{D}^n/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$.

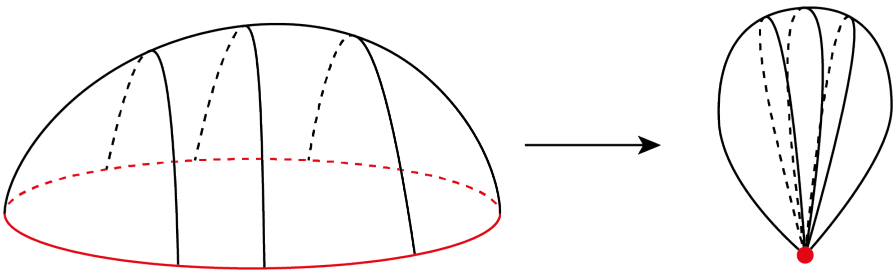


Figure 1: Collapsing the boundary of $\mathbb{D}^n$ to get $\mathbb{S}^n$

Proof. Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we let $i(x) = (x_1, \dots, x_n, 0) \in \mathbb{R}^{n+1}$. We let $u = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Then we define:

$$f : \mathbb{D}^n \to \mathbb{S}^n, \ f(x) \coloneqq \cos(\pi\|x\|)u + \sin(\pi\|x\|)\frac{i(x)}{\|x\|}.$$

Then f is a continuous map which induces a continuous map $\bar{f} : \mathbb{D}^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$. We can then check that $\bar{f}$ is a bijection with a compact source and Hausdorff target, thus it is a homeomorphism. □

1.4.2 Group actions

Definition 1.23. A *topological group* is a space G endowed with a group structure such that the multiplication map $G^2 \to G, (g, h) \mapsto gh$ and the inverse map $G \to G, g \mapsto g^{-1}$ are continuous.

Example 1.24. Any group with the discrete topology is a topological group. Linear groups (i.e., subgroups of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$) are naturally topological groups.

Definition 1.25. Let G be a topological group and X be a topological space. A *continuous group action* of G on X is an action which defines a continuous map $G \times X \to X$. The quotient space X/G is then endowed with the quotient topology.

Example 1.26. The group $\mathbb{Z}$ acts continuously on $\mathbb{R}$ by translations, i.e., $n \cdot x \coloneqq n + x$. Then the quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $\mathbb{S}^1$.

Proof. Let $f\, R \to \mathbb{S}^1$ be defined by $f(t) := (\cos(2\pi t), \sin(2\pi t))$. This is a continuous map which goes through the quotient. We can then check that it defines a bijective continuous map $\mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$. We can also prove that $\mathbb{R}/\mathbb{Z}$ is compact, because it is the image of the compact subspace $[0, 1] \subset \mathbb{R}$. Since $\mathbb{S}^1$ is Hausdorff, we can conclude. □

Example 1.27. Given $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $\mathbb{K}^*$ act on $\mathbb{K}^{n+1}$ by rescaling. The quotient space is called the projective space of dimension n, denoted $\mathbb{P}^n(\mathbb{K})$. Exercise: prove that it is Hausdorff.

1.4.3 Gluing spaces

Definition 1.28. Let A, X, Y be topological spaces and $f : A \to X$, $g : A \to Y$ be continuous maps. We denote by $X \cup_A Y$ the *pushout* of f and g, defined as the quotient space $(X \sqcup Y)/\sim$ of the disjoint union $X \sqcup Y$ by the equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$.

There are natural maps $i : X \to X \cup_A Y$, $j : Y \to X \cup_A Y$. Note that these maps are not necessarily injective.

Lemma 1.29. *For any pair of continuous maps* $(\phi : X \to Z, \psi : Y \to Z)$ *such that* $\phi \circ f = \psi \circ g$*, there exists a unique continuous map* $\theta : X \cup_A Y \to Z$ *such that* $\theta \circ i = \phi$ *and* $\theta \circ j = \psi$*.*

Definition 1.30. Let X, Y be two spaces and choose two base points $x_0 \in X$, $y_0 \in Y$. The *wedge product* $X \vee Y$ is the pushout $X \cup_{\{*\}} Y$ of the maps $f : \{*\} \to X,\ * \mapsto x_0$ and $g : \{*\} \to Y,\ * \mapsto y_0$.

1.5 Topological manifolds

Definition 1.31. A *topological manifold* is a Hausdorff space X such that every point of X admits a neighborhood homeomorphic to $\mathbb{R}^n$ for some n. If the integer n is the same for all points, then X is said to be of *dimension* n.

Remark 1.32. Some authors require that manifolds are paracompact or second-countable.

Example 1.33. Open subsets of $\mathbb{R}^n$ (e.g., general linear groups $\mathrm{GL}_d(\mathbb{R})$ or $\mathrm{GL}_d(\mathbb{C})$), spheres, projective spaces are all topological manifolds.

Example 1.34. Topological manifolds of dimension 0 are discrete spaces. Manifolds of dimension 1 are called curves, those of dimension 2 are called surfaces.

2 Homotopy

From now on, all maps are assumed to be continuous unless mentioned otherwise.

2.1 Homotopy between maps

Definition 2.1. Let $f, g : X \to Y$ be two maps. A *homotopy* between f and g is a map $H : X \times [0,1] \to Y$ such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. If such a homotopy exists, then the maps f and g are said to be *homotopic*, which is denoted $f \simeq g$.

Proposition 2.2. *The relation "being homotopic" is an equivalence relation on* $\mathcal{C}(X,Y)$*.*

Proposition 2.3. *The relation "being homotopic" is compatible with composition, i.e., if* $f, f' : X \to Y$ *and* $g, g' : Y \to Z$ *are maps such that* $f \simeq f'$ *and* $g \simeq g'$*, then* $g \circ f \simeq g' \circ f'$*.*

Definition 2.4. Two spaces X, Y are said to be *homotopy equivalent*, or to have the *same homotopy type*, if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. This is denoted $X \simeq Y$. The maps f and g are then called *homotopy equivalences.*

Example 2.5. Homeomorphisms are homotopy equivalences.

Example 2.6. The sphere $\mathbb{S}^n$ has the same homotopy type as the Euclidean space minus a point, $\mathbb{R}^n \setminus \{0\}$. Indeed, let $f : \mathbb{S}^n \to \mathbb{R}^n \setminus \{0\}$ be the inclusion, and let $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^n$, $x \mapsto x/|x|$. Then $g \circ f$ is equal to the identity of $\mathbb{S}^n$, while $f \circ g$ is homotopic to the identity via:

$$H : (\mathbb{R}^n \setminus \{0\}) \times [0,1] \to \mathbb{R}^n \setminus \{0\},\ (x,t) \mapsto tx + (1-t)\frac{x}{|x|}.$$

Definition 2.7. A space X is *contractible* if it is homotopy equivalent to a singleton.

Remark 2.8. This is equivalent to id_X being homotopic to a constant map.

Example 2.9. The Euclidean space $\mathbb{R}^n$ is contractible via a straight-line homotopy. More generally, nonempty convex subsets of $\mathbb{R}^n$ are contractible.

2.2 Path components

Definition 2.10. Let X be a space and $x, y \in X$. A *path* from x to y is a map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The set of paths from x to y is denoted $\Omega_{x,y}X$. If $x = y$, then γ is called a *loop* and we let $\Omega_x X \coloneqq \Omega_{x,y} X$.

Definition 2.11. Let X be a space and $x \in X$ be a point. The *constant path* at x, denoted $\epsilon_x \in \Omega_x X$, is the path defined by $\epsilon_x(t) \coloneqq x$.

Remark 2.12. The set $\Omega_{x,y}X$ can be made into a topological space by endowing it with the *compact-open topology*, that is, the topology generated by the subbase given by sets of the form

$$V(K,U) = \{\gamma \mid \gamma(K) \subseteq U\}, \text{for } K \text{ compact and } U \text{ open.}$$

If X happens to be a metric space, then this is the same topology as that of uniform convergence, induced by the metric $d_\infty(\gamma, \gamma') = \inf_{t \in [0,1]} d(\gamma(t), \gamma'(t))$.

Definition 2.13. Let X be a topological space and let $\sim$ be the equivalence relation given by $x \sim y$ if and only if there exists a path from x to y. The quotient set of X is denoted $\pi_0(X)$ and called the *set of path components* of X.

Proposition 2.14. *For any spaces X, Y, there is a natural bijection $\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$.*

Definition 2.15. Let $f : X \to Y$ be a map. There is a natural induced map, denoted $\pi_0(f)$ or $f_* : \pi_0(X) \to \pi_0(Y)$ defined by $[x] \mapsto [f(x)]$.

Lemma 2.16. *If $f, g : X \to Y$ are homotopic then $\pi_0(f) = \pi_0(g)$.*

Corollary 2.17. *If X and Y are homotopy equivalent, then $\pi_0(X)$ is in bijection with $\pi_0(Y)$.*

Example 2.18. There is no homeomorphism between $\mathbb{R}$ and $\mathbb{R}^n$ for $n \geq 2$.

Proof. If there were one, say $f : \mathbb{R} \to \mathbb{R}^n$, then this would induce a homeomorphism, and thus a homotopy equivalence, between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^n \setminus \{0\}$. But $\pi_0(\mathbb{R} \setminus \{0\}) = \{[1], [-1]\}$ has two elements, while $\pi_0(\mathbb{R}^n \setminus \{0\})$ is a singleton. This is absurd. □

2.3 Fundamental group

2.3.1 Path homotopies

Definition 2.19. Let X be a space, $x, y \in X$ be points, and $\gamma, \gamma' \in \Omega_{x,y}X$ be paths from x to y. A *homotopy of paths* from γ to γ' is a homotopy $H : [0,1] \times [0,1] \to X$ from γ to γ' such that for all $t \in [0,1]$, we have $H(0,t) = x$ and $H(1,t) = y$. If such a homotopy of paths exists, we write $\gamma \sim_* \gamma'$ and we say that γ and γ' are *path homotopic*.

Lemma 2.20. *Let X be a space and $x, y \in X$ be points. The relation $\sim_*$ defines an equivalence relation on $\Omega_{x,y}X$.*

Definition 2.21. Let X be a space, $x, y, z \in X$ be points, and $\gamma \in \Omega_{x,y}X$, $\mu \in \Omega_{y,z}X$ be paths. We define a path $\gamma \cdot \mu \in \Omega_{x,z}X$, called the *concatenation* of the *product* of γ and μ, by:

$$(\gamma \cdot \mu)(t) := \begin{cases} \gamma(2t), & \text{if } t \leq 1/2; \\ \mu(2t-1), & \text{if } t \geq 1/2. \end{cases}$$

Note that this is well defined at $t = 1/2$ because $\gamma(1) = y = \mu(0)$.

The proof of the following lemma is immediate:

Lemma 2.22. *Let X, x, y, z be as above, $\gamma, \gamma' \in \Omega_{x,y}X$, $\mu, \mu' \in \Omega_{y,z}X$ be paths. If $\gamma \sim_* \gamma'$ and $\mu \sim_* \mu'$ then $\gamma \cdot \mu \sim_* \gamma' \cdot \mu'$.*

To prove the other properties of concatenation, we will use the following operation:

Definition 2.23. Let $\gamma \in \Omega_{x,y}X$ be a path and $\phi : [0,1] \to [0,1]$ be a map such that $\phi(0) = 0$ and $\phi(1) = 1$. The *reparameterization* of γ with respect to ϕ is the path $\phi^*\gamma$ defined by:

$$\phi^*\gamma(t) := \gamma(\phi(t)).$$

Intuitively, $\phi^*\gamma$ is the "same path" as γ, but run through at a possibly different speed, possibly going back and forth, etc.

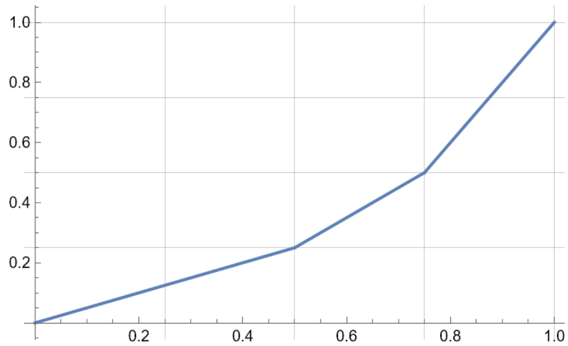
Lemma 2.24. *For any $\gamma \in \Omega_{x,y}X$ and $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$, we have $\phi^*\gamma \sim_* \gamma$.*

Proof. There is a homotopy given by:

$$H : [0,1] \times [0,1] \to X, \qquad (s,t) \mapsto \gamma((1-t)\phi(s) + st).$$ □

Lemma 2.25. *Path concatenation is associative up to path homotopy, that is, for any triple of composable paths γ, μ, ν, we have $\gamma \bullet (\mu \bullet \nu) \sim_* (\gamma \bullet \mu) \bullet \nu$.*

Proof. Consider the piecewise linear map $\phi : [0,1] \to [0,1]$ whose graph is:



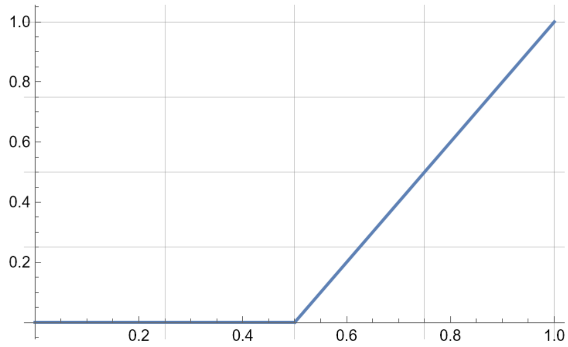
Then a homotopy is given by the "reparameterization:"

$$H(s,t) := ((\gamma \bullet \mu) \bullet \nu)((1-s)\phi(t) + st).$$

□

Lemma 2.26. *Constant paths are units up to path homotopy, that is, for any path $\gamma \in \Omega_{x,y}X$, we have $\gamma \bullet \epsilon_y \sim_* \gamma \sim_* \epsilon_x \bullet \gamma$.*

Proof. Tere is a homotopy defined through the reparameterization associated with the piecewise linear map $\phi : [0,1] \to [0,1]$ whose graph is:



□

Definition 2.27. Let $\gamma \in \Omega_{x,y}X$ be a path in some space X. Its *inverse*, denoted $\gamma^{-1} \in \Omega_{y,x}X$, is the path defined by $\gamma^{-1}(t) := \gamma(1-t)$.

Lemma 2.28. *For any path $\gamma \in \Omega_{x,y}X$, we have $\gamma \bullet \gamma^{-1} \sim_* \epsilon_x \sim_* \gamma^{-1} \bullet \gamma$.*

Proof. For the first relation (the other one is similar), we can define a homotopy by:

$$H(s,t) := \begin{cases} \gamma(2st), & \text{if } s \leq 1/2; \\ \gamma(2(1-s)t), & \text{if } s > 1/2. \end{cases}$$ □

2.3.2 Definition of the fundamental group

Definition 2.29. A *pointed topological space* (also known as a *based space*) is a pair (X, x) where X is a space and $x_0 \in X$ is a base point.

Definition 2.30. Let (X, x) be a pointed topological space. The *fundamental group* of (X, x) is the quotient set of $\Omega_x X$ by the relation $\sim_*$, denoted $\pi_1(X, x)$.

Proposition 2.31. *The fundamental group $\pi_1(X, x)$ has a group structure with multiplication given by $[\gamma] \cdot [\mu] := [\gamma \bullet \mu]$, unit given by $[\epsilon_x]$, and inverses given by $[\gamma]^{-1} = [\gamma^{-1}]$.*

Proposition 2.32. *Let X be a space and $x, y \in X$ be points. Any path $\gamma \in \Omega_{x,y}X$ from x to y induces an isomorphism of groups by conjugation:*

$$\Phi_\gamma : \pi_1(X, x) \to \pi_1(X, y), \ [\mu] \mapsto [\gamma^{-1}\mu\gamma].$$

Remark 2.33. This allows us to "forget" the base point when X is path connected and simply write $\pi_1(X)$. We still must be careful about base points in some circumstances, as the fundamental is only defined "up to conjugation".

Remark 2.34. If γ is a loop at x, then $\Phi_\gamma : \pi_1(X, x) \to \pi_1(X, x)$ is conjugation by $[\gamma]$.

Definition 2.35. A space X is called *simply connected* if it is path connected and its fundamental group is trivial.

Example 2.36. Contractible spaces are simply connected.

2.3.3 Continuous maps

Proposition 2.37. *Let (X, x) be a pointed space and $f : X \to Y$ be a map. Then f induces a group morphism:*

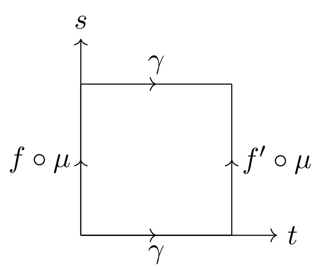
$$\pi_1(f) = f_* : \pi_1(X, x) \to \pi_1(Y, f(x)), \; [\gamma] \mapsto [f \circ \gamma].$$

Theorem 2.38. *If $f : X \to Y$ is a homotopy equivalence, then for all $x \in X$, $\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of groups.*

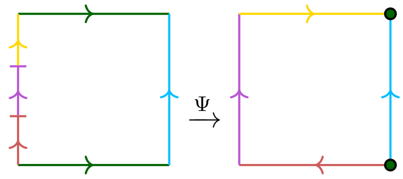
Lemma 2.39. *Let $f, f' : X \to Y$ be two homotopic maps, with a homotopy $H : X \times [0,1] \to Y$. Let $x \in X$ be a base point and let γ be the path from $f(x)$ to $f'(x)$ defined by $\gamma(t) \coloneqq H(x,t)$. Then we have the equality:*

$$\Phi_\gamma \circ \pi_1(f) = \pi_1(g) : \pi_1(X, x) \to \pi_1(Y, f'(x)).$$

Proof. Let $\mu \in \Omega_x X$ be a loop. We want to show that $\gamma^{-1} \cdot (f_0 \circ \mu) \cdot \gamma$ and $f_1 \circ \mu$ are homotopic. We define a first homotopy $K : [0,1]^2 \to Y$ by $K(s,t) = H(\mu(s), t)$. This satisfies $K(s, 0) = (f \circ \mu)(s)$ and $K(s, 1) = (f' \circ \mu)(s)$. The boundary of this homotopy looks like:



It thus remains to pre-compose K with a map $\Psi : [0,1]^2 \to [0,1]^2$ which acts as follows on the boundary:



□

Proof of the theorem. Suppose that $f : X \leftrightarrows Y : g$ is a homotopy equivalence between X and Y. Since the map $g \circ f$ is homotopic to id_X, there exists a path γ such that $\Phi_\gamma \circ \pi_1(g \circ f) = \mathrm{id}_{\pi_1(X,x)}$. Both Φ_γ and the identity are isomorphisms, thus so is $\pi_1(g \circ f)$. It follows that $\pi_1(f)$ is injective. By a mirrored reasoning, $\pi_1(f \circ g)$ is also an isomorphism, and thus $\pi_1(f)$ is also surjective. It follows that $\pi_1(f)$ is an isomorphism. □

2.4 Fundamental group of the circle

Recall that we view the circle $\mathbb{S}^1$ as the set of complex numbers of norm 1:

$$\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Definition 2.40. The exponential map is:

$$E : \mathbb{R} \to \mathbb{S}^1, \ t \mapsto \exp(2i\pi t).$$

Lemma 2.41. *Let $U = \mathbb{S}^1 \setminus \{1\}$ and $V = \mathbb{S}^1 \setminus \{-1\}$. For any integer $n \in \mathbb{Z}$, the exponential map induces homeomorphisms $(n-1, n) \to U$ and $(n-1/2, n+1/2) \to V$.*

Proposition 2.42 (Lifting of paths)**.** *For any map $\gamma : [a,b] \to \mathbb{S}^1$ and any $t \in \mathbb{R}$ be such that $E(t) = a$, there exists a unique* lift *$\tilde{\gamma} : [a,b] \to \mathbb{R}$ such that $E \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = t$.*

Lemma 2.43 (Lebesgue's number lemma)**.** *Let X be a compact metric space and $(U_i)_{i \in I}$ be an open cover of X. There exists $\epsilon > 0$ such that*

$$\forall x \in X, \exists i \in I, B(x, \epsilon) \subseteq U_i.$$

Proof. Since X is compact, we may assume that $(U_i)_{i=1}^n$ is a finite cover. If any of the U_i is equal to X, then we can take any $\epsilon > 0$, e.g., $\epsilon = 1$. Otherwise, we can define:

$$f : X \to \mathbb{R}_+, \qquad x \mapsto \frac{1}{n} \sum_{i=1}^{n} d(x, X \setminus U_i).$$

Note that $f(x) > 0$ for all $x \in X$ (because if we had $f(x) = 0$, then x wouldn't be in any of the U_i). Since X is compact, this map reaches a minimum at some point $x_0 \in X$. Let $\epsilon = f(x_0) > 0$. We thus have $f(x) \geq \epsilon > 0$ for all $x \in X$, and therefore $d(x, X \setminus U_i) \geq \epsilon$ for some i (as otherwise the average would be less than ϵ), i.e., $B(x, \epsilon) \subseteq U_i$. □

Proof of the proposition. If we apply the lemma to the cover $[a,b] = (E \circ \gamma)^{-1}(U) \cup (E \circ \gamma)^{-1}(V)$, we get that there exists $\epsilon > 0$ such that for all $t \in [a,b]$, $B(t, \epsilon) \subseteq (E \circ \gamma)^{-1}(U)$ or $B(t, \epsilon) \subseteq (E \circ \gamma)^{-1}(V)$. We can thus cut $[a,b]$ into $a = a_0 < a_1 < \ldots < a_n = b$ such that for all i, $[a_i, a_{i+1}] \subseteq \gamma^{-1}(U)$ or $\gamma^{-1}(V)$. Let us use this decomposition to build $\tilde{\gamma}$ and prove that it is unique.

We define $\tilde{\gamma}|_{[a,a_0]}$ by $\tilde{\gamma}(a) = t$. Suppose (by induction) that a lift $\tilde{\gamma}|_{[a,a_i]}$ has been built for some $i \geq 0$. WLOG we can assume that $\gamma([a_i, a_{i+1}]) \in U$. Thus, there exists some integer $n \in Z$ such that $\tilde{\gamma}(a_i) \in (n, n+1)$. Since $E : (n, n+1) \to U$ is a homeomorphism, we can define $\tilde{\gamma}$ on $[a_i, a_{i+1}]$ by composing $\gamma|_{[a_i, a_{i+1}]}$ with the inverse of $E|_{(n,n+1)}$. By induction, we eventually get the existence of $\tilde{\gamma}$.

Suppose now that two possibly different lifts $\tilde{\gamma}, \bar{\gamma}$ have been built. Clearly, $\tilde{\gamma}|_{[a,a_0]} = \bar{\gamma}|_{[a,a_0]}$. Suppose that $\tilde{\gamma}|_{[a,a_i]} = \bar{\gamma}|_{[a,a_i]}$ for some $i \geq 0$. WLOG, we can assume that $\tilde{\gamma}(a_i) = \bar{\gamma}(a_i)$ belongs to $(n, n+1) \subseteq U$ for some integer $n \in \mathbb{Z}$. Since $E|_{(n,n+1)}$ is a homeomorphism onto U, we get that $\tilde{\gamma}$ and $\bar{\gamma}$ agree on $[a_i, a_{i+1}]$. □

Definition 2.44. Let $\gamma : [0,1] \to \mathbb{S}^1$ be a loop in $\mathbb{S}^1$. The *degree* of γ, denoted $\deg(\gamma)$, is the number $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ for some lift $\tilde{\gamma}$ of γ.

Lemma 2.45. *The degree of a loop is well-defined and is an integer.*

Proposition 2.46 (Lifting of homotopies)**.** *Let $h : [a,b] \times [0,1] \to \mathbb{S}^1$ be a map and let $t \in \mathbb{R}$ be such that $E(t) = f(a, 0)$. There exists a unique lift $\tilde{h} : [a,b] \times [0,1] \to \mathbb{R}$ such that $E \circ \tilde{h} = f$ and $\tilde{h}(a, 0) = t$.*

Proof. Again, using the Lebesgue lemma, we can cut $[a, b] \times [0, 1]$ into small rectangles of the for $[a_i, a_{i+1}] \times [x_j, x_{j+1}]$ such that $a = a_0 < \ldots < a_n = b$, $0 = x_0 < \ldots < x_m = 1$, and $h([a_i, a_{i+1}] \times [x_j, x_{j+1}])$ is included in U or V. Then as above, we can construct $\tilde{h}$ piece by piece and prove that it is unique. $\square$

Theorem 2.47. *The map deg* : $\pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ *induces a group isomorphism.*

Proof. The degree map goes through the quotient that defines the fundamental group thanks to the lifting of homotopies proposition. If $[\gamma], [\mu] \in \pi_1(\mathbb{S}^1)$, choose some lift $\tilde{\gamma} : [0, 1] \to \mathbb{R}$ of γ, and choose the unique lift $\tilde{\mu} : [0, 1] \to \mathbb{R}$ such that $\tilde{\mu}(0) = \tilde{\gamma}(1)$. Then the concatenation $\tilde{\gamma} \bullet \tilde{\mu}$ lifts $\gamma \bullet \mu$. It follows that deg is a group morphism:

$$
\begin{aligned}
\deg(\gamma \bullet \mu) &= (\tilde{\gamma} \bullet \tilde{\mu})(1) - (\tilde{\gamma} \bullet \tilde{\mu})(0) = \tilde{\mu}(1) - \tilde{\gamma}(0) \\
&= \tilde{\mu}(1) - \tilde{\mu}(0) + \tilde{\gamma}(1) - \tilde{\gamma}(0) = \deg(\gamma) + \deg(\mu).
\end{aligned}
$$

Now, given $n \in \mathbb{Z}$, let $\gamma : [0, 1] \to \mathbb{S}^1$ be the loop defined by $\gamma(t) = E(nt)$. A lift of γ is given by $\tilde{\gamma}(t) = nt$, so $\deg(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0) = n$. It follows that deg is surjective.

Finally, suppose that $[\gamma] \in \pi_1(\mathbb{S}^1)$ satisfies $\deg(\gamma) = 0$. In other words, γ admits a lift $\tilde{\gamma} : [0, 1] \to \mathbb{R}$ such that $\tilde{\gamma}(1) = \tilde{\gamma}(0) = t_0$, i.e., $\tilde{\gamma}$ is a loop in $\mathbb{R}$. Since $\mathbb{R}$ is contractible, this loop is path-homotopic to the constant loop ϵ_{t_0}. Therefore, $\gamma = E \circ \tilde{\gamma}$ is path-homotopic to $E \circ \epsilon_{t_0} = \epsilon_1$, i.e., $[\gamma]$ is the unit of $\pi_1(\mathbb{S}^1)$. It follows that deg is injective. $\square$

2.5 Applications

Theorem 2.48 (D'Alembert). *Any nonconstant complex polynomial admits a root.*

Proof. Let $P \in \mathbb{C}[X]$ be a polynomial and suppose that P has no root. WLOG we can assume that $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ for some $a_i \in \mathbb{C}$. Since P has no root, we can define the following map:

$$
f : \mathbb{S}^1 \to \mathbb{S}^1, \ z \mapsto \frac{P(z)}{|P(z)|}.
$$

This map induces a group morphism $\pi_1(f) : \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^1)$. Since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, this map is completely determined by the integer $k = \pi_1(f)(1) \in \mathbb{Z}$.

On the one hand, since P has no root inside the unit disk, we can define a homotopy:

$$
H : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^1, \ (z, t) \mapsto \frac{P(zt)}{|P(zt)|}.
$$

This map is a homotopy between f and the constant loop at $a_0/|a_0|$. It follows that $\pi_1(f) = 0$ and thus $k = 0$. On the other hand, since P has no root outside the unit disk, be can also define a homotopy:

$$
K : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^1, \ (z, t) \mapsto \frac{t^n P(\frac{z}{t})}{|t^n P(\frac{z}{t})|}.
$$

This defines a homotopy between f and $z \mapsto z^n$, which has degree n. It follows that $k = n$. But since $k = 0$, we get $n = 0$, i.e., P is a constant polynomial. $\square$

Theorem 2.49 (Brouwer)**.** *Any continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ admits a fixpoint.*

Proof. Suppose to the contrary that for all $x \in \mathbb{D}^2$, $f(x) \neq x$. We can thus consider the half-infinity line $[f(x), x)$ which intersects $\partial\mathbb{D}^2 = \mathbb{S}^1$ in exactly one point, denoted $r(x)$.

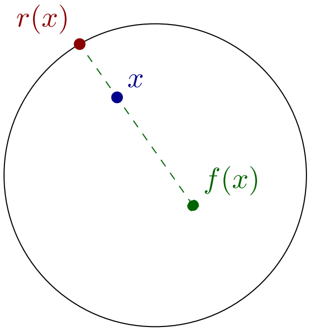


Figure 2: The definition of the retraction in Brouwer's theorem.

It is not hard to show that r is continuous (exercise), and moreover if $x \in \mathbb{S}^1$ then $r(x)$. In other words, r defines a retraction of the inclusion $i : \mathbb{S}^1 \to \mathbb{D}^2$, that is, $r \circ i = \mathrm{id}_{\mathbb{S}^1}$. On the fundamental groups, this induces maps:

$$\pi_1(\mathrm{id}) = \mathrm{id} : \pi_1(\mathbb{S}^1) = \mathbb{Z} \xrightarrow{\pi_1(i)} \pi_1(\mathbb{D}^2) = 0 \xrightarrow{\pi_1(r)} \to \pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

This is absurd: the identity of $\mathbb{Z}$ cannot factor through the trivial group. □

Corollary 2.50. *Let $R = [-1, 1]^2$ be a square and let $\alpha, \beta : [0, 1] \to R$ be paths that join opposite sides of the square. Then* $\mathrm{im}(\alpha) \cap \mathrm{im}(\beta) \neq \varnothing$.

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ such that $\alpha_1(\pm 1) = \pm 1$ and $\beta_2(\pm 1) = \pm 1$. Suppose that α and β do not intersect. We define, for $s, t \in [0, 1]$,

$$N(s, t) := \max_{i=1 \text{ or } 2} \big|\alpha_i(s) - \beta_i(t)\big|.$$

Since α and β do not intersect, we have that $N(s, t) > 0$ for all s, t. This allows us to define:

$$F : [-1, 1]^2 \to [-1, 1]^2, \ (s, t) \mapsto \frac{1}{N(s, t)}(\beta_1(t) - \alpha_1(s), \beta_2(t) - \alpha_2(s)).$$

Note that for any s, t, we have that either $F(s, t) = (\pm 1, u)$ or $F(s, t) = (u, \pm 1)$ for some $u \in [-1, 1]$.

Thanks to Brouwer's theorem, F admits a fixpoint, say $y_0 = (s_0, t_0)$ such that $F(y_0) = y_0$. Since y_0 is in the image of F, WLOG we can assume that $s_0 = 1$ (otherwise, we can mirror the square). Moreover, given that $F(1, t_0) = (1, t_0)$, we get that:

$$(N(1, t_0), t_0 N(1, t_0)) = (\beta_1(t_0) - \alpha_1(1), \beta_2(t_0) - \alpha_2(1)) = (\beta_1(t_0) - 1, \beta_2(t_0) - \alpha_2(1)).$$

In particular, $N(1, t_0) = \beta_1(t_0) - 1$. As we noticed before, $N(1, t_0) > 0$. However, $\beta_1(t_0) \in [-1, 1]$, thus $\beta_1(t_0) - 1 \leq 0$. This is absurd.

2.6 Seifert–Van Kampen theorem

2.6.1 Free groups, free product, abelianization

Definition 2.51. Let G_1 and G_2 be two groups. Their *free product* (or coproduct) is the group $G_1 * G_2$ generated by elements of the form $a_1 * a_2 * \ldots * a_n$ for $n \geq 0$ (the case $n = 0$ corresponds to an empty product, denoted e) and where each a_i is either in G_1 or G_2, subject to the relations:

- If $a_i = 1$ for some i, then $a_1 * \ldots * a_n = a_1 * \ldots * a_{i-1} * a_{i+1} * \ldots * a_n$;
- If two consecutive terms are in the same group, i.e., if a_i and a_{i+1} are both in G_1 or G_2, then $a_1 * \ldots * a_n = a_1 * \ldots * a_{i-1} * a_i a_{i+1} * a_{i+2} * \ldots * a_n$.

Proposition 2.52. *There are natural group morphisms $i_1 : G_1 \to G_1 * G_2$, $i_2 : G_2 \to G_1 * G_2$ satisfying the following universal property: for any pair of morphisms $(\phi_1 : G_1 \to H, \phi_2 : G_2 \to H)$, there exists a unique morphism $\phi : G_1 * G_2 \to H$ such that $\phi \circ i_1 = \phi_1$ and $\phi \circ i_2 = \phi_2$.*

Corollary 2.53. *The natural morphisms i_1 and i_2 are injective.*

Proof. Let $\phi_1 : G_1 \to G_1$ be the identity and $\phi_2 : G_2 \to G_1$ be any morphism (e.g., the trivial one). Then there exists $\phi : G_1 * G_2 \to G_1$ such that $\phi \circ i_1 = \mathrm{id}_{G_1}$. It follows that i_1 is injective. □

Corollary 2.54. *Free product is associative: for any triple of groups G_1, G_2, G_3, there is a natural isomorphism $G_1 * (G_2 * G_3) \cong (G_1 * G_2) * G_3$.*

Definition 2.55. The *free group* on n generators, denoted F_n, is the free product of n copies of $\mathbb{Z}$, that is:

$$F_n = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{n \text{ times}}.$$

We let $t_1, \ldots, t_n$ be the canonical generators of F_n.

Remark 2.56. For any group G and any n-uplet of elements $(g_1, \ldots, g_n) \in G^n$, there exists a unique morphism $\phi : F_n \to G$ such that $\phi(t_i) = g_i$ for all i.

There is a generalization of the free product called the amalgamated sum, or pushout.

Definition 2.57. Let $\phi_1 : K \to G_1$ and $\phi_2 : K \to G_2$ be group morphisms. The *amalgamated sum* of (ϕ_1, ϕ_2), also called the *pushout* of (ϕ_1, ϕ_2) and denoted $G_1 *_K G_2$,, is the quotient of $G_1 * G_2$ by the normal subgroup generated by terms of the form $\phi_1(x) * \phi_2(x)^{-1}$ for $x \in K$.

Proposition 2.58. *Given ϕ_1, ϕ_2 as in the definition, for any pair of morphisms $(f_1 : G_1 \to H, f_2 : G_2 \to H)$ such that $f_1 \circ \phi_1 = f_2 \circ \phi_2$, there exists a unique morphism $\phi : G_1 *_K G_2 \to H$ such that $\phi \circ i_1 = \phi_1$ and $\phi \circ i_2 = \phi_2$.*

Definition 2.59. Let G be a group. The *commutator subgroup* of G, denoted $[G, G]$, is the subgroup generator by elements of the form:

$$[x, y] := xyx^{-1}y^{-1}, \text{ for } x, y \in G.$$

Definition 2.60. The quotient $G/[G, G]$ is called the *abelianization* of G and is denoted G_{ab}.

Proposition 2.61. *The commutator subgroup of a group G is the largest normal subgroup of $K \trianglelefteq G$ such that the quotient G/K is abelian.*

Proof. First, $[G, G]$ is a normal subgroup. Indeed, for any $g \in G$ and $x \in [G, G]$, we have that:

$$gxg^{-1} = \underbrace{gxg^{-1}x^{-1}}_{=[g,x]\in[G,G]} \underbrace{x}_{\in[G,G]} \in [G, G].$$

Moreover, the quotient is abelian: for any $x, y \in G$, if we let their classes be denoted by $\bar{x}, \bar{y} \in G/[G, G]$, we have that:

$$\bar{x} \cdot \bar{y} = \overline{xy} = \overline{xyx^{-1}y^{-1}yx} = \overline{xyx^{-1}y^{-1}} \cdot \overline{yx} = \overline{yx} = \bar{y} \cdot \bar{x}.$$

Finally, if $K \trianglelefteq G$ is a normal subgroup such that G/K is abelian, then for any $x, y \in G$, in the quotient we have $\overline{xy} = \overline{yx}$ and thus $\overline{xyx^{-1}y^{-1}} = 1$, i.e., $xyx^{-1}y^{-1} \in K$. □

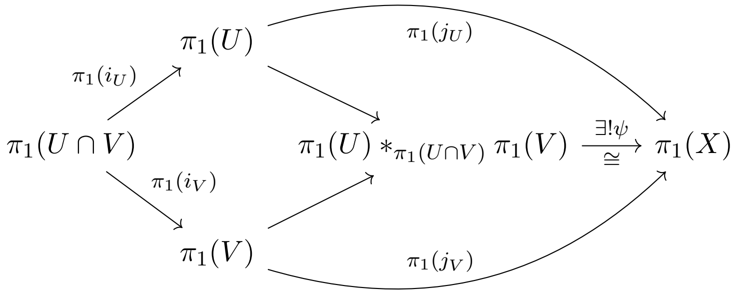
Proposition 2.62. *Let $\phi : G \to A$ be a group morphism where A is abelian. There exists a unique group morphism $\bar{\phi} : G_{ab} \to A$ such that $\bar{\phi}(\bar{x}) = \phi(x)$ for any $x \in G$.*

Proposition 2.63. *The abelianization of the free group on n generators is isomorphic to $\mathbb{Z}^n$, i.e., $(F_n)_{ab} = \mathbb{Z}^n$.*

Proof. The unique morphism $\phi : F_n \to \mathbb{Z}^n$ such that $\phi(t_i) = e_i = (\delta_{ij})_{j=1}^n$ induces a group morphism $\bar{\phi} : (F_n)_{\text{ab}} \to \mathbb{Z}^n$. Moreover, since $\mathbb{Z}^n$ is the free abelian group on n generators, there exists a unique group morphism $\psi : \mathbb{Z}^n \to (F_n)_{\text{ab}}$ such that $\psi(e_i) = \bar{t}_i$. The morphisms $\bar{\phi}$ and ψ are thus inverse to each other. □

2.6.2 The theorem and its proof

Theorem 2.64 (Seifert–Van Kampen)**.** *Let X be a space, $U, V \subseteq X$ opens such that $X = U \cup V$ and $U \cap V$ are path connected and let $x_0 \in U \cap V$. Let $i_U : U \cap V \to U$, $i_V : U \cap V \to V$, $j_U : U \to U \cup V$, and $j_V : V \to U \cup V$ be the inclusions. Let $\pi_1(U, x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V, x_0)$ be the pushout of $\pi_1(i_U)$ and $\pi_1(i_V)$, and let $\psi : \pi_1(U, x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the unique morphism induced by $\pi_1(j_U)$ and $\pi_1(j_V)$.*



Then ψ is an isomorphism of groups.

Proof. Let us first prove that ψ is surjective. It suffices to show that $\psi' : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ is surjective.

Let $[\alpha] \in \pi_1(X, x_0)$ be some element, where α is a loop in X based at x_0. Thanks to Lebesgue's lemma, there exists a decomposition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$ such that for all i, we have that $\alpha([t_i, t_{i+1}]) \subseteq U$ or V. Let us merge two consecutive parts of the decomposition if the image of α on these parts is in the same open set (U or V), such that WLOG:

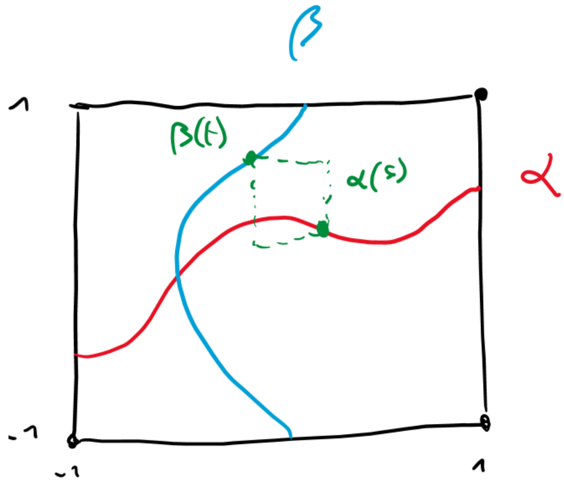
$$\alpha([t_0, t_1]) \subseteq U, \ \alpha([t_1, t_2]) \subseteq V, \ \alpha([t_2, t_3]) \subseteq U, \ \text{etc.}$$

It follows that for all j, $\alpha(t_j) \in U \cap V$. Since $U \cap V$ is assumed to be path connected, we can choose paths β_j from $\alpha(t_j)$ to x_0 in $U \cap V$. Moreover, let us denote:

$$\alpha_j \coloneqq \beta_j^{-1} \bullet \alpha|_{[t_j, t_{j+1}]} \bullet \beta_{j+1} \in \Omega_{x_0} U \text{ or } \Omega_{x_0} V.$$

It follows that each class $[\alpha_j]$ is in the image of ψ'. Moreover, we have that:

$$[\alpha] = [\alpha|_{[t_0,t_1]}] \bullet \ldots \bullet [\alpha|_{[t_{n-1},t_n]}] = [\alpha_0] \bullet \ldots \bullet [\alpha_n] \in \operatorname{im}(\psi').$$

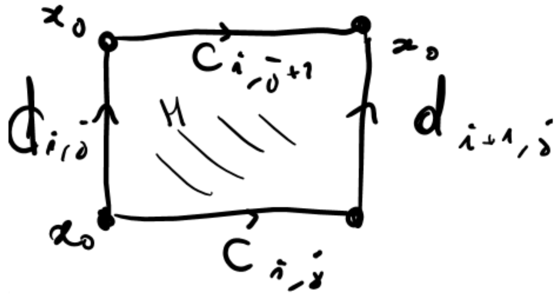


Let us now prove that ψ is injective. Suppose that $[\alpha_1] * \ldots * [\alpha_n] \in \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ is such that $[\alpha] = [\alpha_1 \bullet \ldots \bullet \alpha_n] = 1 \in \pi_1(X)$. There exists a homotopy $H : [0,1]^2 \to X$ between α and ϵ_{x_0}. Thanks to Lebesgue's lemma, we can cut $[0,1]^2$ into rectangles $r_{i,j} := [s_i, s_{i+1}] \times [t_j, t_{j+1}]$ such that $H(r_{i,j})$ is contained entirely either in U or V. WLOG, by subdividing the square further, we can assume that $(s_0 < s_1 < \ldots < s_k)$ is a finer subdivision of $[0,1]$ than the one used to define the concatenation $\alpha_1 \bullet \ldots \bullet \alpha_n$.

The theorem will follow from the next two lemmas. □

Lemma 2.65. *If H is equal to x_0 at each point of the grid, i.e., $H(s_i, t_j) = x_0$ for all i, j, then $[\alpha_1] * \ldots * [\alpha_n]$ is trivial in the pushout $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$.*

Proof. On each rectangle $r_{i,j}$, the map H defines four loops $c_{i,j}, c_{i,j+1}, d_{i,j}, d_{i+1,j}$ and a homotopy $c_{i,j} \bullet d_{i+1,j} \sim d_{i,j} \bullet c_{i,j+1}$:



Moreover, in the pushout, we have that:

$$[\alpha_1] * \ldots * [\alpha_n] = [c_{0,0}] * [c_{1,0}] * \ldots * [c_{k,0}].$$

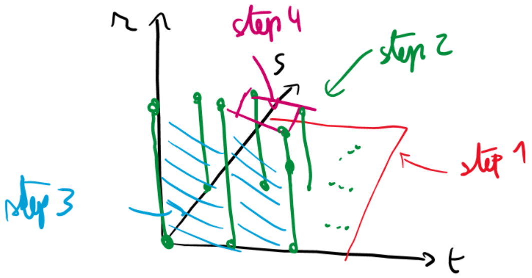
By using the restriction of the homotopy H on each of the rectangles $r_{i,j}$, we find that in the pushout:

$$[c_{0,0}] * [c_{1,0}] * \ldots * [c_{k,0}] = [\epsilon_{x_0} c_{0,1} d_{1,0}^{-1}] * [d_{1,0} c_{1,1} d_{1,1}^{-1}] * \ldots = \ldots = [c_{0,l}] * \ldots * [c_{k,l}].$$

But the last element in the equality is a subdivision of the constant path ϵ_{x_0}, which lives in $\pi_1(U \cap V)$, so it becomes trivial in the pushout. □

Lemma 2.66. *We can replace H by a homotopy H' which satisfies the hypothesis of the previous lemma.*

Proof. Our goal is to construct a new map $\tilde{H} : [0,1]^3 \to X$ such that for all s, t, $\tilde{H}(0, s, t) = H(s, t)$ and $\tilde{H}(1, s, t)$ is the desired $H'(s, t)$. We will define $\tilde{H}(r, s, t)$ in several steps.

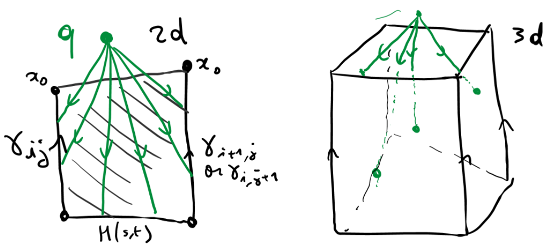


1. First, we set $\tilde{H}(0, s, t) := H(s, t)$ for all s, t.

2. Next, we define $\tilde{H}$ on "stakes" going upwards at all (s_i, t_j). For all (i, j), we choose a path $\gamma_{i,j}$ from $H(s_i, t_j)$ to x_0 which remains entirely either in U or V, depending on whether $H(r_{i,j})$ is contained in U or V. We then define $\tilde{H}(r, s_i, t_j) := \gamma_{i,j}(r)$.

3. Then, we extend $\tilde{H}$ on rectangles of the form $[0, 1] \times \{s_i\} \times [t_j, t_{j+1}]$ or $[0, 1] \times [s_i, s_{i+1}] \times \{t_j\}$, i.e., the faces "between the stakes." For this, we use the stereographic projection q defined in the picture, and we let $\tilde{H}(r, s_i, t) := \tilde{H}(q_1(r, t), s_i, q_2(r, t))$ and similarly for the face of the other type.

4. Finally, we extend $\tilde{H}$ to the "bricks between the faces" again by choosing a stereographic projection and extending $\tilde{H}$ by pre-composition.



We then let $H'(s, t) := \tilde{H}(1, s, t)$. Thanks to our choice of $\gamma_{i,j}$, this new homotopy satisfies the hypotheses of the previous lemma. Moreover, the "front face" of the cube $\tilde{H}$ defines a homotopy between $[\alpha_1] * \ldots * [\alpha_n]$ and $[\alpha'_1] * \ldots * [\alpha'_n]$ (the path $H'(1, s, 0)$) in the pushout. We can thus apply the lemma to show that $[\alpha_1] * \ldots * [\alpha_n] = [\alpha'_1] * \ldots * [\alpha'_n]$ is trivial in the pushout. □

2.6.3 Applications

Proposition 2.67. *For all $n \geq 2$, we have that $\pi_1(\mathbb{S}^n)$ is trivial.*

Proof. We can decompose $\mathbb{S}^n$ into two hemispheres U and V, each homeomorphic to an open disk $\mathbb{D}^n$, and such that the intersection $U \cap V$ is homeomorphic to $\mathbb{S}^{n-1} \times (0, 1)$ (which is homotopy equivalent to $\mathbb{S}^{n-1}$). Since both U and V are open and the intersection is path connected (because $n - 1 \geq 1$), we can apply the theorem to show that $\pi_1(\mathbb{S}^n) \cong$

$\pi_1(\mathbb{D}^n) *_{\pi_1(\mathbb{S}^1)} \pi_1(\mathbb{D}^n)$. Since $\mathbb{D}^n$ is convex and thus contractible, its fundamental group is trivial, and the pushout of trivial groups is trivial, so $\pi_1(\mathbb{S}^n) = 1$. ☐

Remark 2.68. This shows that the assumption that the intersection is path connected is mandatory in the statement of the theorem. In the previous argument, for $n = 1$, both U and V are contractible, but their intersection is not path connected and the theorem does not apply.

Definition 2.69. A pointed space (X, x_0) is said to be *well-pointed* if there exists an open neighborhood $U \subseteq X$ of x_0 and a map $H : U \times [0,1] \to U$ such that:

$$H(x,0) = x_0,\ H(x,1) = x,\ H(x_0,t) = x_0,\ \forall x \in U, t \in [0,1].$$

Remark 2.70. Such a map H is called a *deformation retract* of U onto $\{x_0\}$.

Example 2.71. If x_0 admits a neighborhood homeomorphic to $\mathbb{D}^n$, then (X, x_0) is well-pointed.

Example 2.72. The space $(X = \{0\} \cup \{1/n \mid n \in \mathbb{N}^*\}, 0)$ is not well-pointed.

Theorem 2.73. *Let (X, x_0) and (X', x'_0) be two well-pointed spaces. Consider the wedge sum $X \vee X'$ with its canonical base point $\chi \in X \vee X'$. The group morphisms $\pi_1(X, x_0) \to \pi_1(X \vee X', \chi)$ and $\pi_1(X', x'_0) \to \pi_1(X \vee X', \chi)$ induced by the natural inclusions induce an isomorphism*

$$\pi_1(X, x_0) * \pi_1(X', x'_0) \xrightarrow{\cong} \pi_1(X \vee X', \chi).$$

Proof. Let U, U' be neighborhood of x, x' that deformation retract onto $\{x\}, \{x'\}$. The subspaces $X \cup U'$ and $U \cup X'$ of $X \vee X'$ are open, and their intersection $U \cup U'$ is contractible, as it deformation retracts onto χ. By the Seifert–Van Kampen theorem, we have that:

$$\pi_1(X \vee X') \cong \pi_1(X \cup U') * \pi_1(U \cup X').$$

Since $X \cup U'$ and $U \cup X'$ are respectively homotopy equivalent to X and X', we obtain the result of the theorem. ☐

Example 2.74. The fundamental group of a wedge sum of k circles, $\bigvee^k \mathbb{S}^1$, is isomorphic to the free group F_k.

some knot complements

2.7 Cellular complexes and surfaces

2.7.1 Cell gluing

Definition 2.75. Let X be a space and let $f : \mathbb{S}^{n-1} \to X$ be a map. The *attachment of an n-cell along f*, denoted $X \cup_f \mathbb{D}^n$, is the pushout $X \cup_{\mathbb{S}^{n-1}} \mathbb{D}^n$ where the map $\mathbb{S}^{n-1} \to X$ is f and the map $\mathbb{S}^{n-1} \to \mathbb{D}^n$ is the natural inclusion.

In what follows, we fix f and we denote $j_X : X \to X \cup_f \mathbb{D}^n$ and $j_{\mathbb{D}} : \mathbb{D}^n \to X \cup_f \mathbb{D}^n$ the natural map.

Proposition 2.76. *The process of attaching a cell satisfies the following properties:*

1. *The space $X \cup_f \mathbb{D}^n$ is covered by the disjoint subspaces $j_X(X)$ and $j_{\mathbb{D}}(int(\mathbb{D}^n))$. The first is closed, and the second is open.*
2. *The map j_X induces a homeomorphism of X onto its image.*
3. *The map $j_{\mathbb{D}}$ induces a homeomorphism of $int(\mathbb{D}^n)$ onto its image.*
4. *The space X is Hausdorff if and only if $X \cup_f \mathbb{D}^n$ is Hausdorff.*

The proof is left as an exercise and follows from the basic properties of the pushout construction.

Proposition 2.77. *Let X be a path connected space and $f : \mathbb{S}^{n-1} \to X$ be a map $(n \geq 1)$ and let $Y = X \cup_f \mathbb{D}^n$.*

1. If $n \geq 3$, then $j_X : X \to Y$ induces an isomorphism $\pi_1(X) \to \pi_1(Y)$.
2. If $n = 2$, then $j_X : X \to Y$ induces a surjective morphism $\pi_1(X) \twoheadrightarrow \pi_1(Y)$ whose kernel is the normal subgroup generated by the image of $\pi_1(f) : \pi_1(\mathbb{S}^1) = \mathbb{Z} \to \pi_1(X)$.

Proof. Let U and V be the subspaces of Y given by:

$$U = j_{\mathbb{D}}(\mathbb{D}(0, 1/2)), \; V = j_X(X) \cup (\mathbb{D}^n \setminus \mathbb{D}^n(0, 1/4)).$$

Their intersection $U \cap V$ is a connected annulus homeomorphic to $\mathbb{S}^{n-1} \times (1/4, 1/2)$, which is path connected (as $n \geq 2$), so Seifert–Van Kampen's theorem applies. ☐

1. If $n \geq 3$, then the open subset U is contractible, V deformation retracts onto X, and $U \cap V$ is simply connected, thus $\pi_1(Y) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \pi_1(X) *_{\{1\}} \{1\} = \pi_1(X)$.
2. If, however, $n = 2$, the reasoning is the same but $U \cap V$ is no longer path connected. The pushout becomes $\pi_1(X) *_{\mathbb{Z}} \{1\} = \pi_1(X)/\langle \text{im}(\pi_1(f)) \rangle$, as desired.

2.7.2 Finite cellular complexes

Proposition 2.78. *A (finite)* CW complex*, or (finite)* cellular complex*, is a topological space endowed with a filtration $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$ (for some $n \geq 0$) such that:*

- *The subspace X_0 is a finite discrete subspace.*
- *For all $k \geq 0$, the space X_{k+1} is obtained from X_k by attaching a finite number of $(k+1)$-cells, i.e., X_{k+1} is the pushout of X_k and $\bigsqcup^{c_k} \mathbb{D}^{k+1}$ along $\bigsqcup^{c_k} \mathbb{S}^k$ for some $c_k \geq 0$ and some map $\bigsqcup^{c_k} \mathbb{S}^k \to X_k$.*

The subspace X_k is called the *k-skeleton* of X; note that it depends on the chosen CW decomposition of X. The top degree in which cells are attached is called the *dimension* of X.

Example 2.79. Finite CW complexes of dimension 0 are discrete spaces.

Example 2.80. Finite CW complexes of dimension 1 are graphs. The vertices correspond to 0-cells and the edges correspond to 1-cells.

Example 2.81. The sphere $\mathbb{S}^n$ admits many CW decompositions. For example, we can let $X_0 = X_1 = \ldots = X_{n-1} = \{*\}$ be a single point and attach an n-cell to that point in the only possible way. We can also let $X_0 = \ldots = X_{n-2}$ be a point, attach an $(n-1)$-cell to that point (the equator of the sphere), then attach two n-cells to the equator (the two hemispheres).

Proposition 2.82. *Let X be a finite CW complex. The inclusion $X_2 \to X$ induces an isomorphism on π_1.*

Proof. This follows from the main result of the previous section. □

Definition 2.83. A group G is *finitely presented* if it is the quotient of a free group by a normal subgroup generated by a finite number of elements. Such a presentation is usually denoted by $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$, where the x_i are generators of the free group and the r_j generate the normal subgroup that we mod out.

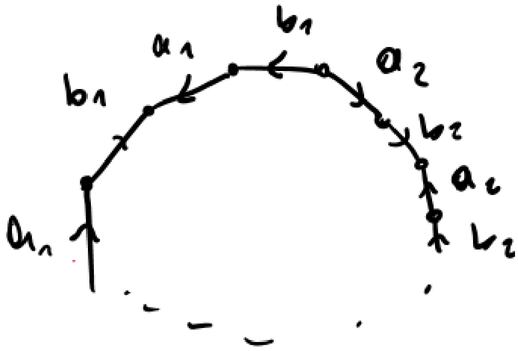
Example 2.84. The dihedral group $D_k = \langle \sigma, \rho \mid \sigma^2 = 1, \rho^k = 1, \sigma\rho\sigma = \rho^{-1} \rangle$ is finitely presented.

Proposition 2.85. *Let G be a finitely presented group. There exists a finite CW complex of dimension 2 whose fundamental group is isomorphic to G.*

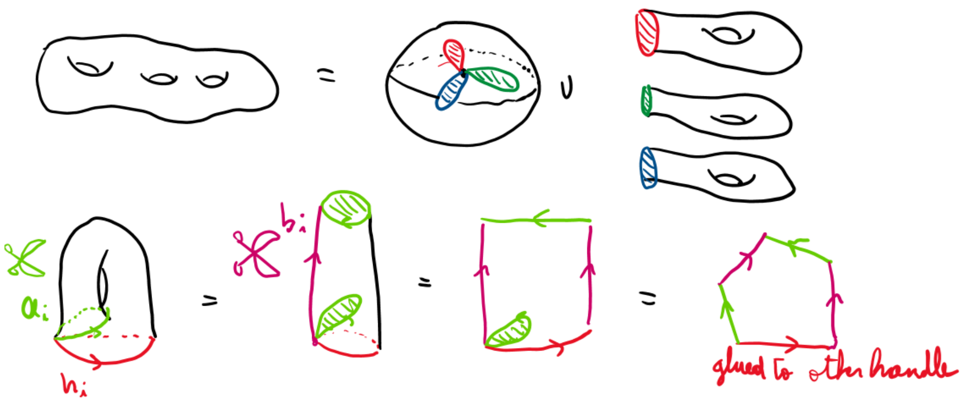
Proof. Suppose that $G = \langle t_1, \ldots, t_k \mid r_1, \ldots, r_l \rangle$. Then G is isomorphic to the fundamental group of the pushout $\bigvee^k \mathbb{S}^1 \cup_{\bigsqcup_{i=1}^l f_j} (\bigsqcup_{j=1}^l \mathbb{D}^2)$, where $f_j : \mathbb{S}^1 \to \bigvee^k \mathbb{S}^1$ is such that $\pi_1(f_j)(1) = r_j$. □

2.7.3 Surfaces

Definition 2.86. Let $g \geq 0$ be an integer. The *compact orientable surface of genus* g, denoted S_g, is the quotient of the regular $4g$-gone P_{4g} by the identification relation in the following picture:



Note that this corresponds to the usual pictures. Here is an example for $g = 3$:



Proposition 2.87. *The surface S_g, for $g \geq 1$, is homeomorphic to the pushout $\bigvee^{2g} \mathbb{S}^1 \cup_f \mathbb{D}^2$, where $f : \mathbb{S}^1 \to \bigvee^g \mathbb{S}^1$ is a map which induces on π_1:*

$$\mathbb{Z} \to F_{2g} = \langle a_1, b_1, \ldots, a_g, b_g \rangle, \ 1 \mapsto a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}.$$

Corollary 2.88. *The fundamental group of S_g is isomorphic to the finitely presented group:*

$$\pi_1(S_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

3 Homology

For simplicity, we will work in this chapter over a field $\mathbb{K}$. We can also define homology over a ring R, but this requires some more work on homological algebra.

Fundamental group, as we defined in the previous chapter, is a powerful homotopy invariant of spaces. It has a few issues, however. It cannot "see" what happens in dimension higher than 2. It is also difficult to effectively compute. For example, it is easy to distinguish between the sphere $\mathbb{S}^2 = S_0$ (whose fundamental group is trivial), the torus $S_1 = \mathbb{S}^1 \times \mathbb{S}^1$ (whose fundamental group is $\mathbb{Z}^2$) and the compact orientable surfaces of genus higher than 2 (whose fundamental groups are not abelian). But it is surprisingly difficult to prove that e.g., $\pi_1(S_2)$ and $\pi_1(S_3)$ are not isomorphic.

The goal of this chapter is to define another homotopy invariant of topological spaces, called homology. This invariant is typically much easier to compute than the fundamental group, and it "see" what happens in high dimension. There is, of course, a tradeoff: there are spaces that can be distinguished by their fundamental group, but not by their homology.

3.1 Chain complexes & homology

Definition 3.1. A *chain complex* (C_*, d) is the data of a family of $\mathbb{K}$-vector spaces $(C_n)_{n\geq 0}$ and linear maps $(d_n : C_{n+1} \to C_n)_{i\geq 0}$ such that for all $i \geq 0$, $d_{n+1} \circ d_n = 0$. A chain complex is usually written as:

$$\ldots \to C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \to 0.$$

The linear maps d_n are called the *differentials* and the elements of C_n are called the *n-chains*.

Remark 3.2. When clear from the context, we will usually drop the indices of the differentials, so that the equation $d_{n+1} \circ d_n = 0$ could be written as $d \circ d = 0$. It is also usual to speak of "the chain complex C_*" with the differentials being implicit from the context.

Definition 3.3. Let C_* and D_* be chain complexes. A *chain map* $f : C_* \to D_*$ is a collection of linear maps $(f_n : C_n \to D_n)_{i\geq 0}$ such that for all $i \geq 1$, $f_{n-1} \circ d_n = d_n \circ f_n$.

Remark 3.4. As for the differentials, it is usual to drop the indices from f_n when clear from the context, so that the equation becomes $f \circ d = d \circ f$.

Definition 3.5. Let C_* be a chain complex and $n \geq 0$. An *n-cycle* (or *cycle* for short) is an element $x \in C_n$ such that $dx = 0$. The subspace of i-cycles is denoted $Z_n(C) \coloneqq \ker(d : C_n \to C_{n-1})$. For consistency, we also define $Z_0 \coloneqq C_0$, i.e., every 0-chain is a 0-cycle.

Definition 3.6. Let C_* be a chain complex and $n \geq 0$. An n-boundary is an element $x \in C_n$ such that there exists $y \in C_{n+1}$ with $dy = x$. The subspace of i-boundaries is denoted $B_n(C) \coloneqq \mathrm{im}(d : C_{n+1} \to C_n)$.

Lemma 3.7. *Let C_* be a chain complex. For every $n \geq 0$, we have $B_n(C) \subseteq Z_i(C)$.*

Definition 3.8. Let C_* be a chain complex and $n \geq 0$. The *nth homology group* of C_*, denoted $H_n(C)$, is the quotient $H_n(C) \coloneqq Z_n(C)/B_n(C)$. Given some cycle $z \in Z_n(C)$, we denote $[z] \in H_n(C)$ its homology class.

Definition 3.9. Let C_* be a chain complex and $i \geq 0$. Elements $x, x' \in C_n$ are called homologous if there exists $y \in C_{n+1}$ such that $dy = x' - x$.

Note that the class of a cycle $z \in Z_n(C)$ is the set of elements of C_n homologous to z.

Lemma 3.10. *Let $f : C_* \to D_*$ be a chain map and $n \geq 0$. There is a natural induced linear map $f_* : H_n(C) \to H_n(D)$ (also denoted $H_n(f)$) defined by $f_*[z] \coloneqq [f(z)]$ for all $[z] \in H_i(C)$.*

Definition 3.11. Let $f, g : C_* \to D_*$ be chain maps. A *homotopy* between f and g is a collection of linear maps $(h_n : C_n \to D_{n+1})_{i \geq 0}$ such that for all $n \geq 0$,

$$g_n - f_n = h_{n-1} \circ d_n + d_{n+1} \circ h_n, \text{ also written as } g - f = h \circ d + d \circ h.$$

If such a homotopy exists, then f and g are said to be *homotopic*.

Lemma 3.12. *Let $f, g : C_* \to D_*$ be chain maps. If f and g are homotopic, then for all $n \geq 0$, we have $H_n(f) = H_n(g)$.*

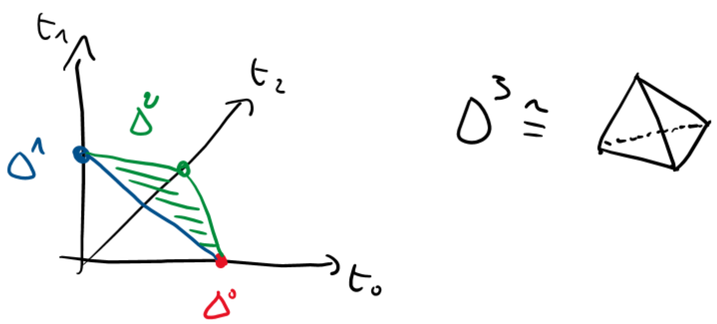
Proof. For any cycle $z \in C_n$, we have:

$$g_*[z] - f_*[z] = [(g - f)(z)] = [h(dz) - d(h(z))] = [h(0) - d(h(z))] = -[d(h(z))] = 0. \qquad \square$$

3.2 Singular homology

Definition 3.13. Let $n \geq 0$ be an integer. The standard simplex of dimension n, denoted Δ^n, is the following subspace of $\mathbb{R}^{n+1}$:

$$\Delta^n \coloneqq \{(t_0, \ldots t_n) \in \mathbb{R}^{n+1} \mid t_0 + \ldots t_n = 1, \ \forall i, t_i \geq 0\}.$$



Remark 3.14. The space Δ^n is the convex hull of the vertices $e_i = (\delta_{ij})_{j=0}^n \in \mathbb{R}^{n+1}$ for $0 \leq i \leq n$.

Definition 3.15. Let $0 \le i \le n$ be integers. The *ith face of the standard simplex* Δ^n, denoted $\partial_i \Delta^n$, is the convex hull of $(e_0, \dots \widehat{e}_i, \dots e_n)$.

Definition 3.16. We let $d^i : \Delta^{n-1} \to \Delta^n$ be the restriction of the unique linear map such that:

$$d^i(e_k) = \begin{cases} e_k, & \text{if } k < i; \\ e_{k-1}, & \text{if } k \ge i. \end{cases}$$

Lemma 3.17. *The map $d^i : \Delta^{n-1} \to \Delta^n$ defines a homeomorphism of Δ^{n-1} onto $\partial_i \Delta^n$.*

Lemma 3.18. *Given integers $0 \le i < j \le n$, we have the relation $d^i \circ d^{j+1} = d^j \circ d^i$.*

3.2.1 Singular chains

Definition 3.19. Let X be a space. The *singular chain complex* of X is the chain complex $C_*(X; \mathbb{K})$ (or $C_*(X)$ when $\mathbb{K}$ is clear) such that:

- For all $n \ge 0$, the space $C_n(X)$ of n-chains is the $\mathbb{K}$-vector space with basis $\mathcal{C}(\Delta^n, X)$, to explain i.e., the set of all continuous maps $\sigma : \Delta^n \to X$. Such an element σ is called a *singular simplex* of X.
- For $n \ge 0$ and a basis element $(\sigma : \Delta^{n+1} \to X) \in C_{n+1}(X)$, we define the differential by:

$$d_n(\sigma) := \sum_{i=0}^{n} (-1)^i (\sigma \circ d^i).$$

Lemma 3.20. *The singular chain complex is a chain complex.*

Proof. It suffices to check that $d^2 = 0$ on basis elements. Let $\sigma : \Delta^{n+2} \to X$ be a map seen as an element of $C_{n+2}(X)$. Then we have:

$$\begin{aligned} d_n(d_{n+1}(\sigma)) &= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n+1} (-1)^j \sigma d^j d^i \\ &= \sum_{0 \le i < j \le n+1} (-1)^{i+j} \sigma d^j d^i + \sum_{0 \le j \le i \le n} (-1)^{i+j} \sigma d^j d^i. \end{aligned}$$

In the first sum, we can use the relation $d^j d^i = d^i d^{j+1}$ for $i < j$ to get:

$$\begin{aligned} d_n(d_{n+1}(\sigma)) &= \sum_{0 \le i < j \le n+1} (-1)^{i+j} \sigma d^i d^{j+1} + \sum_{0 \le j \le i \le n} (-1)^{i+j} \sigma d^j d^i \\ &= \sum_{0 \le i \le j' \le n} (-1)^{i+j'+1} \sigma d^i d^{j'} + \sum_{0 \le j \le i \le n} (-1)^{i+j} \sigma d^j d^i = 0. \end{aligned}$$

∎

Definition 3.21. The *singular homology* of X (with coefficients $\mathbb{K}$) is the homology of the chain complex $C_*(X)$, denoted $H_*(X)$.

Example 3.22. If $X = \varnothing$, then $C_n(\varnothing) = 0$ for all n. It follows that $H_n(\varnothing) = 0$ for all n.

Example 3.23. If $X = \{x\}$ is a singleton, then $C_n(\{x\}) = \mathbb{K}$ for all n, with basis element the unique map $\Delta^n \to \{x\}$. The differential $d_n : C_{n+1}(\{x\}) \to C_n(\{x\})$ is zero if n is even, and $\mathrm{id}_{\mathbb{K}}$ if n is odd. In other words, the chain complex can be written as:

$$\ldots \to \mathbb{K} \xrightarrow{=} \mathbb{K} \xrightarrow{0} \mathbb{K} \xrightarrow{=} \mathbb{K} \xrightarrow{0} \mathbb{K} \to 0.$$

It follows that:

$$H_n(\{x\}) = \begin{cases} \mathbb{K}, & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases}$$

Lemma 3.24. *For all spaces X, we have a decomposition compatible with the differentials:*

$$C_*(X) = \bigoplus_{\alpha \in \pi_0(X)} C_*(X_\alpha).$$

It follows that for all n, $H_n(X) = \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha)$.

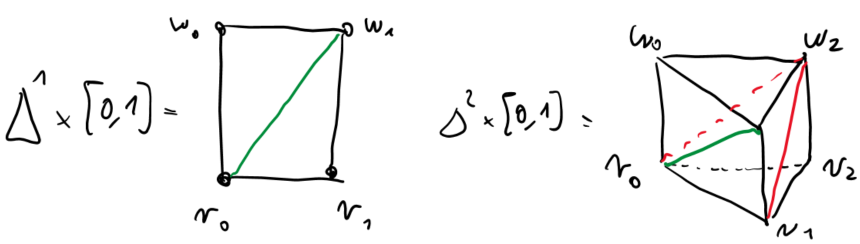
Definition 3.25. Let $f : X \to Y$ be a continuous map. It induces a natural chain map denoted $f_* : C_*(X) \to C_*(Y)$, which in turns induces a collection of linear maps $f_* : H_*(X) \to H_*(Y)$, defined on basis elements $\sigma : \Delta^n \to X$ by $f_*(\sigma) \coloneqq f \circ \sigma : \Delta^n \to Y$. These maps are moreover compatible with composition and identities (i.e., homology is functorial).

Theorem 3.26. *If two continuous maps $f, g : X \to Y$ are homotopic, then they induce the same map on homology.*

Proof. Let $H : X \times [0,1] \to Y$ be a homotopy from f to g, i.e., $H(\_, 0) = f$ and $H(\_, 1) = g$. If we let, for $t \in [0,1]$, the map $i_t : X \to X \times [0,1], x \mapsto (x,t)$, then we have $H \circ i_0 = f$ and $H \circ i_1 = g$. Since homology is functorial, it suffices to prove that $(i_0)_*, (i_1)_* : C_*(X) \to C_*(X \times [0,1])$ are homotopic as chain maps.

Geometrically, our goal is to decompose the prism $\Delta^n \times [0,1]$ into standard simplices and "glue" homotopies together. Let $v_i = (e_i, 0), w_i = (e_i, 1) \in \Delta^n \times [0,1] \subset \mathbb{R}^{n+1} \times [0,1]$ be the vertices of the prism. Then $\Delta^n \times [0,1]$ can be decomposed as a union of convex hulls:

$$\Delta^n \times [0,1] = \bigcup_{i=0}^{n} \langle v_0, \ldots, v_i, w_i, \ldots w_n \rangle.$$



Let us denote by $[v_0, \dots, v_i, w_i, \dots, w_n]$ the unique affine map $\Delta^{n+1} \to \Delta^n \times [0,1]$ such that:

$$[v_0, \dots, v_i, w_i, \dots, w_n] : e_j \mapsto \begin{cases} v_j, & \text{if } j \leq i; \\ w_{j-1} & \text{if } j > i. \end{cases}$$

Let us also define the n-chain:

$$P_n \coloneqq \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_i, w_i, \dots, w_n] \in C_{n+1}(\Delta^n \times [0,1]).$$

Lemma 3.27. *We have the following relation:*

$$dP_n = [w_0, \dots, w_n] - [v_0, \dots, v_n] + \sum_{j=0}^{n} (-1)^{j+1} (d^j \times \mathrm{id}_{[0,1]})_*(P_{n-1}).$$

Proof. In the computation of dP_n, the interior faces of the prism contribute twice each with opposite signs. All that remains is thus the contribution of $\Delta^n \times \{0\}$, the contribution of $\Delta^n \times \{1\}$, as well as the exterior faces with the signs in the lemma. More concretely, we have that:

$$dP_n = \sum_{i=0}^{n} (-1)^i d[v_0, \dots, v_i, w_i, \dots w_n] = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} [v_0, \dots, v_i, w_i, \dots, w_n] \circ d^j.$$

Each map $[v_0, \dots, v_i, w_i, \dots, w_n] \circ d^j$ is affine, so we can check on the affine basis that:

$$[v_0, \dots, v_i, w_i, \dots, w_n] \circ d^j = \begin{cases} [v_0, \dots, \hat{v}_j \dots, v_i, w_i, \dots, w_n], & \text{if } j \leq i \\ [v_0, \dots, v_i, w_i, \dots \hat{w}_{j-1}, \dots, w_n], & \text{if } j > i. \end{cases}$$

It follows that:

$$\begin{aligned} dP_n = & \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} [v_0, \dots, \hat{v}_j \dots, v_i, w_i, \dots, w_n] \\ & + \sum_{0 \leq i \leq j' \leq n} (-1)^{i+j'+1} [v_0, \dots, v_i, w_i, \dots \hat{w}_{j'}, \dots, w_n] \\ = & [w_0, \dots, w_n] - [v_0, \dots, v_n] + \sum_{0 \leq j < i \leq n} (-1)^{i+j} [v_0, \dots, \hat{v}_j \dots, v_i, w_i, \dots, w_n] \\ & + \sum_{0 \leq i = j \leq n} (-1)^{i+j} [v_0, \dots, \hat{v}_j \dots, v_i, w_i, \dots, w_n] \\ & + \sum_{0 \leq i < j' \leq n} (-1)^{i+j'+1} [v_0, \dots, v_i, w_i, \dots \hat{w}_{j'}, \dots, w_n] \\ & + \sum_{0 \leq i = j' \leq n} (-1)^{i+j'+1} [v_0, \dots, v_i, w_i, \dots \hat{w}_{j'}, \dots, w_n]. \end{aligned}$$

The second and fourth sum cancel out. Moreover,

$$
\begin{aligned}
(d^j \times \mathrm{id}_{[0,1]}) \circ P_{n-1} &= \sum_{i=0}^{n-1} (d^j \times \mathrm{id}_{[0,1]}) \circ [v_0, \ldots, v_i, w_i, \ldots, w_n] \\
&= \sum_{i=0}^{j} [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n] + \sum_{i=j+1}^{n} [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n].
\end{aligned}
$$

And thus, we can identify the remained of dP_n with the lemma. □

Equipped with this lemma, we can now define the following homotopy between $(i_0)_*$ and $(i_1)_*$:

$$h_n : C_n(X) \to C_{n+1}(X \times [0,1]),\ \sigma \mapsto (\sigma \times \mathrm{id}_{[0,1]}) \circ P_n.$$

We can then compute that:

$$
\begin{aligned}
(dh + hd)(\sigma) &= d((\sigma \times \mathrm{id}_{[0,1]}) \circ P_n) + h(\sum_{i=0}^{n} \sigma \circ d^i) \\
&= (\sigma \times \mathrm{id}_{[0,1]}) \circ dP_n + \sum_{i=0}^{n} (-1)^i ((\sigma \circ d^i) \times \mathrm{id}_{[0,1]}) \circ P_{n-1} \\
&= (\sigma \times \mathrm{id}_{[0,1]}) \circ [w_0, \ldots, w_n] - (\sigma \times \mathrm{id}_{[0,1]}) \circ [v_0, \ldots, v_n] \\
&\quad + \sum_{j=0}^{n} (-1)^{j+1} (\sigma \times \mathrm{id}_{[0,1]}) \circ (d^j \times \mathrm{id}_{[0,1]}) \circ P_{n-1} \\
&\quad + \sum_{i=0}^{n} (-1)^i (\sigma \circ d^i \times \mathrm{id}_{[0,1]}) \circ P_{n-1}.
\end{aligned}
$$

The last two sums cancel out, and we have that:

$$(i_0)_*(\sigma) = (\sigma \times \mathrm{id}_{[0,1]}) \circ [v_0, \ldots, v_n],\ (i_1)_*(\sigma) = (\sigma \times \mathrm{id}_{[0,1]}) \circ [w_0, \ldots, w_n]. \quad \square$$

Corollary 3.28. *If $f : X \to Y$ is a homotopy equivalence, then for all $n \geq 0$, $H_n(f)$ is an isomorphism.*

3.3 Relative homology

3.3.1 Exact sequences

Definition 3.29. A sequence of linear maps $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \ldots \xrightarrow{f_n} V_{n+1}$ is called *exact* if, for possible i, we have $\ker(f_{i+1}) = \mathrm{im}(f_i)$.

Example 3.30. A *short exact sequence* is an exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. This means that f is injective, g is surjective, and $\ker(g) = \mathrm{im}(f)$.

Remark 3.31. In a short exact sequence, $\dim(A) - \dim(B) + \dim(C) = 0$.

Definition 3.32. A *short exact sequence of chain complexes* is a sequence of chain maps $0 \to A_* \to B_* \to C_* \to 0$ such that for all n, the sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact.

Theorem 3.33. *Let $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$ be a short exact sequence of complexes. There is a natural exact sequence, called the* long exact sequence *associated to the short exact sequence:*

$$\dots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \to \dots$$
$$\dots \xrightarrow{\partial_1} H_0(A) \xrightarrow{f_*} H_0(B) \xrightarrow{g_*} H_0(C) \to 0.$$

Proof. Let us first build the connecting morphism $\partial_n : H_n(C) \to H_{n-1}(A)$. Given some cycle $x \in Z_n(C)$, since $g : B_n \to C_n$ is surjective, there exists $y \in B_n$ such that $g(y) = x$. Moreover, $g(dy) = dg(y) = dx = 0$, i.e., dy is in the kernel of g. By exactness, this means that dy is also in the image of f, i.e., there exists a unique (as f is injective) $z \in A_{n-1}$ such that $f(z) = dy$. ☐

The above proof is summarized by the following "diagram chase:"

$$\begin{array}{ccccc} A_n & \xhookrightarrow{f} & B_n & \xrightarrow{g} & C_n \\ \downarrow d & & \downarrow d & & \downarrow d \\ A_{n-1} & \hookrightarrow & B_{n-1} & \twoheadrightarrow & C_{n-1} \end{array} \qquad \begin{array}{ccccc} & & \exists y & \longmapsto & x \\ & & \downarrow & & \downarrow \\ \exists z & \longmapsto & dy & \longmapsto & 0. \end{array}$$

Lemma 3.34. *The element z defined above is a cycle and $[z] \in H_{n-1}(A)$ only depends on $[x] \in H_n(C)$.*

Proof. First, $f(dz) = d(f(z)) = d^2y = 0$ and f is injective, so $dz = 0$, i.e., z is a cycle. Next, suppose that $\tilde{y} \in B_n$ is another choice of element such that $g(y) = x$, with $d\tilde{y} = f(\tilde{z})$. We then have that $g(y - \tilde{y}) = 0$, i.e., $y - \tilde{y} \in \ker(g)$. By exactness, there exists $w \in B_n$ such that $f(w) = y - \tilde{y}$. It follows that

$$f(z - \tilde{z}) = dy - d\tilde{y} = df(w) = f(dw),$$

which implies that $z - \tilde{z} = dw$ and thus, $[z] = [\tilde{z}]$. We thus get a well-defined map $\partial : Z_n(C) \to H_{n-1}(A)$. This map is linear: if $x, x' \in Z_n(C)$ and $\lambda, \lambda' \in \mathbb{K}$, with $\partial(x) = [z]$ and $\partial(x') = [z']$, we get that $f(\lambda y + \lambda' y') = \lambda x + \lambda' x'$ and $g(\lambda z + \lambda' z') = \lambda\ dy + \lambda'\ dy' = d(\lambda y + \lambda' y')$ and so $\partial(\lambda x + \lambda' x') = \lambda[z] + \lambda'[z']$.

Finally, we need to check that ∂ vanishes on boundaries so that we get a map well-defined on $H_n(C) = Z_n(C)/B_n(C)$. If $x \in B_n(C)$, with $da = x$, then there exists $b \in B_{n+1}$ such that $g(b) = a$, and we can choose $y = db$. But then $dy = d^2b = 0$ and so we can choose $z = 0$ such that $f(z) = dy$. We thus get $\partial(x) = [0]$, as desired.

Let us now check that the long sequence is exact. We have three kinds of exactness to check.

First, we need to check that $\mathrm{im}(f_*) = \ker(g_*)$. Since $g \circ f = 0$, the same holds on homology, i.e., $g_* \circ f_* = 0$, which implies that $\mathrm{im}(f_*) \subseteq \ker(g_*)$. Conversely, if

$[b] \in H_n(B)$ is such that $g_*[b] = 0$, we get that there exists some $c \in C_n$ such that $g(b) = dc$. Given that g is surjective, there exists $b' \in C_{n+1}$ such that $g(b') = c$. We thus have $g(b - db') = dc - dc = 0$, i.e., there exists $a \in A_n$ such that $f(a) = b - db'$. Note that $f(da) = db - d^2 b' = 0$, and f is injective, so $da = 0$. Thus $f_*[a] = [b]$, so that $[b] \in \mathrm{im}(f_*)$ as desired.

Next, we must prove that $\mathrm{im}(\partial) = \ker(f)$. Suppose $[x] \in H_n(C)$, and $\partial[x] = [z]$ with $f(z) = dy$. Then we have that:

$$f_*(\partial[x]) = f_*[z] = [dy] = 0.$$

This shows that $\partial[x] \in \ker(f)$. Conversely, suppose $f_*[z] = 0$ for some $[z] \in H_{n-1}(A)$. This means that there exists $y \in B_{n-1}$ such that $dy = f(z)$, and if we choose $x = g(y)$, then we have $\partial[x] = [z]$ by definition, so $[z] \in \mathrm{im}(\partial)$.

The proof of the last kind of exactness, i.e., that $\mathrm{im}(g) = \ker(\partial)$ is left as an exercise. □

3.3.2 Definition of relative homology and long exact sequence

Definition 3.35. A *pair of spaces* is a couple (X, A) where X is a space and $A \subseteq X$ is a subspace.

Definition 3.36. Let (X, A) be a pair of spaces, with $i : A \to X$ being the inclusion. The *relative singular chain complex* of (X, A) is the quotient:

$$C_*(X, A) := C_*(X)/C_*(A).$$

The *relative homology* of (X, A), denoted $H_*(X, A)$, is the homology of the chain complex $C_*(X, A)$.

Remark 3.37. In general, $H_n(X, A) \neq H_n(X)/H_n(A)$.

Example 3.38. Let X be a space. Then for all n, $H_n(X, \varnothing) = H_n(X)$, while $H_n(X, X) = 0$.

Theorem 3.39. *Let (X, A) be a pair of spaces. There is a natural long exact sequence:*

$$\dots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \dots$$

Proof. This is the long exact sequence associated to the short exact sequence of complexes:

$$0 \to C_*(A) \to C_*(X) \to C_*(X)/C_*(A) \to 0.$$

□

3.4 Computations

3.4.1 Low degrees

Theorem 3.40. *For any space X, $H_0(X)$ is the vector space with basis $\pi_0(X)$.*

Proof. Given that $H_0(X) = \bigoplus_{\alpha \in \pi_0(X)} H_0(X_\alpha)$, it suffices to show that if X is path connected, then $H_0(X) = \mathbb{K}$. Let us thus assume that X is path connected and define a linear map on the natural basis of $C_0(X)$:

$$\epsilon : C_0(X) \to \mathbb{K}, \quad \sum_{x \in X} \lambda_x [x] \mapsto \sum_{x \in X} \lambda_x.$$

This map is surjective, since $X \neq \varnothing$ (the empty space is not connected). Let us prove that $\ker(\epsilon) = B_0(X)$. First, if $\sigma : \Delta^1 \to X$ is a 1-simplex then $d\gamma = \gamma(1) - \gamma(0)$ and thus $\epsilon(d\gamma) = 1 - 1 = 0$, so that $B_0(X) \subseteq \ker(\epsilon)$. Conversely, suppose that $\sigma = \sum_x \lambda_x x \in \ker(\epsilon)$. Let $x_0 \in X$ be any point and choose, for all $x \in X$, a path γ_x from x_0 to x (which exists because X is path connected). Then:

$$d(\sum_x \lambda_x \gamma_x) = \sum_x \lambda_x (x - x_0) = \sum_x \lambda_x x - (\sum_x \lambda_x) x_0 = \sigma - \epsilon(\sigma) x_0 = \sigma.$$

It follows that $\sigma \in B_0(X)$, and thus $\ker(\epsilon) \subseteq B_0(X)$. □

In the following theorem, we will temporarily use homology with integral coefficients, which forms abelian groups ("$\mathbb{Z}$-modules").

tensor product?

Theorem 3.41 (Hurewicz)**.** *Let X be a path connected space and let $x_0 \in X$ be any point. There exists a natural isomorphism:*

$$\pi_1(X, x_0)_{ab} \cong H_1(X; \mathbb{Z}).$$

Proof. Let us define a map:

$$h : \Omega_{x_0} X \to H_1(X; \mathbb{Z}), \ \gamma \mapsto [\gamma].$$

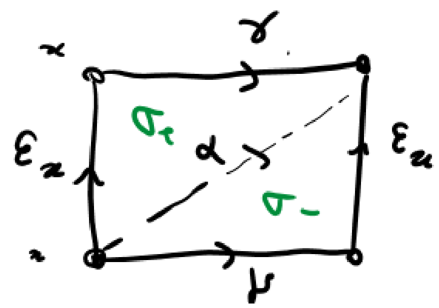
In other words, given a loop $\gamma : [0,1] \to X$ at x_0, we may view γ as a map $\gamma : \Delta^1 \to X$. Since γ is a loop, $d\gamma = \gamma(1) - \gamma(0) = 0 \in C_0(X)$, i.e., $[\gamma]$ defines a homology class.

Lemma 3.42. *The map h induces a map $h : \pi_1(X, x_0) \to H_1(X)$.*

Proof. To prove that h passes through the quotient defining $\pi_1(X, x_0)$, we need to check that if two loops $\gamma, \mu \in \Omega_{x_0} X$ are path homotopic, then $h(\gamma) = h(\mu)$. Suppose that we have a path homotopy $H : [0,1] \times [0,1] \to X$ from γ to μ. We can cut $[0,1]^2$ into two triangles (i.e., 2-simplices) σ_+ and σ_- such that:

$$d\sigma_+ = \mu - \alpha + \epsilon_{x_0}, \quad d\sigma_- = \epsilon_{x_0} - \alpha + \gamma,$$

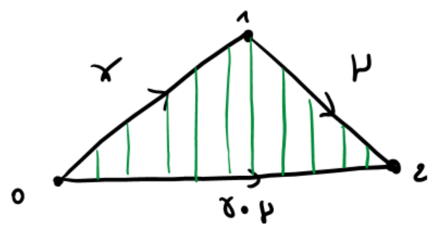
Where $\alpha : [0,1] \to X$ is the path given by the diagonal ($\alpha(t) = H(t,t)$).



Then we have that $\mu - \gamma = d(\sigma_+ - \sigma_-)$, i.e., $h(\gamma) = h(\mu)$. It follows that h induces a map $h : \pi_1(X, x) \to H_1(X)$. $\square$

Lemma 3.43. *The map $h : \pi_1(X, x_0) \to H_1(X)$ is a group morphism.*

Proof. Let $\gamma, \mu \in \Omega_{x_0} X$ be loops. We must show that $h(\gamma \bullet \mu) = h(\gamma) + h(\mu)$. Let us define a 2-simplex $\sigma \in C_2(X)$ from the following picture, such that σ is constant on the vertical lines and such that $d\sigma = \gamma \bullet \mu - \gamma - \mu$:



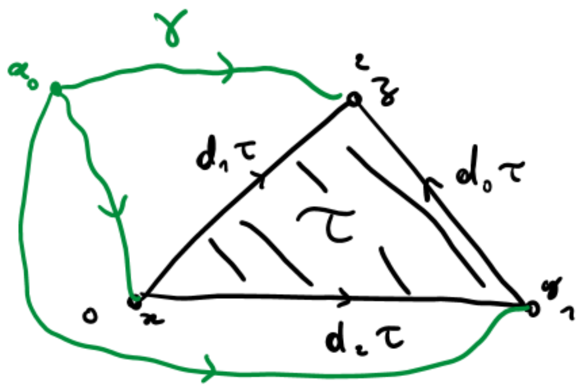
Since $H_1(X)$ is abelian, the morphism h induces a morphism $h : \pi_1(X, x_0)_{\mathrm{ab}} \to H_1(X)$. Let us show that this defines a group isomorphism by building an inverse of h.

For every $x \in X$, since X is path connected, we may choose a path $\gamma_x \in \Omega_{x_0,x} X$ from x_0 to x. We then define a map on the canonical basis of $C_1(X)$ given by maps $\sigma : \Delta^1 \to X$ viewed as paths:

$$\psi : C_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}, \quad (\sigma : \Delta^1 \to X) \mapsto [\gamma_{\sigma(0)} \bullet \sigma \bullet (\gamma_{\sigma(1)})^{-1}]. \qquad \square$$

Lemma 3.44. *The map ψ vanishes on boundaries, i.e., if $\tau \in C_2(X)$ then $\psi(d\tau) = 0$.*

Proof. Let $\tau : \Delta^2 \to X$ be a generator of $C_2(X)$ and $d\tau = d_0\tau - d_1\tau + d_2\tau$, where $d_i\tau$ is the restriction of τ to the segment $\partial_i \Delta^2 = \Delta^1$.



Let us denote $x = d_1\tau(0) = d_2\tau(0)$, $y = d_0\tau(0) = d_2\tau(1)$, and $z = d_0\tau(1) = d_1\tau(1)$. Then we have:

$$\psi(d\tau) = [\gamma_y \cdot d_0\tau \cdot \gamma_z^{-1}] \cdot [\gamma_z \cdot (d_1\tau)^{-1} \cdot \gamma_x^{-1}] \cdot [\gamma_x \cdot d_2\tau \cdot \gamma_y^{-1}] = [\gamma_y \cdot d_0\tau \cdot (d_1\tau)^{-1} \cdot d_2\tau \cdot \gamma_y^{-1}].$$

Since we have abelianized the fundamental groups, γ_y and γ_y^{-1} cancel out. Moreover, the path $d_0\tau \cdot (d_1\tau)^{-1} \cdot d_2\tau$ is simply the path that goes around the boundary of the simplex once. This path is homotopic to the constant path through the homotopy τ, so that we get $\psi(d\tau) = 0$.

The map ψ thus induces a map $\psi' : H_1(X) \to \pi_1(X, x_0)_{\text{ab}}$. □

Lemma 3.45. *We have* $\psi' \circ h = \text{id}$.

Proof. Let $[\gamma] \in \pi_1(X, x_0)_{\text{ab}}$ be the class of some loop $\gamma \in \Omega_{x_0}X$. Then:

$$(\psi' \circ h)([\gamma]) = [\gamma_{x_0}] \cdot [\gamma] \cdot [\gamma_{x_0}^{-1}].$$

Again, since we are in the abelianization, $[\gamma_{x_0}]$ and $[\gamma_{x_0}^{-1}]$ cancel, and thus $(\psi \circ h)([\gamma]) = [\gamma]$. □

Lemma 3.46. *We have* $h \circ \psi' = \text{id}$.

Proof. For any 1-simplex $\sigma : \Delta^1 \to X$, $h(\psi(\sigma))$ is the homology class of the 1-cycle $\sigma + \gamma_{\sigma(0)} - \gamma_{\sigma(1)}$. It follows that if $\alpha \in H_1(X)$ is the homology class of a linear combination of 1-simplices, the terms of the form $\gamma_{\sigma(0)} - \gamma_{\sigma(1)}$ in $h(\psi'(\alpha))$ cancel out (since α is closed) so that $h(\psi'(\alpha)) = \alpha$. □

This concludes the proof of the theorem. □

3.4.2 Mayer–Vietoris theorem

Theorem 3.47 (Mayer–Vietoris)**.** *Let X be a space, $U, V \subseteq X$ open subsets such that $X = U \cup V$. Denote the inclusions by $i_U : U \cap V \to U$, $i_V : U \cap V \to V$, $j_U : U \to X$, $j_V : V \to X$. Then there is a natural long exact sequence:*

$$\dots \xrightarrow{\partial} H_k(U\cap V) \xrightarrow{(i_U)_*\oplus(i_V)_*} H_k(U)\oplus H_k(V) \xrightarrow{(j_U)_*-(j_V)_*} H_k(X) \xrightarrow{\partial} H_{k-1}(U\cap V) \to \dots$$
$$\dots \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to\to H_0(X) \to 0.$$

The proof of this theorem is quite involved and relies upon the following result, that we will not prove.

Definition 3.48. Let X be a space and $\mathcal{U} = (U_i)_{i\in I}$ be a family of subsets of X such that $X = \bigcup_i int(U_i)$. A singular n-simplex $\sigma : \Delta^n \to X$ is called *$\mathcal{U}$-small* if $\mathrm{im}(\sigma)$ is contained in one of the U_i. The complex of $\mathcal{U}$-small chains is the subcomplex $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ spanned by $\mathcal{U}$-small singular simplices.

Theorem 3.49 (Small chains)**.** *The inclusion $C_*^{\mathcal{U}}(X) \to C_*(X)$ induces an isomorphism on homology.*

proof of small chains theorem

Proof of the Mayer–Vietoris theorem. Let $\mathcal{U} = \{U, V\}$. There is a short exact sequence of complexes:

$$0 \to C_*(U \cap V) \xrightarrow{i_U\oplus i_V} C_*(U) \oplus C_*(V) \xrightarrow{j_U-j_V} C_*^{\mathcal{U}}(X) \to 0.$$

This induces the long exact sequence of the theorem. □

Using this theorem, we can finally perform nontrivial computations in homology.

Example 3.50. Let $n \geq 1$ be an integer. Then the homology groups of $\mathbb{S}^n$ are given by:

$$H_k(\mathbb{S}^n) = \begin{cases} \mathbb{K}, & \text{if } k = 0 \text{ or } k = n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Recall that $\mathbb{S}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 = 1\}$. Let $U = \{x \in \mathbb{S}^n \mid x_0 > -\frac{1}{2}\}$ and $V = \{x \in \mathbb{S}^n \mid x_0 < 1/2\}$ be open hemispheres that cover $\mathbb{S}^n$. The Mayer–Vietoris sequence reads:

$$\dots \to H_{k+1}(U) \oplus H_{k+1}(V) \to H_{k+1}(\mathbb{S}^n) \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to \dots.$$

Both U and V are contractible, and the intersection $U \cap V$ is homotopy equivalent to $\mathbb{S}^{n-1}$. Therefore, for $k > 0$, the exact sequence becomes:

$$\dots \to 0 \to H_{k+1}(\mathbb{S}^n) \to H_k(\mathbb{S}^{n-1}) \to 0 \to \dots$$

Thus, $H_{k+1}(\mathbb{S}^n) \cong H_k(\mathbb{S}^{n-1})$.

For $k = 0$, we must distinguish the case $n = 1$ from the case $n > 1$. If $n = 1$, then the exact sequence looks like:

$$\cdots \to 0 \to H_1(\mathbb{S}^1) \to \underbrace{H_0(\mathbb{S}^0)}_{=\mathbb{K}^2} \xrightarrow{\phi} \underbrace{H_0(U) \oplus H_0(V)}_{=\mathbb{K}^2} \xrightarrow{\psi} \underbrace{H_0(\mathbb{S}^1)}_{=\mathbb{K}} \to 0$$

The maps ϕ, ψ are given by $\psi(x, y) = (x - y, y - x)$ and $\psi(x, y) = x + y$. Given that the sequence is exact, we have an isomorphism $H_1(\mathbb{S}^1) \cong \ker(\phi) = \mathbb{K}$, as expected. If, however, $n \geq 2$, then the exact sequence becomes:

$$\cdots \to 0 \to H_1(\mathbb{S}^n) \to \underbrace{H_{0(\mathbb{S}^{n-1})}}_{=\mathbb{K}} \xrightarrow{\phi} \underbrace{H_0(U) \oplus H_0(V)}_{=\mathbb{K}^2} \xrightarrow{\psi} \underbrace{H_0(\mathbb{S}^1)}_{=\mathbb{K}} \to 0$$

Now $\phi(x) = (x, -x)$ and $H_1(\mathbb{S}^n) = \ker(\phi) = 0$. With all these equalities, we can now prove the formula by induction on k and n. □

3.4.3 Excision

Theorem 3.51 (Excision)**.** *Let (X, A) be a pair of spaces and $U \subseteq X$ a subset such that $\bar{U} \subset int(A)$. Then the inclusion of pairs $(X \smallsetminus U, A \smallsetminus U) \to (X, A)$ induces an isomorphism in homology:*

$$H_*(X \smallsetminus U, A \smallsetminus U) \to H_*(X, A).$$

Proof. Let $B = X \smallsetminus U$, such that $A \cap B = A \smallsetminus U$. We must check that the following inclusion induces an isomorphism on homology:

$$j : C_*(B)/C_*(A \cap B) \to C_*(X)/C_*(A).$$

fix proof

Let $\mathcal{U} = \{A, B\}$, which satisfies the hypothesis of the small chains' theorem. If we denote $C_*(A + B) := C_*^{\mathcal{U}}(X)$, then j factors as:

$$C_*(B)/C_*(A \cap B) \to C_*(A + B)/C_*(A) \to C_*(X)/C_*(A).$$ □

Exercise: prove that both maps induce isomorphisms on homology (use the small chains theorem and the five lemma).

Definition 3.52. Let X be a space and $x \in X$. The *local homology* of X at x is $H_*(X, X \smallsetminus \{x\})$.

Proposition 3.53. *If X is a topological manifold of dimension n, then for all $x \in X$ and $k \geq 0$,*

$$H_k(X, X \smallsetminus \{x\}) \cong \begin{cases} \mathbb{K}, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since X is a manifold of dimension n, there exists a neighborhood W of x homeomorphic to $\mathbb{D}^n$. Let $U = X \setminus W$. By excision, $H_*(X, X \setminus \{x\}) \cong H_*(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\})$. Using the long exact sequence for relative homology and the fact that $\mathbb{D}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ gets us the result. □

Definition 3.54. A pair (X, A) is called *good* if A is a nonempty subspace of X and there exists a neighborhood V of A which deformation retracts onto A.

Example 3.55. If X is a CW complex and $A \subseteq X$ is a nonempty subcomplex, then (X, A) is a good pair. proof

Definition 3.56. Let X be a nonempty space and $x_0 \in X$ be a point. The reduced homology of X is the relative homology $\tilde{H}_*(X) \coloneqq H_*(X, x_0)$.

Proposition 3.57. *If (X, A) is a good pair, then the quotient map $\pi : (X, A) \to (X/A, A/A)$ induces an isomorphism $H_*(X, A) \cong H_*(X/A, A/A) = \tilde{H}(X/A)$.*

Proof. Suppose that V is a neighborhood of A which deformation retracts onto A. We have a commutative diagram:

$$\begin{array}{ccccc} H_n(X, A) & \to & H_n(X, V) & \leftarrow & H_n(X \setminus A, V \setminus A) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(X/A, A/A) & \to & H_n(X/A, V/A) & \leftarrow & H_n(X/A \setminus A/A, V/A \setminus A/A). \end{array}$$

The upper left map is an isomorphism because $H_n(V, A) = 0$. Similarly, the bottom left map is an isomorphism. The other two horizontal maps are isomorphisms thanks to excision. Finally, the right vertical map is an isomorphism because π is a homeomorphism on the complement of A. It follows that the left vertical map is an isomorphism. □

3.5 Applications

Theorem 3.58 (Brouwer)**.** *Any continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ (where $n \geq 1$) admits a fixpoint.*

Proof. Let us assume to the contrary that f has no fixpoint. For any $x \in \mathbb{D}^n$, we consider the intersection $r(x) \in \partial\mathbb{D}^n = \mathbb{S}^{n-1}$ of the half line $[f(x), x)$ with the boundary of $\mathbb{D}^n$. This defines a continuous retraction $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ of the inclusion $i : \mathbb{S}^{n-1} \to \mathbb{D}^n$, i.e., $r \circ i = \mathrm{id}$. On homology, we thus get that the identity of $H_{n-1}(\mathbb{S}^{n-1})$ factors through the space $H_{n-1}(\mathbb{D}^n) = 0$, which is absurd. □

Theorem 3.59 (Invariance of dimension)**.** *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be nonempty open subsets. If there exists a homeomorphism $f : U \cong V$, then $m = n$.*

Proof. If such a homeomorphism exists, then for any $x \in U$, we get an isomorphism on local homology $H_*(U, U \setminus \{x\}) = H_*(V, V \setminus \{f(x)\})$. Given that the first one is concentrated in degree m, while the second one is concentrated in degree n, this implies that $m = n$. □

3.6 Cellular homology

Proposition 3.60. *Let X be a space, $n \geq 1$ be an integer, and $f : \mathbb{S}^{n-1} \to X$ be a map. Let $Y = X \cup_f \mathbb{D}^n$.*

- If $k \neq n-1, n$, then the inclusion $i : X \to Y$ induces an isomorphism $H_k(X) \to H_k(Y)$.
- There is an exact sequence:

$$0 \to H_n(X) \xrightarrow{i_*} H_n(Y) \to H_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{f_*} H_{n-1}(Y) \to H_{n-1}(X) \to 0.$$

Proof. Apply the long exact sequence for relative homology to the map of pairs (Y, X). By excision, $H_*(Y, X) = H_*(\mathbb{D}^n, \mathbb{S}^{n-1})$. If we apply again the long exact sequence to the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$ and the fact that $\mathbb{D}^n$ is contractible, we get that $H_*(\mathbb{D}^n, \mathbb{S}^{n-1}) = H_{*-1}(\mathbb{S}^{n-1})$. □

Recall that $S_g = (\bigvee^{2g} \mathbb{S}^1) \cup_f \mathbb{D}^2$ is the compact oriented surface of genus g.

Corollary 3.61. *The homology of S_g is given by:*

$$H_k(S_g) = \begin{cases} \mathbb{K}, & \text{if } k = 0; \\ \mathbb{K}^{2g}, & \text{if } k = 1; \\ \mathbb{K}, & \text{if } k = 2; \\ 0, & \text{if } k \geq 3. \end{cases}$$

Proof. Let $X = \bigvee^{2g} \mathbb{S}^1$. Using the Mayer–Vietoris sequence, we get that $H_0(X) = \mathbb{K}$, $H_1(X) = \mathbb{K}^{2g}$, and $H_i(X) = 0$ for $i \geq 2$. To apply the previous proposition, we must determine the effect of the attaching map $f : \mathbb{S}^1 \to X$ on homology. By the Hurewicz theorem, $H_1(X) = \pi_1(X)_{\mathrm{ab}}$. The map f sends a generator of $\pi_1(\mathbb{S}^1)$ to $[a_1 b_1 a_1^{-1} b_1^{-1} \ldots]$ which belongs to the commutator subgroup of $\pi_1(X)$, and thus vanishes in $H_1(X)$. It follows that $H_1(f) = 0$, and thus we get the result of the corollary by applying the previous proposition. □

Recall that if X is a CW complex, then X_n (for $n \geq 0$) is its n-skeleton, i.e., the space obtained by gluing cells until dimension n.

Lemma 3.62. *Let X be a CW complex. Then:*

1. *For $k \neq n$, we have $H_k(X_n, X_{n-1}) = 0$, while $H_n(X_n, X_{n-1})$ has a basis given by the n-cells of X.*
2. *For $k > n$, we have $H_k(X_n) = 0$.*
3. *The inclusion of X_n into X induces an isomorphism $H_k(X_n) \to H_k(X)$ for $k < n$.*

Proof. The first statement follows from the fact that (X_n, X_{n-1}) is a good pair, and X_n/X_{n-1} is a wedge of n-spheres – one for each n-cell of X. The second statement follows from the first and the long exact sequence of the pair (X_n, X_{n-1}).

For the last statement, note first that (a) implies that $H_k(X_n) \to H_k(X_{n+d})$ is an isomorphism for $k < n$ and $d > 0$. Now, let us note that any simplex $\sigma : \Delta^k \to X$ has compact image; therefore, it can only meet finitely many cells, and thus its image lies in X_{n+d} for some d. A cycle is a finite linear combination of simplices, so any homology class lies in the image of $H_k(X_{n+d})$ for some d, and thus $H_k(X_n) \to H_k(X)$ is surjective. Conversely, if a cycle in X_n is homologous to zero in X, i.e., it bounds a combination of simplices in X, then that combination of simplices lies in X_{n+d} for some $d > 0$. Since $H_k(X) \cong H_k(X_{n+d})$, it follows that the cycle was trivial in $H_k(X_n)$ to begin with, i.e., $H_k(X_n) \to H_k(X)$ is injective. □

Definition 3.63. Let X be a CW complex. The *cellular chain complex* of X, denoted $C_*^{\mathrm{cell}}(X)$, is defined by $C_n^{\mathrm{cell}}(X) := H_n(X_n, X_{n-1})$ with differential:

$$H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{j_*} H_{n-1}(X_{n-1}, X_{n-2}).$$

Proposition 3.64. *The cellular chain complex is a chain complex (i.e., its differential squares to zero) and there is an isomorphism $H_n^{\mathrm{cell}}(X) \cong H_n(X)$.*

Proof. The differential squares to zero because $d \circ d = j_* \circ \partial \circ j_* \circ \partial$ where $j_* : H_n(X_n) \to H_n(X_n, X_{n-1})$, and $\partial \circ j_* = 0$ as these are two consecutive maps in the long exact sequence of the pair (X_n, X_{n-1}).

Thanks to the previous lemma, the long exact sequence of (X_{n+1}, X_n) gives:

$$\cdots \to H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial_{n+1}} H_n(X_n) \xrightarrow{i_n} H_n(X_{n+1}) = H_n(X) \to 0.$$

It follows that $H_n(X) = H_n(X_n)/\operatorname{im}(\partial_{n+1})$. Moreover, from the long exact sequence of (X_n, X_{n+1}), we get that:

$$0 \to H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}) \xrightarrow{\partial_n} H_{n-1}(X_{n-1}) \to \cdots$$

It follows that j_n is injective and carries isomorphically $\operatorname{im}(\partial)$ to $\operatorname{im}(j_n\partial_{n+1}) = \operatorname{im}(d_n)$, and $H_n(X_n)$ to $\operatorname{im}(j_n) = \ker(\partial_n)$. Moreover, j_{n-1} is also injective, thus $\ker(\partial_n) = \ker(j_{n-1}\partial_n) = \ker(d_n)$. It follows that

$$H_n(X) = H_n(X_n)/\operatorname{im}(\partial_{n+1}) \cong \operatorname{im}(j_n)/\operatorname{im}(\partial_{n+1}) \cong \ker(d_n)/\operatorname{im}(d_n).$$ □

invariance of domain, hopf theorem

next steps: covering space theory, cohomology, categorical language

References

[1] Tammo tom Dieck. *Algebraic topology.* EMS Textbooks in Mathematics. Zürich: EMS, 2008. 567 pp. ISBN: 978-3-03719-048-7. DOI: 10.4171/048.

[2] Allen Hatcher. *Algebraic topology.* Cambridge: Cambridge Univ. Press, 2002. 544 pp. ISBN: 0-521-79160-X. URL: http://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[3] Edwin Spanier. *Algebraic Topology.* Berlin: Springer, 1995. 528 pp. ISBN: 978-1-4684-9322-1. DOI: 10.1007/978-1-4684-9322-1.

[4] Antoine Touzé. *Topologie Algébrique.* Lecture notes. École Normale Supérieure. URL: https://pro.univ-lille.fr/antoine-touze/enseignements.